Complexity Results in Graph Reconstruction

Stanisław Radziszowski,

Rochester Institute of Technology,

MFCS 2004

List of Coauthors

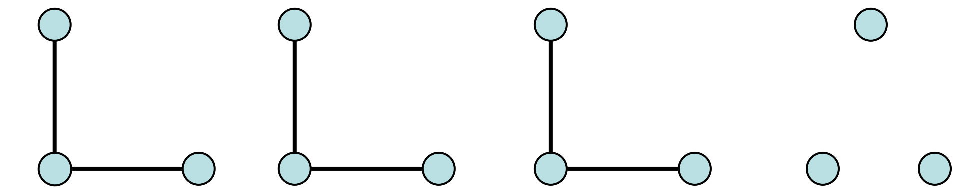
- Edith Hemaspaandra
  - Rochester Institute of Technology, NY
- Lane Hemaspaandra
  - University of Rochester, NY
- Rahul Tripathi
  - University of Rochester, NY

Reconstruction Problems in Graph Theory

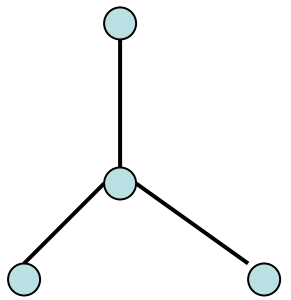
Reconstruction Conjecture [Kel42,Ula60]. Every finite simple undirected graph on ≥ 3 vertices is determined uniquely (up to isomorphism) by its collection of 1-vertex-deleted subgraphs.

Edge-Reconstruction Conjecture [Har64]. Every finite simple undirected graph with ≥ 4 edges can be reconstructed from its collection of 1-edge-deleted subgraphs.

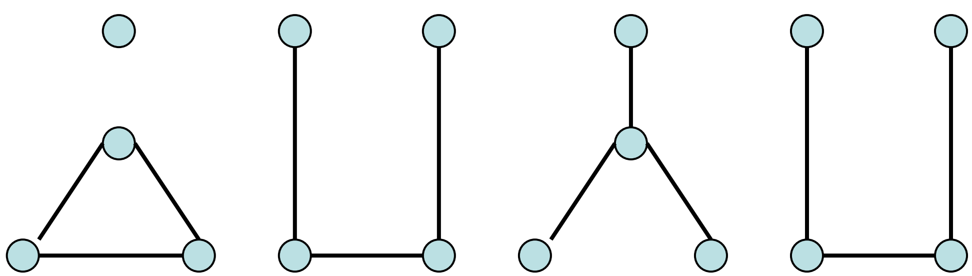
Examples of Vertex- and Edge-Reconstructions



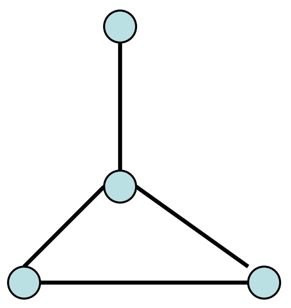
G_1 G_2 G_3 G_4



Unique Vertex-preimage G



G_1 G_2 G_3 G_4



Unique Edge-preimage G

Basic Questions on Reconstruction of Graphs

- Vertex/Edge-Deck Checking Problem (VDC/EDC).
  - Given a graph G and a collection D of graphs, whether G is a preimage of D?

- Legitimate Vertex/Edge-Deck Problem (LVD/LED).
  - Given a collection of graphs, whether the collection is a legitimate?

More General Questions on Graph Reconstruction

- VDC_c : Given $\langle G; [G_1, \ldots, G_n]\rangle$, is it the case that $[G_1, \ldots, G_n] = \text{vertex-deck}_c(G)$.

- EDC_c : Given $\langle G; [G_1, \ldots, G_m]\rangle$, is it the case that $[G_1, \ldots, G_m] = \text{edge-deck}_c(G)$.

More General Questions on Graph Reconstruction

- LVD_c : Given $\langle[G_1, \ldots, G_n]\rangle$, is there a graph G such that $[G_1, \ldots, G_n]$ = vertex-$deck_c(G)$.

