Wheel and Star-critical Ramsey Numbers for Quadrilateral

Wu Yali, Sun Yongqi*

School of Computer and Information Technology
Beijing Jiaotong University
Beijing 100044, P. R. China

yqsun@bjtu.edu.cn

and

Stanisław Radziszowski†

Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623, USA

spr@cs.rit.edu

Abstract

The star-critical Ramsey number $r_*(H_1, H_2)$ is the smallest integer k such that every red/blue coloring of the edges of $K_n - K_{1,n-k-1}$ contains either a red copy of H_1 or a blue copy of H_2, where n is the graph Ramsey number $R(H_1, H_2)$. We study the cases of $r_*(C_4, C_n)$ and $R(C_4, W_n)$. In particular, we prove that $r_*(C_4, C_n) = 5$ for all $n \geqslant 4$, obtain a general characterization of Ramsey-critical (C_4, W_n)-graphs, and establish the exact values of $R(C_4, W_n)$ for 9 cases of n between 18 and 44.

Keywords: Ramsey number; wheel; cycle; Hamiltonian graph

Mathematics Subject Classifications: 05C55, 05C38

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph $G = (V(G), E(G))$, we denote the order of G by $p(G) = |V(G)|$. The Ramsey *arrowing* operator $\rightarrow$ is a logical predicate, which holds for graphs G, H_1 and H_2, written $G \rightarrow (H_1, H_2)$, if and only if for all partitions $E(G) = E_1 \cup E_2$ into two sets (colors) E_1 contains H_1 or E_2 contains H_2. The *Ramsey number* $R(H_1, H_2)$ is the smallest n such that $K_n \rightarrow (H_1, H_2)$. Any edge 2-coloring witnessing $K_n \not\rightarrow (H_1, H_2)$ will be called an $(H_1, H_2; n)$-*coloring*, which can be seen as a graph not containing H_1 and without H_2 in the complement. The *star-critical Ramsey number* $r_*(H_1, H_2)$ is the smallest k such that $K_n - K_{1,n-k-1} \rightarrow (H_1, H_2)$, where $n = R(H_1, H_2)$ [12].

If $V(G) \cap V(H) = \emptyset$, then the graph $G + H$ on vertices $V(G) \cup V(H)$ has the edges $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced in G by S, and $G \setminus S = G[V(G) \setminus S]$. For $v \in S$, let $N_{G[S]}(v) = \{u : u \in S \wedge uv \in E(G)\}$ and $d_{G[S]}(v) = |N_{G[S]}(v)|$. If $S = V(G)$, we simply write $N(v)$, $d(v)$, and $N[v] = N(v) \cup \{v\}$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees in G, respectively. $\alpha(G)$ denotes the order of the maximum independent set in G, $\kappa(G)$ is the vertex connectivity of G. P_k is the path on k vertices, C_k is the cycle of length k, T_k is a k-vertex tree, and W_{k+1} is the wheel graph, where a hub is connected by k spokes to C_k. $K_{m,n}$ is the complete $m \times n$ bipartite graph, in particular $K_{1,n}$ is the star graph. K_n^m is the complete m-partite graph with each part of order n.

It is known that $R(C_4, W_4) = 10$, $R(C_4, W_5) = 9$ and $R(C_4, W_6) = 10$ (cf. [18]). Tse [21] determined the values of $R(C_4, W_m)$ for $7 \leqslant m \leqslant 13$. Dybizbański and Dzido [7] proved that $R(C_4, W_m) = m + 4$ for $14 \leqslant m \leqslant 16$, and $R(C_4, W_{q^2+1}) = q^2 + q + 1$ for prime powers $q \geqslant 4$. They also gave an upper bound on $R(C_4, W_m)$ for $m \geqslant 11$. The concept of star-critical Ramsey numbers was introduced by Hook and Isaak [12]. They proved that $r_*(C_4, C_3) = 5$, $r_*(T_n, K_m) = (n-1)(m-2) + 1$, $r_*(nK_2, mK_2) = m$ for $n \geqslant m$, and $r_*(C_4, P_n) = 3$ for $n \geqslant 3$.

Recall that $R(C_4, C_n) = n + 1$ for $n \geqslant 6$ [14]. The main results of this paper are as follows:

Theorem 1. *For all $n \geqslant 6$, any $(C_4, C_n; n)$-graph is in one of the graph sets $\mathcal{F}_i$, $1 \leqslant i \leqslant 4$, as in Definition 4.*

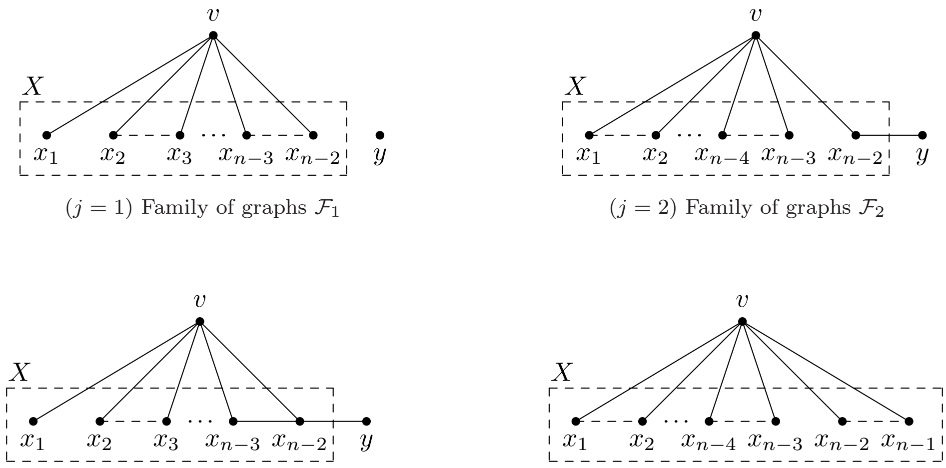
Theorem 2. $r_*(C_4, C_n) = 5$ *for all* $n \geqslant 4$.

