Lower Bounds on Classical Ramsey Numbers
constructions, connectivity, Hamilton cycles

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Previous Work

Our Contributions
- General lower bound constructions
- Connectivity of Ramsey graphs
- Hamiltonian cycles in Ramsey graphs
- Concrete lower bound constructions

What to do next?
- Lower bound on $R(3, k) - R(3, k - 1)$
- Find new smart constructions
Ramsey Numbers

- \( R(G, H) = n \) iff minimal \( n \) such that in any 2-coloring of the edges of \( K_n \) there is a monochromatic \( G \) in the first color or a monochromatic \( H \) in the second color.

- 2-colorings \( \cong \) graphs, \( R(m, n) = R(K_m, K_n) \)

- Generalizes to \( k \) colors, \( R(G_1, \cdots, G_k) \)

- Theorem (Ramsey 1930): Ramsey numbers exist
• Bounds (Erdős 1947, Spencer 1975, Thomason 1988)

\[
\frac{\sqrt{2}}{e} 2^{n/2} n < R(n, n) < \left( \frac{2n - 2}{n - 1} \right) n^{-1/2 + c/\sqrt{\log n}}
\]

• Newest upper bound (Conlon, 2010)

\[
R(n + 1, n + 1) \leq \left( \frac{2n}{n} \right) n^{-c \frac{\log n}{\log \log n}}
\]

• Conjecture (Erdős 1947, $100)

\[
\lim_{n \to \infty} R(n, n)^{1/n} \text{ exists.}
\]

If it exists, it is between $\sqrt{2}$ and 4 ($250 for value).
Recursive construction yielding
\[ R(3, 4k + 1) \geq 6R(3, k + 1) - 5 \]
\[ \Omega(k^{\log_6^4}) = \Omega(k^{1.29}) \]
Chung-Cleve-Dagum 1993

Explicit \( \Omega(k^{3/2}) \) construction
Alon 1994, Codenotti-Pudlák-Giovanni 2000

Kim 1995, lower bound
Ajtai-Komlós-Szemerédi 1980, upper bound
Bohman 2009, triangle-free process

\[ R(3, k) = \Theta \left( \frac{k^2}{\log k} \right) \]
Off-Diagonal Cases

Fixing small $k$

- McKay-R 1995, $R(4, 5) = 25$
- Bohman triangle-free process - 2009

$$R(4, n) = \Omega(n^{5/2} / \log^2 n)$$

- Kostochka, Pudlák, Rödl - 2010
  constructive lower bounds

$$R(4, n) = \Omega(n^{8/5}), \quad R(5, n) = \Omega(n^{5/3}), \quad R(6, n) = \Omega(n^2)$$

(vs. probabilistic 5/2, 6/2, 7/2 with /logs)
Values and Bounds on $R(k, l)$

two colors, avoiding cliques

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[ElJC survey Small Ramsey Numbers, revision #12, 2009]
General lower bound constructions aren’t that good

**Theorem** Burr, Erdős, Faudree, Schelp, 1989

\[ R(k, n) \geq R(k, n - 1) + 2k - 3 \quad \text{for} \ k \geq 2, \ n \geq 3 \ (\text{not} \ n \geq 2) \]

**Theorem** (Xu-Xie-Shao-R 2004, 2010)

*If 2 \leq p \leq q and 3 \leq k, then* \[ R(k, p + q - 1) \geq R(k, p) + R(k, q) + \begin{cases} 
  k - 3, & \text{if } 2 = p \\
  k - 2, & \text{if } 3 \leq p \text{ or } 5 \leq k \\
  p - 2, & \text{if } 2 = p \text{ or } 3 = k \\
  p - 1, & \text{if } 3 \leq p \text{ and } 4 \leq k 
\end{cases} \]

For \( p = 2, \ n = q + 1 \), we have \( R(k, p) = k \), which implies BEFR’89
Previous Work

