Reconstruction Numbers of Small Graphs

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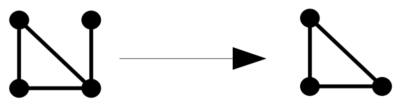
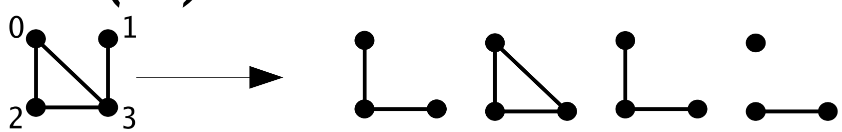
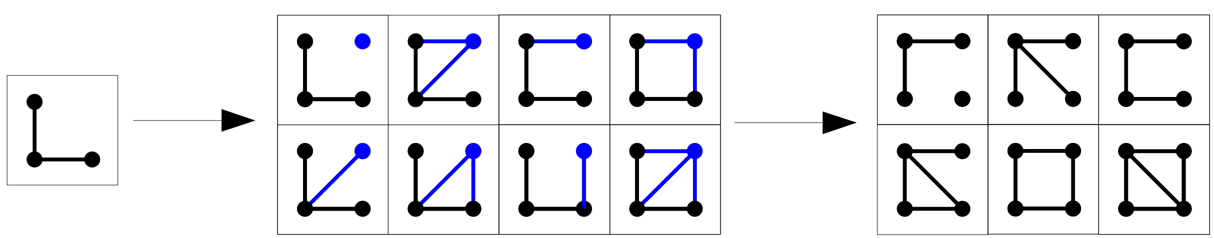
Definitions

- Card of *G*: one vertex removed
  - 
- Deck*(G)*: multiset of all cards of *G*
  - 

- Extensions*(G)*: all graphs on |V*(G)*|+1 vertices with induced *G*
  - 
- Edge-deleted versions: εCard, εDeck, εExtensions

Reconstruction Conjectures

- Any graph on 3 or more vertices can be reconstructed its 1-vertex-deleted subgraphs (Kelly, Ulam - 1941)
- Any graph on 4 or more edges can be uniquely identified by the multiset of its 1-edge-deleted subgraphs (Harary - 1964)
- Reconstruction Numbers: (Harary, Plantholt - 1985)
  - ∃rn is the smallest number of cards required to reconstruct
  - ∀rn is the minimum number such that any set of ∀rn cards can reconstruct

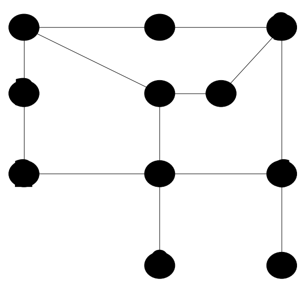
Recent Results

- $\exists vrn(G) = \exists vrn(\overline{G})$, $\forall vrn(G) = \forall vrn(\overline{G})$
  (Harary, Plantholt - 1985)
- Almost every graph has $\exists vrn = \forall vrn = 3$
  (Myrvold - 1988; Bollobás - 1990)
- Almost every graph has $\exists ern = \forall ern = 2$
  (Lauri - 1992)
- All $\exists vrn$ and $\forall vrn$ up to 10 vertices
  (McMullen, Radziszowski - 2005, 2007)
- Families of graphs with $\forall vrn = 2\lfloor \frac{1}{3}(n-1) \rfloor$
  (Bowler, Brown, Fenner - 2006)

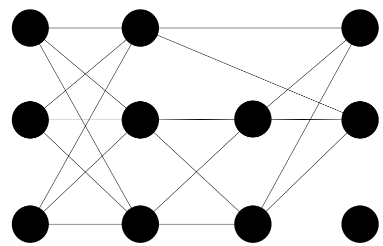
1-Vertex-Deletion Results

		graph order								
		3	4	5	6	7	8	9	10	11
$\exists vrn_1$	3	4	8	34	150	1,044	12,334	274,666	12,005,156	1,018,997,864
	4		3		4		8		6	
	5				2		2	2	4	
	6						2			
	7								2	
$\forall vrn_1$	3	4	2	7	8	16	266	45,186	6,054,148	815,604,300
	4		9	19	56	496	8,308	199,247	5,637,886	199,382,868
	5			8	90	520	3,584	28,781	301,530	3,922,130
	6				2	12	284	1,434	10,686	83,730
	7						4	20	914	4,824
	8								4	12