Theorem 3. $$R(C_4, W_m) = \begin{cases} m+4, & \textit{for } 18 \leqslant m \leqslant 21, \\ m+5, & \textit{for } m = 27, \\ m+6, & \textit{for } 35 \leqslant m \leqslant 37, \textit{ and} \\ m+7, & \textit{for } m = 44. \end{cases}$$

Definition 4. Graph sets $\mathcal{F}_j$, $1 \leqslant j \leqslant 4$, are defined on vertices $\{v, x_1, \ldots, x_{n-2}, y\}$. We present them in Figure 1. In each case the distinguished vertex $v \in V(F_j^i)$ is of maximum degree, $X = N(v)$, and X induces i disjoint edges iK_2 in F_j^i. We describe these graphs in detail as follows.

(1) $F_1^i \in \mathcal{F}_1$, $d(v) = n-2$, and $N(y) = \emptyset$;
$F_1^i[X] = (n-2i-2)K_1 \cup iK_2$ for $0 \leqslant i \leqslant (n-2)/2$.
(2) $F_2^i \in \mathcal{F}_2$, $d(v) = n-2$, $N(y) = \{x_{n-2}\}$, and $d_{F_2^i[X]}(x_{n-2}) = 0$;
$F_2^i[X] = (n-2i-2)K_1 \cup iK_2$ for $0 \leqslant i \leqslant (n-3)/2$.
(3) $F_3^i \in \mathcal{F}_3$, $d(v) = n-2$, $N(y) = \{x_{n-2}\}$, and $d_{F_3^i[X]}(x_{n-2}) = 1$;
$F_3^i[X] = (n-2i-2)K_1 \cup iK_2$ for $1 \leqslant i \leqslant (n-2)/2$.
(4) $F_4^i \in \mathcal{F}_4$, $y = x_{n-1}$, and $d(v) = n-1$;
$F_4^i[X] = (n-2i-1)K_1 \cup iK_2$ for $0 \leqslant i \leqslant (n-1)/2$.

In all cases (i,j), one can easily see that the graphs F_j^i have no C_4, their complements have no C_n, and thus all of them are $(C_4, C_n; n)$-graphs.



($j=1$) Family of graphs $\mathcal{F}_1$ ($j=2$) Family of graphs $\mathcal{F}_2$

($j=3$) Family of graphs $\mathcal{F}_3$ ($j=4$) Family of graphs $\mathcal{F}_4$, $y = x_{n-1}$

Figure 1: Structure of graphs in $\mathcal{F}_j$ for $1 \leqslant j \leqslant 4$.

Some of the known results which will be used in our proofs are summarized in the next two theorems.

Theorem 5. [14] $R(C_4, C_n) = \begin{cases} 7, & for\ n = 3,\ 5, \\ 6, & for\ n = 4,\ and \\ n+1, & for\ n \geqslant 6. \end{cases}$

Theorem 6. [6, 2, 3, 1] *Let G be any graph of order $n \geqslant 3$. If G satisfies any of the following conditions, then it is Hamiltonian:*
(a) $\delta(G) \geqslant \lceil n/2 \rceil$,
(b) For all $i < n/2$, either $d_i \geqslant i+1$ or $d_{n-i} \geqslant n-i$, where $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$ is the degree sequence,
(c) $\alpha(G) \leqslant \kappa(G)$, *or*
(d) G is 2-connected and $\sigma_3(G) \geqslant n + \kappa(G)$, where

$$\sigma_3(G) = \min\Big\{ \sum_{i=1}^{3} d(v_i) : \{v_1, v_2, v_3\} \text{ is an independent set in } G \Big\}.$$

2 Proof of Theorem 1

Lemma 7. *For a graph G of order $n+m+1$ for $n \geqslant m \geqslant 2, n \geqslant 4$, such that $C_4 \nsubseteq \overline{G}$, let v be a vertex of degree $\delta(G) = m$, $Y = N(v)$ and $X = V(G) - N[v]$, so $|X| = n$. If $K_2^t \subseteq G[X]$ for even n or $(K_1 + K_2^t) \subseteq G[X]$ for odd n ($t = \lfloor \frac{n}{2} \rfloor$), and each vertex of Y is adjacent to at least $n-1$ vertices of X, then G is Hamiltonian.*

Proof. Note that since $\delta(G) = m$ and $G \setminus Y$ is disconnected, we have $\kappa(G) = m$, and $C_4 \nsubseteq \overline{G}$ implies $\alpha(G) \leqslant 3$. If $m \geqslant 3$, then G is Hamiltonian by Theorem 6(c). So assume that $m = 2$, $Y = \{y_1, y_2\}$ and $X = \{x_1, x_2, \ldots, x_n\}$. We can see that $d(v) = 2$, $d(y_1), d(y_2) \geqslant n$, and $d(x_i) \geqslant n-2$ for $1 \leqslant i \leqslant n$. We will consider two cases: $n = 4$ and $n \geqslant 5$.