Our Contributions

General lower bound constructions

Connectivity of Ramsey graphs

Hamiltonian cycles in Ramsey graphs

Concrete lower bound constructions

Proof by construction

Given

$(k, p)$-graph $G$, $(k, q)$-graph $H$, $k \geq 3$, $p, q \geq 2$

$G$ and $H$ contain induced $K_{k-1}$-free graph $M$

construct

$(k, p + q - 1)$-graph $F$, $n(F) = n(G) + n(H) + n(M)$

$V_G = \{v_1, v_2, ..., v_{n_1}\}$, $V_H = \{u_1, u_2, ..., u_{n_2}\}$

$V_M = \{w_1, ..., w_m\}$, $m \leq n_1, n_2$, $K_{k-1} \not\subset M$

$G[\{v_1, ..., v_m\}], H[\{u_1, ..., u_m\}] \cong M$

$\phi(w_i) = v_i$, $\psi(w_i) = u_i$ isomorphisms

$V_F = V_G \cup V_H \cup V_M$

$E(G, H) = \{\{v_i, u_i\} \mid 1 \leq i \leq m\}$

$E(G, M) = \{\{v_i, w_j\} \mid 1 \leq i \leq n_1, 1 \leq j \leq m, \{v_i, v_j\} \in E(G)\}$

$E(H, M) = \{\{u_i, w_j\} \mid 1 \leq i \leq n_2, 1 \leq j \leq m, \{u_i, u_j\} \in E(H)\}$
In 1980, Paul Erdős wrote

Faudree, Schelp, Rousseau and I needed recently a lemma stating

$$\lim_{n \to \infty} \frac{\binom{n+1}{2} \times \binom{n}{2}}{n} = \infty.$$  

We could prove it without much difficulty, but could not prove that \( r(n + 1, n) - r(n, n) \) increases faster than any polynomial of \( n \). We of course expect

$$\lim_{n \to \infty} \frac{r(n + 1, n)}{r(n, n)} = C^\frac{1}{2},$$

where \( C = \lim_{n \to \infty} r(n, n)^{1/n} \).

The best known lower bound for \( (r(n + 1, n) - r(n, n)) \) is \( \Omega(n) \).
Theorem 1

If $k \geq 5$ and $l \geq 3$, then the connectivity of any Ramsey-critical $(k, l)$-graph is no less than $k$.

This improves by 1 the result by Beveridge/Pikhurko from 2008.
Theorem 2

If \( k \geq l - 1 \geq 1 \) and \( k \geq 3 \), except \((k, l) = (3, 2)\), then any Ramsey-critical \((k, l)\)-graph is Hamiltonian.

In particular, for \( k \geq 3 \), all diagonal Ramsey-critical \((k, k)\)-graphs are Hamiltonian.
Using the best known bounds for $R(k, s)$ we get:

**Theorem 3**

\[
\begin{align*}
R(6, 12) & \geq R(6, 11) + 2 \times 6 - 2 \geq 263, \\
R(7, 8) & \geq R(7, 7) + 2 \times 7 - 2 \geq 217, \\
R(7, 12) & \geq R(7, 11) + 2 \times 7 - 2 \geq 417, \\
R(9, 10) & \geq R(9, 9) + 2 \times 9 - 2 \geq 581, \\
R(11, 12) & \geq R(11, 11) + 2 \times 11 - 2 \geq 1617, \\
R(12, 12) & \geq R(12, 11) + 2 \times 12 - 2 \geq 1639.
\end{align*}
\]
Theorem 4

\[
\begin{align*}
R(5, 17) & \geq 388, \\
R(5, 19) & \geq 411, \\
R(5, 20) & \geq 424, \\
R(6, 8) & \geq 132, \\
R(7, 9) & \geq 241, \\
R(8, 17) & \geq 961, \\
R(8, 8, 8) & \geq 6079.
\end{align*}
\]
Erdős and Sós, 1980, asked about

\[ 3 \leq \Delta_k = R(3, k) - R(3, k - 1) \leq k: \]

\[ \Delta_k \xrightarrow{k} \infty ? \quad \Delta_k / k \xrightarrow{k} 0 ? \]

Challenges

- improve lower bound for \( \Delta_k \)
- generalize beyond triangle-free graphs
SPR’s papers to pick up


- Revision #12 of the survey paper *Small Ramsey Numbers* at the EIJC, August 2009. Revision #13 coming in the summer 2011 ...