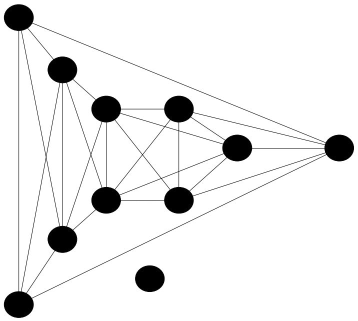
Maximizing ∀vrn on 11 vertices



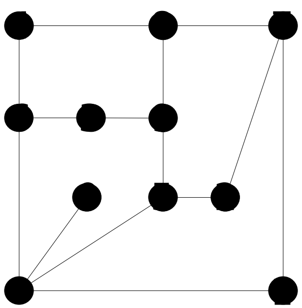
$|E(G)|=13$



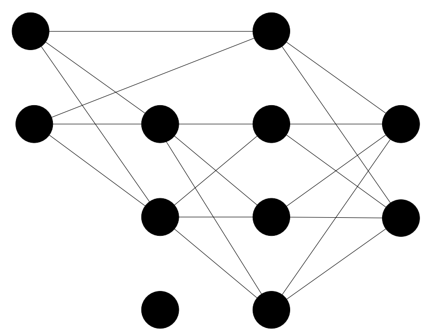
$|E(G)|=19$



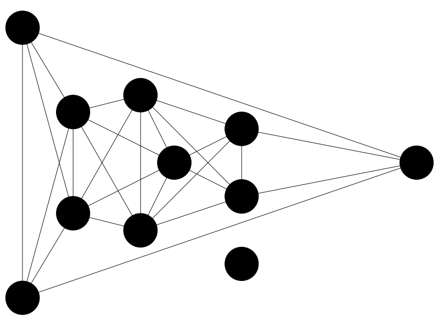
$|E(G)|=25$



$|E(G)|=14$



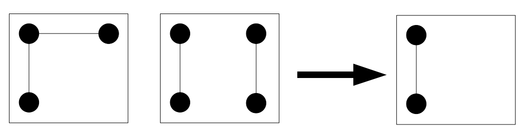
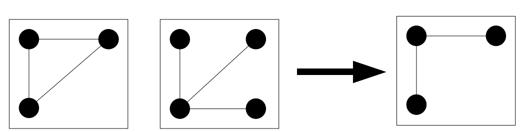
$|E(G)|=20$



$|E(G)|=26$

1-Edge-Deletion Results

		graph order								
		3	4	5	6	7	8	9	10	11
not reconstructible		0	4	4	4	4	4	4	4	4
$\exists ern_1$	1	3	5	9	18	23	35	46	64	71
	2			14	115	980	12242	274523	12004951	1018997596
	3		1	6	16	31	57	81	130	167
	4				2	5	4	9	10	15
	5						3	3	5	6
	6							1	2	2
	7								1	1
	8									1