Suppose that $n = 4$, so $|V(G)| = 7$. If there is a vertex in X, say x_1, which is nonadjacent to y_1 or y_2, then y_1 (or y_2) is adjacent to each vertex in $\{x_2, x_3, x_4\}$, and we can easily find a Hamiltonian cycle in G. If each vertex of X is adjacent to y_1 or y_2, then the degree sequence of G is 2334444, and G is Hamiltonian by Theorem 6(b).

Finally, we can assume that $n \geqslant 5$. If T is an independent set of order 3 in G , then there are two subcases, say $T = \{x_1, y_1, y_2\}$ and $T = \{v, x_1, x_2\}$. If $T = \{x_1, y_1, y_2\}$, then $d(x_1) + d(y_1) + d(y_2) \geqslant 3n - 2$. If $T = \{v, x_1, x_2\}$, then we have $d(v) + d(x_1) + d(x_2) \geqslant 2n$, and hence $\sigma_3(G) = 2n$. Now, we conclude that G is Hamiltonian by Theorem 6(d). □

Proof of Theorem 1. First we prove that any $(C_4, C_n; n)$-graph G for $n \geqslant 8$ is isomorphic to one of the graphs in $\mathcal{F}_j$, $1 \leqslant j \leqslant 4$. Since $C_n \nsubseteq \overline{G}$, we have that $\overline{G}$ is not Hamiltonian. By Theorem 6(a), we have $\delta(\overline{G}) < \lceil \frac{n}{2} \rceil$ which implies $\Delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$. Let v be a vertex of maximum degree and $X = N_G(v) = \{x_1, x_2, \ldots, x_k\}$, $k \geqslant 4$. Since $C_4 \nsubseteq G$, we have that $G[X]$ is isomorphic to $(k-2i)K_1 \cup iK_2$ for some $i \leqslant t = \lfloor \frac{k}{2} \rfloor$. Hence we have $K_2^t \subseteq \overline{G}[X]$ for even k or $(K_1 + K_2^t) \subseteq \overline{G}[X]$ for odd k. Let $Y = N_{\overline{G}}(v)$, and observe that $|X| \geqslant |Y|$. Since $C_4 \nsubseteq G$, each vertex $y \in Y$ is adjacent to at most one vertex in X in G, that is, it is adjacent to at least $k-1$ vertices in X in $\overline{G}$. If $d_{\overline{G}}(v) \geqslant 2$, then $\overline{G}$ is Hamiltonian by Lemma 7. Hence we need to consider $d_{\overline{G}}(v) \leqslant 1$, that is, $d_G(v) = n-2$ or $d_G(v) = n-1$.

For $d_G(v) = n-2$, $Y = \{y\}$, since $C_4 \nsubseteq G$, y is adjacent to at most one vertex in X. In this situation $G[X]$ is isomorphic to $(n-2i-2)K_1 \cup iK_2$ for some $i \leqslant t = \lfloor \frac{n-2}{2} \rfloor$, which is F_1^i for $0 \leqslant i \leqslant (n-2)/2$, F_2^i for $0 \leqslant i \leqslant (n-3)/2$, or F_3^i for $1 \leqslant i \leqslant (n-2)/2$.

If $d_G(v) = n-1$, then $Y = \emptyset$. Now $G[X]$ is isomorphic to $(n-2i-1)K_1 \cup iK_2$, which is one of the graphs F_4^i for $0 \leqslant i \leqslant (n-1)/2$.

It remains to complete the proof for $n = 6, 7$. Using `geng` of `nauty` [15], we found that there are exactly 44 C_4-free graphs of order 6 and 117 C_4-free graphs of order 7. Among them, we found 10 $(C_4, C_6; 6)$-graphs and 12 $(C_4, C_7; 7)$-graphs, respectively, and we checked that all of them are isomorphic to one of the graphs in $\mathcal{F}_j$, $1 \leqslant j \leqslant 4$. □

3 Proof of Theorem 2

In 1963, Ore [17] defined a graph to be *Hamiltonian-connected* if there is a Hamiltonian path between every pair of distinct vertices (see also an early survey by Dean et al. [5]). Theorem 8 will be used in the proof of the following Lemma 9.

Theorem 8. [17] *Let G be a 2-connected graph with n vertices. If for every pair of nonadjacent vertices u and v we have $d(u) + d(v) \geqslant n + 1$, then G is Hamiltonian-connected.*

Hook and Isaak [12] proved that $r_*(C_4, C_3) = 5$. We will extend their result to $r_*(C_4, C_n)$ for all $n \geqslant 4$. Let $(K_1 + K_2^m)^-$ be the graph obtained by dropping one of the $2m$ edges between K_1 and K_2^m.

Lemma 9. *The graphs K_2^m, $(K_1 + K_2^m)^-$ and $K_1 + (K_1 + K_2^{m-1})^-$ are Hamiltonian-connected for all $m \geqslant 3$.*

Proof. Let u and v be any two nonadjacent vertices of G as in Lemma 9. If $G = K_2^m$, then $d(u) + d(v) = 4m - 4 \geqslant 2m + 1$. If $G = (K_1 + K_2^m)^-$, then $d(u) + d(v) \geqslant 4m - 4 \geqslant 2m + 2$. For $G = K_1 + (K_1 + K_2^{m-1})^-$, we notice that there is only one vertex of degree $\delta(G) = 2m-3$. Hence, we have $d(u) + d(v) \geqslant 4m - 5 \geqslant 2m + 1$. In all cases, these graphs are Hamiltonian-connected by Theorem 8. $\square$