- LED_c : Given $\langle[G_1, \ldots, G_m]\rangle$, is there a graph G such that $[G_1, \ldots, G_m]$ = edge-$deck_c(G)$.

More General Questions on Graph Reconstruction

- For any fixed $k \geq 2$, problems k-VDC_c, k-EDC_c, k-LVD_c, and k-LED_c can be defined.

- k-VDC_c : Given $\langle G; [G_1, \ldots, G_k]\rangle$, is it the case that $[G_1, \ldots, G_k] \subseteq$ vertex-$deck_c(G)$.

Summary of Our Results

- We show that, for all suitable choices of parameters c and k, these problems are either logspace/polynomial-time isomorphic to the Graph Isomorphism Problem (GI) or, in some cases, many-one hard for GI.
  - Strengthen a result of Mansfield [Man82].
  - Extend the results of Kratsch and Hemaspaandra [KH94].
  - Obtain new complexity results on reconstruction of graphs.

A Sample of Our Results

Theorem. For all $c \geq 1$ and $k \geq 2$, GI is polynomial-time isomorphic to k-LED$_\mathrm{C}$.

Key Steps:

1. We first show that k-LED$_\mathrm{C} \leq^p_{dtt}$ GI and then conclude that k-LED$_\mathrm{C} \leq^p_m$ GI, since $R^p_{dtt}(\mathrm{GI}) = R^p_m(\mathrm{GI})$.

2. We show that GI $\leq^p_m$ k-LED$_\mathrm{C}$:

$$(G, H) \xrightarrow{\sigma} \cup_{i=1}^{k-1} \; [G \cup (K_\ell - S_{\ell,i}) \cup K_{\ell+1}] \\ \cup \; [H \cup K_\ell \cup (K_{\ell+1} - S_{\ell+1,1})].$$

Here, G connected and $\ell > \max\{n, k\}$.

A Result on Legitimate Vertex-Deck

Theorem. For all $c \geq 1$ and $k \geq 2$, GI $\leq_m^l$ k-$\mathrm{LVD_C}$. In particular, for all $c \geq 1$, GI $\equiv_{iso}^p$ 2-$\mathrm{LVD_C}$.

Reconstruction Number of Undirected Graphs

- Ally-reconstruction Number [HP85,Myr89]: the minimum number of one-vertex-deleted subgraphs of a graph G that identify G uniquely (up to isomorphism).
- We call this number $vrn_{\exists}(G)$ and define analogous reconstruction numbers $ern_{\exists}(G)$, $vrn_{\forall}(G)$, and $ern_{\forall}(G)$.
- For instance, $ern_{\forall}(G)$ is the minimum number (k) of one-edge-deleted subgraphs (cards) of G such that every collection of k one-edge-cards of G identify G uniquely (up to isomorphism).

Number of Reconstructions

Lemma. For all $n \geq 4$, there is a disconnected graph G_n such that $|V(G_n)| = n$ and $\mathrm{vrn}_{\exists}(G_n) < \mathrm{vrn}_{\forall}(G_n)$.

Theorem. For all $k \geq 2$ and $n \geq 1$, there is a deck of k vertex-cards on $(2^{k-1} + 1)n + k$ vertices with at least 2^n one-vertex-preimages.

Problems

- Characterize the hardness of the following problems about reconstruction numbers:

a) $\{\langle G, k\rangle \mid vrn_{\exists}(G) \leq k\} \in \Sigma_2^p$.
b) $\{\langle G, k\rangle \mid vrn_{\forall}(G) \leq k\} \in \mathsf{coNP}^{\mathsf{GI}}$.
c) $\{\langle G, k\rangle \mid ern_{\exists}(G) \leq k\} \in \mathsf{NP}^{\mathsf{GI}}$.
d) $\{\langle G, k\rangle \mid ern_{\forall}(G) \leq k\} \in \mathsf{coNP}^{\mathsf{GI}}$.

Thank You