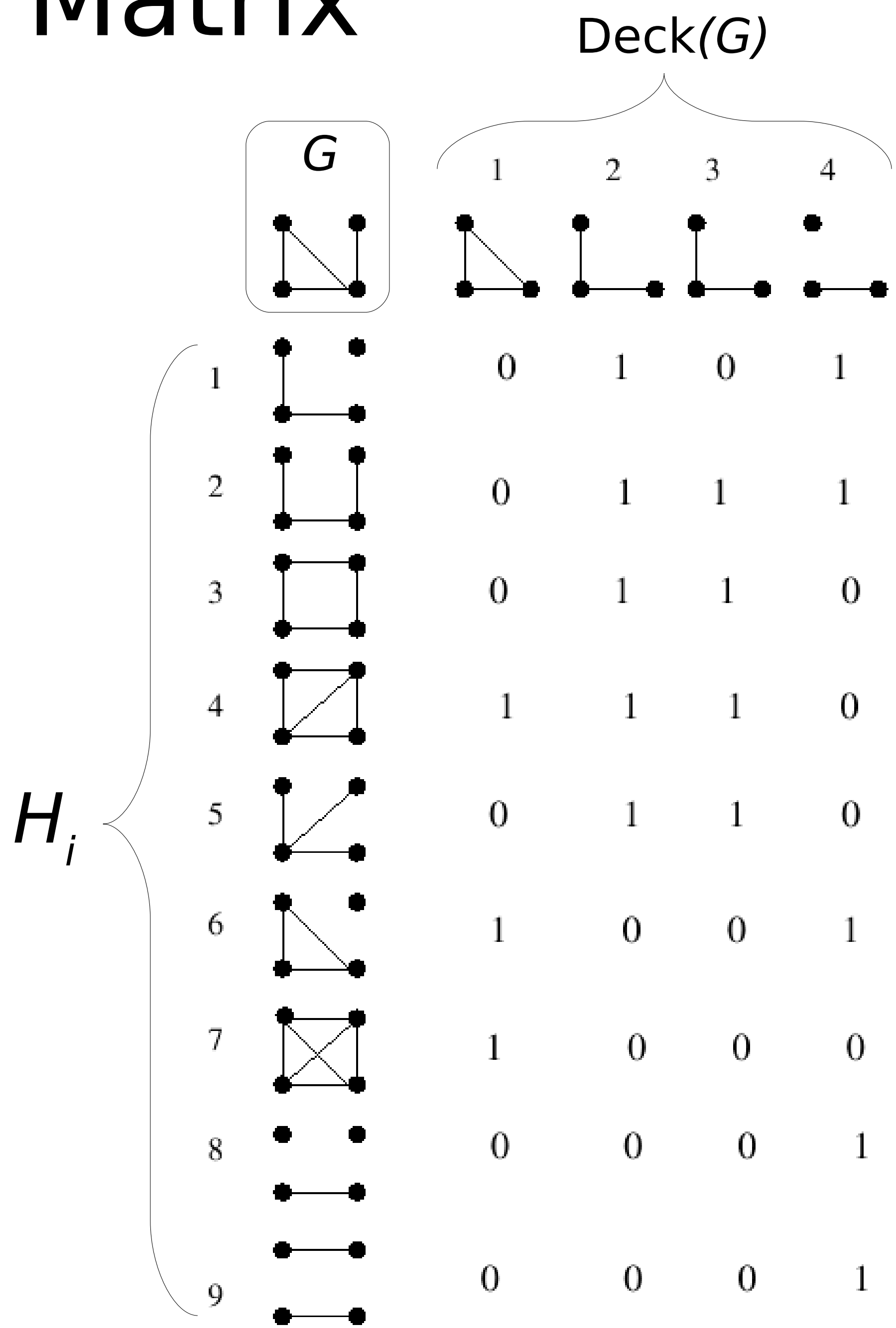
- The non-reconstructible graphs are:
  - P_3 and $2K_2$  x 2
  - K_3 and $K_{1,3}$  x 3

1-Edge-Deletion Results

		graph order								
		3	4	5	6	7	8	9	10	11
$\forall ern$	1	3	3	3	3	3	3	3	3	3
	2			2	14	19	51	152	1591	2479879
	3		3	8	28	131	1,622	65,814	5,895,154	748,858,136
	4			6	36	285	5,059	141,767	4,976,002	239,960,040
	5			8	46	394	3,880	50,196	925,253	24,213,068
	6			2	15	128	952	10,379	138,350	2,533,007
	7				5	41	520	4,171	47,953	711,284
	8				4	20	136	1,228	11,382	141,498
	9					12	55	521	5,704	67,083
	10					4	26	202	1,854	18,352
	11					2	21	110	1070	9050
	12						8	57	359	2615
	13						4	37	292	2562
	14						2	10	68	512
	15						2	10	66	376
	16							2	23	188
	17								19	106
	18							2	8	30
	19								2	26
	20							2	2	10
	21								2	6
	22								2	8
	23								2	4
	24									2
	25									6
	26								2	2
	28									4
	33									2