Proof of Theorem 2. We first prove that $r_*(C_4, C_n) = 5$ for all $n \geqslant 7$. Let $\mathcal{G}$ denote the graph $K_{n+1} - K_{1,n-k}$ in this proof, $V(\mathcal{G}) = \{v_i : 1 \leqslant i \leqslant n + 1\}$, and $E(\mathcal{G}) = E(K_n) \cup \{v_i v_{n+1} : 1 \leqslant i \leqslant k\}$. Since $R(C_4, C_n) = n + 1$, hence it is sufficient to show that $\max\{k : \mathcal{G} \not\to (C_4, C_n)\} = 4$. For a red/blue coloring of the edges of $\mathcal{G}$ witnessing $\mathcal{G} \not\to (C_4, C_n)$, we use $\mathcal{G}^r$ and $\mathcal{G}^b$ to denote its red and blue subgraphs. Hence $C_4 \nsubseteq \mathcal{G}^r$ and $C_n \nsubseteq \mathcal{G}^b$. Let $H = \mathcal{G}^r[\{v_1, v_2, \ldots, v_n\}]$, and v_n be a vertex of maximum degree in H. By Theorem 1, we know that H is isomorphic to one of the graphs in $\mathcal{F}_j$, $1 \leqslant j \leqslant 4$.

We first consider the case $H = F_1^0$, and suppose $E(H) = \{v_i v_n : 1 \leqslant i \leqslant n - 2\}$. Since $C_4 \nsubseteq \mathcal{G}^r$, v_{n+1} is adjacent to at most one vertex v_i for $1 \leqslant i \leqslant n-2$. Together with $v_{n-1}v_{n+1}, v_n v_{n+1} \in E(\mathcal{G}^r)$, there are at most three red edges between v_{n+1} and $V(H)$. Since $F_1^0 \subseteq H$ for any $H \in \mathcal{F}_j$, then in all cases there are also at most three red edges between v_{n+1} and $V(H)$.

Next we consider the graph $\overline{H}$, and set $W = \overline{H} \setminus \{v_n\}$ and $m = \lfloor (n-1)/2 \rfloor$. If n is even, then $(K_1 + K_2^m)^- \subseteq W$. Lemma 9 and $C_n \nsubseteq \mathcal{G}^b$ imply that v_{n+1} is adjacent to at most one vertex of $V(W)$ in $\mathcal{G}^b$. If n is odd, then $K_2^m \subseteq W$ or $(K_1 + (K_1 + K_2^{m-1})^-) \subseteq W$. By Lemma 9 and $C_n \nsubseteq \mathcal{G}^b$, we also see that v_{n+1} is adjacent to at most one vertex of $V(W)$ in $\mathcal{G}^b$. So, $\max\{k : \mathcal{G} \not\to (C_4, C_n)\} = 4$, and the theorem holds for all $n \geqslant 7$.

For the special cases of $n = 4, 5, 6$, we have $R(C_4, C_n)$ equal to 6, 7 and 7, respectively. Hence we need to show that $K_6 - e \not\to (C_4, C_4)$, $K_7 - P_3 \not\to (C_4, C_n)$ and $K_7 - e \to (C_4, C_n)$ for $n = 5, 6$. The number of potential counterexamples (similarly as in the proof of Theorem 1) is very small, and we checked that none exist. Hence, $r_*(C_4, C_n) = 5$ for all $n \geqslant 4$. $\square$

4 Proof of Theorem 3

The *girth* of a graph G is the length of its shortest cycle. A k-regular graph with girth g is called a (k, g)-*graph*. When the number of vertices in the (k, g)-graph is minimized then we call it a (k, g)-cage. We use $ex(n, C_4)$ to denote the maximum size of a C_4-free graph of order n. The graph of size $ex(n, C_4)$ is called an *extremal* graph, and let $EX(n, C_4)$ denote the set of all corresponding extremal graphs. Clapham, Flockhart and Sheehan [4] gave the exact values of $ex(n, C_4)$ for $n \leqslant 21$ and the graphs in $EX(n, C_4)$. Yang and Rowlinson [23] determined the exact values of $ex(n, C_4)$ for $22 \leqslant n \leqslant 31$ and the corresponding extremal graphs. Recently, Shao, Xu and Xu [20] established that $ex(32, C_4) = 92$. It was conjectured by Erdős that for $n = q^2 + q + 1$, where q is a prime power, $ex(n, C_4) = \frac{1}{2}q(q + 1)^2$. That is, the Erdős-Renyi graph ER_q has the optimal number of edges and is a witness for $ex(n, C_4)$. In 1996, Füredi [10]

Table 1. The values of $ex(n, C_4)$ for $n \leqslant 32$

n	$ex(n, C_4)$	n	$ex(n, C_4)$	n	$ex(n, C_4)$
3	3	13	24	23	56
4	4	14	27	24	59
5	6	15	30	25	63
6	7	16	33	26	67
7	9	17	36	27	71
8	11	18	39	28	76
9	13	19	42	29	80
10	16	20	46	30	85
11	18	21	50	31	90
12	21	22	52	32	92

proved this conjecture for all $q > 13$. All known nontrivial values of $ex(n, C_4)$ for $n \leqslant 32$ are shown in Table 1.

Theorem 10. [7] *$R(C_4, W_m) \leqslant m + \sqrt{m-2} + 1$ for $m \geqslant 11$.*

Lemma 11. *(a) If G is a graph of order n and $\delta(G) > n - m$, then $W_m \nsubseteq \overline{G}$.*
(b) If there exists a $(k, 5)$-graph of order n, then $R(C_4, W_m) \geqslant n + 1$ for $m > n - k$.
(c) If G is a $(C_4, C_n; n)$-graph for $n \geqslant 6$, then $(K_1 \cup K_{1,n-2}) \subseteq G$.

