Bounds on Some Ramsey Numbers Involving Quadrilateral

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40th SE CCGTC, Boca Raton, March 2009

Outline

1. Previous Work
   - Ramsey numbers avoiding C_4
2. Our Contributions
   - Summary of old and new results
   - Upper bounds
   - Lower bounds
3. What to do next?

Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.

- $2 - colorings \cong graphs, \quad R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$
- Avoiding C_4, $|N(v) \cap N(u)| \leq 1$

- Theorem (Ramsey 1930): Ramsey numbers exist

Asymptotics

Ramsey numbers avoiding C_4

- Spencer - 1977

$$c_1\left(\frac{n}{\log n}\right)^{3/2} \leq R(C_4, K_n)$$

- Caro, Li, Rousseau, Zhang - 2000
credit to Erdös, Szemerédi - 1980 (unpublished)

$$R(C_4, K_n) \leq c_2\left(\frac{n}{\log n}\right)^2$$

- Kim - 1995

$$R(C_3, K_n) = \Theta\left(\frac{n^2}{\log n}\right)$$

Basic cases and connections

C_4 versus K_n

- $R(C_4, K_n) = 7, 10, 14, 18, 22, 26$ for $n = 3, \cdots, 8$
- First open cases:
  $30 \leq R(C_4, K_9) \leq 32$, $34 \leq R(C_4, K_{10}) \leq 39$
- This is the OTHER end of the
  Erdös-Faudree-Rousseau-Schelp conjecture (1978)

$$R(C_n, K_m) = (n-1)(m-1) + 1$$

for all $n \geq m \geq 3$

Basic cases and connections

Irving, Chung, Graham, Parsons, Lortz, Mengersen, Monte Carmelo, and many others ...

- C_4 versus stars, trees, books, wheels
- Connects to projective planes
- Connects to Hadamard matrices
- Connects to much studied case $R(K_{2,k}, K_{m,n})$

Multicolor cases

- $k^2 + 2 \le R_k(C_4) \le k^2 + k + 1$
  lower bound for prime power k

  Irving, Chung, Graham (1970's)
  Lazebnik, Woldar, Ling, Mubayi (2000's)

- $R_3(C_4) = 11$
  Bialostocki/Schönheim 1984, Clapham 1987
- $R_4(C_4) = 18$
  amazing computation by Sun/Yang/Lin/Zheng 2007
- $27 \le R_5(C_4) \le 29$
  just math, Lazebnik/Woldar 2000

Strange multicolor asymptotics

- Sun/Yang/Lin/Zheng 2007 (computations)

$$R(C_4, C_4, C_n) = n + 2 \text{ for } n \geq 11$$

- Shiu/Lam/Li 2003

$$c_3 \left(\frac{n}{\log n}\right)^{3/2} \leq R(C_4, C_4, K_n) \leq c_4 \left(\frac{n}{\log n}\right)^2$$

- Alon/Rödl 2005

$$R(C_4, C_4, K_n) = \tilde{\Theta}(n^2)$$

$$R(C_4, C_4, \cdots, C_4, K_n) = \Theta(n^2)$$

Three colors

$R(C_4, G_1, G_2)$	value/bounds	reference
C_4, C_4, C_4	11	[BiaSch]
C_4, C_4, C_3	12	[Schul]
C_4, C_4, K_4	19-22	
C_4, C_3, C_3	17	[ExRe]
C_4, C_3, K_4	25-32	
C_4, K_4, K_4	52-72	

Table 1. $R(C_4, G_1, G_2)$ for $G_1, G_2 \in \{C_4, C_3, K_4\}$

Four colors

$R(C_4, C_4, G_1, G_2)$	value/bounds	reference
C_4, C_4, C_4, C_4	18	[SYLZ]
C_4, C_4, C_4, C_3	21-27	[XuRad]
C_4, C_4, C_4, K_4	31-50	
C_4, C_4, C_3, C_3	28-36	[XuRad]
C_4, C_4, C_3, K_4	42-76	
C_4, C_4, K_4, K_4	87-179	

Table 2. $R(C_4, C_4, G_1, G_2)$ for $G_1, G_2 \in \{C_4, C_3, K_4\}$

Counting edges

Definition: $t_4(n)$ = max# edges in n-vertex C_4-free graphs

Lemma: For any n-vertex C_4-free graph G, $n > 3$,
(1) $|E(G)| \leq t(n) < \frac{1}{4}n(1 + \sqrt{4n - 3})$,
(2) $\delta(G) < \frac{1}{2}(1 + \sqrt{4n - 3})$.

- $t_4(n)$ known for $n \leq 32$, hard to go any further
- $R(C_4, K_9) \leq 32$
- $R(C_4, C_4, K_4) \leq 22$

Lower bound constructions

Two means of improving lower bounds

- Explicit computer constructions
  e.g. $19 \leq R(C_4, C_4, K_4)$
- Extensions of known constructions
  e.g. $28 \leq R(C_4, C_4, K_3, K_3)$

Summary

- Closing in on several small cases
- C_4 seems easier than K_3

- Next tasks - compute exactly
  - $19 \leq R(C_4, C_4, K_4) \leq 22$ doable
  - $30 \leq R(C_4, K_9) \leq 32$ hard
  - $27 \leq R_5(C_4) \leq 29$ very hard
  - Asymptotics for $R(C_4, K_n)$ nice

Papers

SPR's C_4-papers to pick up

- Kung-Kuen Tse, SPR. A Computational Approach for the Ramsey Numbers $R(C_4, K_n)$, JCMCC 42 (2002) 195-207.
- Xu Xiaodong, SPR. $28 \leq R(C_4, C_4, C_3, C_3) \leq 36$, to appear in Utilitas Mathematica.
- Xiaodong Xu, Zehui Shao, SPR. Bounds ... (this talk), Ars Combinatoria, 90 (2009) 337-344.

- Revision #12 of the survey paper *Small Ramsey Numbers* at the ElJC coming in the summer 2009 ...