Relation Matrix

- Graphs H_i are extensions of card in Deck(G)
- Reconstruction Numbers can be computed by relating each H_i to cards it shares with G
- Construction requires canonically labeling $O(n^2 2^n)$ graphs ($n=|V(G)|$)



Special Sauce

- If *H* is an extension of *C* then *C* must be in Deck*(H)* at least once
- Only the subset of Deck*(H)* that intersects Deck*(G)* matters
- if *C* is in Deck(*G*) exactly once, then computing Deck(Extensions(*C*)) is unnecessary
  - This is true over 97% of the time for |V(*G*)|=11
- Brenden McKay's *Naughty* used to canonically label graphs efficiently

Algorithm

- $\mathcal{D}_G \leftarrow \text{Deck}(G)$
- For each $C_i \in \mathcal{D}_G$:
  - $\mathcal{H}_i \leftarrow \text{Extensions}(C_i) - G$
  - For each $H \in \mathcal{H}_i$

    $m(\mathcal{H}_i; H) \leftarrow \min(\, m(\text{Deck}(H); C_i), m(\mathcal{D}_G; C_i)\,)$

    note: $m(\mathcal{D}_G; C_i) = 1 \;\Rightarrow\; m(\mathcal{H}_i, H) = 1$
- $\forall\text{rn}(G) \leftarrow 1 + \max(\, m(\mathcal{H}; H); H \in \mathcal{H})$
  - where $\mathcal{H} = \biguplus_{C_i \in \mathcal{D}_G} \mathcal{H}_i$
- $\exists\text{rn}(G) \leftarrow \min(\, |S|; (S \subseteq \mathcal{D}_G) \wedge (\bigcap_{C_i \in S} \mathbb{B}(\mathcal{H}_i; m(S; C_i)))\,)$
  - where $\mathbb{B}(\mathcal{H}_i; m) = \{H \mid m(\mathcal{H}_i; H) \geq m\}$

Computation Time

- Most computations on Opteron 248 (2.2GHz) CPUs
- Larger computations used RIT CASCI Cluster (94 1.4GHz P-III CPUs)

		1-vertex deletion		1-edge deletion	
|V(G)|	unique graphs	total CPU time	ms per graph	total CPU time	ms per graph
6	156	0.02 seconds	0.26	0.04 seconds	0.25
7	1044	0.52 seconds	1.07	0.74 seconds	0.71
8	12346	16.8 seconds	2.45	16.3 seconds	1.32
9	274668	14.0 minutes	5.52	10 minutes	2.18
10	12005168	20.9 hours	12.6	10.7 hours	3.21
11†	1018997864	174 days	29.5	23.7 days	4.02

†Computations performed on the CASCI cluster and normalized to Opteron 248 performance

- 165,091,172,592 graphs on 12 vertices
  - Est. vertex-deletion: ~200 equivalent CPU-years
  - Est. edge-deletion: ~12 equivalent CPU-years

Some Open Questions

- If $\exists\mathrm{rn}(G) < \forall\mathrm{rn}(G)$, then how many subdecks of size *s* reconstruct G?
- Reconstruction from other types of cards:
  - add vertex in every possible way
  - complement an edge in every possible way
  - add *or* subtract a vertex in every possible way
- What is the Meta-graph of graphs on n vertices with edges colored by number of shared cards?
- What is the relationship between $\forall\mathrm{vrn}(G)$ and $\forall\mathrm{vrn}(G \cup K_1)$?

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