Proof. For any graph G as in (a), $\Delta(\overline{G}) < m - 1$, hence $W_m \nsubseteq \overline{G}$, and (a) holds. For any $(k, 5)$-graph G of order n, since $\delta(G) = k$ and $C_4 \nsubseteq G$, G is a $(C_4, W_m; n)$-graph, and thus (b) holds by (a). Theorem 1 implies (c) which is equivalent to $\Delta(G) \geqslant n - 2$. $\square$

Lemma 12. *If G is a $(C_4, W_m; n)$-graph for $7 \leqslant m \leqslant n - 4$, then $\delta(G) > n - m$.*

Proof. Suppose that $\delta(G) \leqslant n - m$. Let v be a vertex with $d(v) = \delta(G)$ and $H = G[V(G) - N[v]]$. There are two cases to consider depending on $d(v)$.
Case 1. If $d(v) \leqslant n - m - 1$, then $d_{\overline{G}}(v)$ and $p(H) \geqslant m$. Since $C_4 \nsubseteq H$ and $R(C_4, C_{m-1}) = m$, we have $C_{m-1} \subseteq \overline{H}$. Then v together with some $m - 1$ vertices of $V(H)$ contains W_m in $\overline{G}$, a contradiction.
Case 2. If $d(v) = n - m$, then $p(H) = m - 1$, and let $N(v) = \{v_1, v_2, \ldots, v_{n-m}\}$. Note that $C_{m-1} \nsubseteq \overline{H}$, since otherwise $W_m \subseteq \overline{G}$. Therefore, since $C_4 \nsubseteq H$, H is a $(C_4, C_{m-1}; m-1)$-graph, and by Lemma 11(c), we have $(K_1 \cup K_{1,m-3}) \subseteq H$. Let x be the center of $K_{1,m-3}$, y the isolated vertex of $K_1 \cup K_{1,m-3}$, and $Z = V(H) \setminus \{x, y\} = \{z_1, z_2, \ldots, z_{m-3}\}$. Since $d(z_1) \geqslant n - m \geqslant 4$ and $C_4 \nsubseteq G$, z_1 has to be adjacent to y, one vertex of $N(v)$ and one vertex of Z, say $z_1v_1, z_1z_2 \in E(G)$. However, since $C_4 \nsubseteq G$, z_2 is adjacent to at most one vertex in $N(v) \setminus \{v_1\}$, which is a contradiction.

Cases 1 and 2 imply that $\delta(G) > n - m$. $\square$

Proof of Theorem 3. There are four sets of cases in the proof using Constructions 1, 4 and 5 in the Appendix.

(1) Cases $18 \leqslant m \leqslant 21$. The graphs $H_n, 21 \leqslant n \leqslant 24$, defined in Construction 1, and Lemma 11(a), imply $R(C_4, W_m) \geqslant m + 4$ for $18 \leqslant m \leqslant 21$. To prove the upper bounds, assume

that $R(C_4, W_m) > m + 4$ for some m, $18 \leqslant m \leqslant 21$, and let G be any $(C_4, W_m; m + 4)$-graph. By Lemma 12 we have $\delta(G) > 4$. However, the values of $ex(n, C_4)$ for $22 \leqslant n \leqslant 24$ (see Table 1) imply that $\delta(G) \leqslant 4$, which is a contradiction. Yang and Rowlinson [23] showed that there are exactly nine graphs H in $EX(25, C_4)$ (we obtained them from the authors). We checked that $\delta(H) = 4$ for all of them, a contradiction.

(2) Case $m = 27$. It is known that there are four $(5, 5)$-cages [9], and one of them is shown in Figure 2, denoted by H^a_{30}. Note that u_i is nonadjacent to u_j, and u_i is adjacent to $v_{i,j}$ for $0 \leqslant i, j \leqslant 4$ in H^a_{30}. We extend H^a_{30} to a $(C_4, W_{27}; 31)$-graph H_{31} by setting

$$V(H_{31}) = V(H^a_{30}) \cup \{w\} \text{ and}$$
$$E(H_{31}) = E(H^a_{30}) \cup \{wu_i : 0 \leqslant i \leqslant 4\}.$$

Note that $\delta(H_{31}) = 5$. By Lemma 11(a) we have $R(C_4, W_{27}) \geqslant 32$. For the upper bound

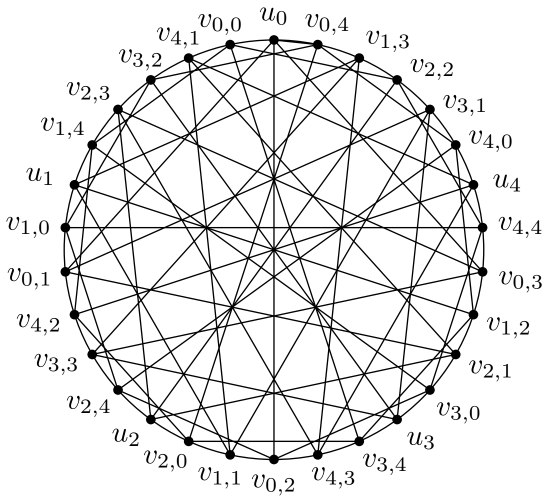


Figure 2: H^a_{30} [9].

assume that G is any $(C_4, W_{27}; 32)$-graph. By Lemma 12, we have $\delta(G) > 5$, a contradiction with $ex(32, C_4) = 92$. Hence $R(C_4, W_{27}) = 32$.

(3) Cases $35 \leqslant m \leqslant 37$. The $(6, 5)$-cage H_{40} (cf. [9]) and Lemma 11(a) imply $R(C_4, W_{35}) \geqslant 41$. The graphs H_{41} and H_{42} in Constructions 4 and 5 (in the Appendix), and Lemma 11(a) give $R(C_4, W_m) \geqslant m + 6$ for $m = 36$ and 37. We obtain $R(C_4, W_m) \leqslant m + 6$ for $35 \leqslant m \leqslant 37$ by Theorem 10, and thus $R(C_4, W_m) = m + 6$.

(4) Case $m = 44$. The $(7, 5)$-cage H_{50} (cf. [9]) and Lemma 11(a) imply $R(C_4, W_{44}) \geqslant 51$. Theorem 10 implies $R(C_4, W_{44}) \leqslant 51$, which gives $R(C_4, W_{44}) = 51$. □

We note that Lemmas 11(a) and 12 can be stated together as:

Theorem 13. *A C_4-free graph G is a $(C_4, W_m; n)$-graph for $n - m \geqslant 4$, $m \geqslant 7$ iff $\delta(G) > n - m$.*

5 Summary of results on $R(C_4, W_m)$

We briefly review some results on $(k, 5)$-graphs relevant for the estimates of $R(C_4, W_m)$. Wang [22] constructed a $(5, 5)$-graph of order 32 using a complete set of Latin squares of order 4. An

(8, 5)-graph of order 84 and a (9, 5)-graph of order 98 were constructed by O'Keefe and Wong [16]. An (8, 5)-graph of order 80 was constructed by Royle [19]. Exoo gave (10, 5)-graphs of order 124 and 126, an (11, 5)-graph, a (12, 5)-graph, and (13, 5)-graphs of order 230 and 240 [8]. Jørgensen constructed an (11, 5)-graph of order 156, and $(k, 5)$-graphs for $k = 9, 12, 14, 15, 16$ and 20 [13]. The $(k, 5)$-graphs for $17 \leqslant k \leqslant 19$ were constructed by Schwenk (cf. [9]). Using these $(k, 5)$-graphs and Constructions 2, 3 and 5 in the Appendix, we obtain the lower bounds on $R(C_4, W_m)$ for various m by Lemma 11(a) or 11(b). These and other previously known results are summarized in Table 2.

Table 2. The values and bounds on $R(C_4, W_m)$

m	value/bounds	reference
4	10	cf. [18]
5	9	cf. [18]
6	10	cf. [18]
7	9	[21]
$8-11$	$m+3$	[21]
$12-13$	$m+4$	[21]
$14-17$	$m+4$	[7]
$18-21$	$m+4$	Cons. 1/Thm. 3
$22-25$	$m+4/m+5$	Cons. 2/[7]
26	31	[7]
27	32	Thm. 3
$28-34$	$m+5/m+6$	[22], Cons. 3/[7]
$35-37$	$m+6$	Cons. 4, 5/[7]
$38-43$	$m+6/m+7$	Cons. 5/[7]
44	51	Thm. 3/[7]
73	81/82	[8]
77	85/86	[16]
88	97/98	[13]
90	99/100	[16]
115	125/126	[8]
117	127/128	[8]
144	155/156	[8]
146	157/159	[13]
192	204/206	[8] ...
205	217/220	[13] /[7]
218	231/233	[8] ...
228	241/244	[8]
275	289/292	[13]
298	313/316	[13]
321	337/339	[13]
432	449/453	cf. [9]
463	481/485	cf. [9]
494	513/517	cf. [9]
557	577/581	[13]

Note: Thm. refers to Theorem in this paper, Cons. refers to Construction in the Appendix. All upper bounds for $m \geqslant 73$ are implied by Theorem 10 [7].

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Appendix 1

The following graph constructions are sorted by the number of vertices n. Constructions 1, 4 and 5 are used in the proof of Theorem 3 in section 4, Constructions 2, 3 and 5 are used in the Summary in section 5.

Construction 1 ($21 \leqslant n \leqslant 24$). The graph H_{20} of order 20 is a $(4,5)$-graph shown in Figure 3, where $V(H_{20}) = \{v_{i,j}, w_k : 0 \leqslant i,j,k \leqslant 3\}$. Based on H_{20}, we construct the graphs H_i of order i, such that $\delta(H_i) = 4$ and $C_4 \nsubseteq H_i$, for $21 \leqslant i \leqslant 24$. Let

$$E_0 = \{v_{0,0}v_{1,0}, v_{2,0}v_{3,2}\}, E_1 = \{v_{0,2}v_{2,1}, v_{1,1}v_{3,0}\},$$
$$E_2 = \{v_{0,1}v_{3,1}, v_{1,2}v_{2,2}\}, E_3 = \{v_{0,3}v_{3,3}, v_{1,3}v_{2,3}\},$$

and let u_j be the vertex added to $V(H_{21+j})$, for $0 \leqslant j \leqslant 3$. Then $V(H_i) = V(H_{i-1}) \cup \{u_{i-21}\}$, and $E(H_i) = (E(H_{i-1}) \setminus E_{i-21}) \cup \{u_{i-21}v_{s,t} : v_{s,t}$ is an endvertex of an edge in $E_{i-21}\}$, and their matrices are shown in Tables 3-7, respectively.

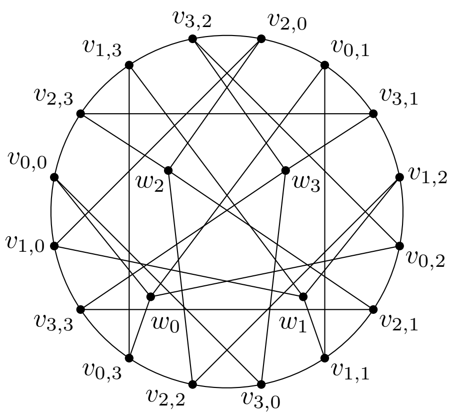

Figure 3: The graph H_{20}

Table 3. Matrix of graph H_{20}

$v_{0,0}$	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0
$v_{0,1}$	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0
$v_{0,2}$	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0
$v_{0,3}$	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0
$v_{1,0}$	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0
$v_{1,1}$	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0
$v_{1,2}$	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
$v_{2,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
$v_{2,1}$	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0
$v_{2,2}$	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
$v_{2,3}$	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0
$v_{3,0}$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1
$v_{3,1}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1
$v_{3,2}$	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1
$v_{3,3}$	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
w_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
w_3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0

Table 4. Matrix of graph H_{21}

$v_{0,0}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	1
$v_{0,1}$	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0
$v_{0,2}$	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0
$v_{0,3}$	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0	0
$v_{1,0}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1
$v_{1,1}$	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0
$v_{1,2}$	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0
$v_{2,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
$v_{2,1}$	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
$v_{2,2}$	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
$v_{2,3}$	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0
$v_{3,0}$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
$v_{3,1}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0
$v_{3,2}$	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
$v_{3,3}$	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
w_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
w_3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0
u_0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0

Table 5. Matrix of graph H_{22}

$v_{0,0}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0
$v_{0,1}$	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0
$v_{0,2}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1
$v_{0,3}$	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0	0	0
$v_{1,0}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0
$v_{1,1}$	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1
$v_{1,2}$	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0
$v_{2,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
$v_{2,1}$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1
$v_{2,2}$	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
$v_{2,3}$	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
$v_{3,0}$	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1
$v_{3,1}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
$v_{3,2}$	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$v_{3,3}$	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
w_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
w_3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
u_0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
u_1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0

Table 6. Matrix of graph H_{23}

$v_{0,0}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0
$v_{0,1}$	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
$v_{0,2}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0
$v_{0,3}$	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
$v_{1,0}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0	0
$v_{1,1}$	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0
$v_{1,2}$	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1
$v_{1,3}$	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0
$v_{2,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
$v_{2,1}$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0
$v_{2,2}$	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1
$v_{2,3}$	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
$v_{3,0}$	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0
$v_{3,1}$	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1
$v_{3,2}$	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
$v_{3,3}$	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
w_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
w_3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0
u_0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
u_1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
u_2	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0

Table 7. Matrix of graph H_{24}

$v_{0,0}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0	0
$v_{0,1}$	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
$v_{0,2}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0
$v_{0,3}$	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1
$v_{1,0}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0
$v_{1,1}$	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0
$v_{1,2}$	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
$v_{2,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
$v_{2,1}$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
$v_{2,2}$	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0
$v_{2,3}$	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
$v_{3,0}$	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0
$v_{3,1}$	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0
$v_{3,2}$	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
$v_{3,3}$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1
w_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w_2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
w_3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
u_0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
u_1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
u_2	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
u_3	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0

Appendix 2

It is known that Hoffman-Singleton graph is the unique (7,5)-cage [9], and let us denote it by H_{50}. The construction of H_{50} based on Robertson's pentagon-pentagram was described in [11], where $V(H_{50}) = \{u_{i,j}, v_{i,j} : 0 \leqslant i, j \leqslant 4\}$, and the edge set $E(H_{50})$ is defined by

$$\begin{aligned} u_{i,j}u_{i,j'} \in E(H_{50}) &\Leftrightarrow j - j' = \pm 1; \\ v_{i,j}v_{i,j'} \in E(H_{50}) &\Leftrightarrow j - j' = \pm 2; \\ u_{i,j}v_{i',j'} \in E(H_{50}) &\Leftrightarrow j = ii' + j'. \end{aligned}$$

Construction 2 ($25 \leqslant n \leqslant 28$)**.** Let $H^b_{30} = H_{50} \setminus S$, where $|S| = 20$ and $S = \{u_{i,j}, v_{i,j} : 3 \leqslant i \leqslant 4,\ 0 \leqslant j \leqslant 4\}$. Then H^b_{30} shown in Figure 4 is one of the four (5,5)-cages, and its matrix is given in Table 8. We construct graphs H_i of order i, $25 \leqslant i \leqslant 29$, such that $\delta(H_i) = 4$ and $C_4 \nsubseteq H_i$. The graphs H_i are obtained by removing one vertex from H_{i+1} (starting from H^b_{30}) as follows.

$$\begin{aligned} &H_{29} = H^b_{30} \setminus \{u_{0,0}\},\ H_{28} = H_{29} \setminus \{u_{0,1}\},\ H_{27} = H_{28} \setminus \{u_{0,2}\}, \\ &H_{26} = H_{27} \setminus \{v_{0,1}\},\ H_{25} = H_{26} \setminus \{v_{1,1}\}. \end{aligned}$$

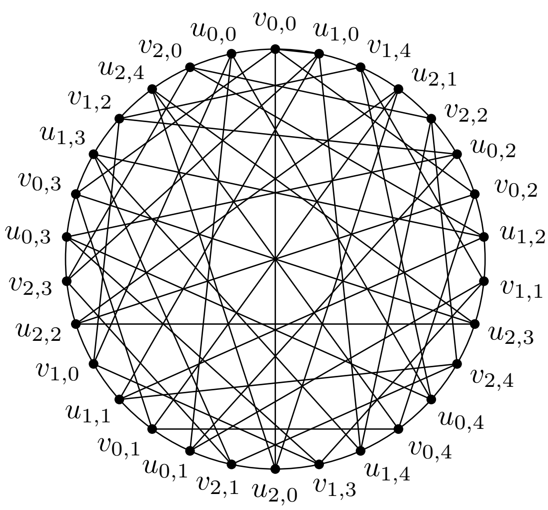

Figure 4: H^b_{30} [9].

Table 8. Matrix of graph H^b_{30}

$u_{0,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
$u_{0,1}$	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
$u_{0,2}$	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
$u_{0,3}$	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
$u_{0,4}$	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
$u_{1,0}$	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
$u_{1,1}$	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1
$u_{1,2}$	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
$u_{1,3}$	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
$u_{1,4}$	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0
$u_{2,0}$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0
$u_{2,1}$	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0
$u_{2,2}$	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0
$u_{2,3}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1
$u_{2,4}$	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0
$v_{0,0}$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
$v_{0,1}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$v_{0,2}$	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$v_{0,3}$	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{0,4}$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$v_{1,0}$	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
$v_{1,1}$	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
$v_{1,2}$	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
$v_{1,4}$	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
$v_{2,0}$	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$v_{2,1}$	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$v_{2,2}$	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
$v_{2,3}$	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
$v_{2,4}$	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0

Construction 3 ($33 \leqslant n \leqslant 38$). First we remove a copy of the Petersen graph from H_{50}, and obtain the unique $(6,5)$-cage, denoted by H_{40}. We have $H_{40} = H_{50} \setminus S$, where $|S| = 10$ and $S = \{u_{4,j}, v_{4,j} : 0 \leqslant j \leqslant 4\}$. We construct graphs H_i of order i, $33 \leqslant i \leqslant 39$, such that $\delta(H_i) = 5$ and $C_4 \nsubseteq H_i$. The graphs H_i are obtained by removing one vertex from H_{i+1} as follows.

$$
\begin{aligned}
&H_{39} = H_{40} \setminus \{u_{0,0}\},\ H_{38} = H_{39} \setminus \{u_{0,1}\},\ H_{37} = H_{38} \setminus \{u_{0,2}\},\\
&H_{36} = H_{37} \setminus \{v_{0,1}\},\ H_{35} = H_{36} \setminus \{v_{1,1}\},\ H_{34} = H_{35} \setminus \{v_{2,1}\},\\
&H_{33} = H_{34} \setminus \{v_{3,1}\}.
\end{aligned}
$$

Construction 4 ($n = 41$). We construct a 6-regular graph H_{41} of order 41 from the $(6,5)$-cage H_{40} by adding a new vertex w and removing certain edges. As in Construction 3, $H_{40} = H_{50} \setminus \{u_{4,j}, v_{4,j} : 0 \leqslant j \leqslant 4\}$. Let

$$
\begin{aligned}
V(H_{41}) =& V(H_{40}) \cup \{w\},\\
E(H_{41}) =& (E(H_{40}) \setminus \{u_{0,0}v_{1,0}, u_{0,1}v_{2,1}, u_{3,2}u_{3,3}\})\\
&\cup \{wu_{0,0}, wv_{1,0}, wu_{0,1}, wv_{2,1}, wu_{3,2}, wu_{3,3}\}.
\end{aligned}
$$

The matrix of H_{41} is shown in Table 9.

Table 9. Matrix of graph H_{41}

$u_{0,0}$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
$u_{0,1}$	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1
$u_{0,2}$	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
$u_{0,3}$	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
$u_{0,4}$	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0
$u_{1,0}$	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
$u_{1,1}$	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
$u_{1,2}$	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0
$u_{1,3}$	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0
$u_{1,4}$	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
$u_{2,0}$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0
$u_{2,1}$	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0
$u_{2,2}$	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
$u_{2,3}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0
$u_{2,4}$	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0
$u_{3,0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0
$u_{3,1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0
$u_{3,2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1
$u_{3,3}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1
$u_{3,4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0
$v_{0,0}$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{0,1}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{0,2}$	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{0,3}$	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{0,4}$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{1,0}$	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1
$v_{1,1}$	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
$v_{1,2}$	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$v_{1,3}$	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{1,4}$	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{2,0}$	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
$v_{2,1}$	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1
$v_{2,2}$	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
$v_{2,3}$	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
$v_{2,4}$	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
$v_{3,0}$	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
$v_{3,1}$	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$v_{3,2}$	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
$v_{3,3}$	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
$v_{3,4}$	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
w	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

Construction 5 ($42 \leqslant n \leqslant 48$)**.** As in Construction 3, we start with the unique $(7,5)$-cage H_{50}. We construct graphs H_i of order i, $42 \leqslant i \leqslant 49$, such that $\delta(H_i) = 6$ and $C_4 \nsubseteq H_i$. The graphs H_i are obtained by removing one vertex from H_{i+1} as follows.

$$H_{49} = H_{50} \setminus \{u_{0,0}\},\ H_{48} = H_{49} \setminus \{u_{0,1}\},\ H_{47} = H_{48} \setminus \{u_{0,2}\},$$
$$H_{46} = H_{47} \setminus \{v_{0,1}\},\ H_{45} = H_{46} \setminus \{v_{1,1}\},\ H_{44} = H_{45} \setminus \{v_{2,1}\},$$
$$H_{43} = H_{44} \setminus \{v_{3,1}\},\ H_{42} = H_{43} \setminus \{v_{4,1}\}.$$