On the Most Wanted Folkman Graph

(K_4-free graph which is not a union of two triangle-free graphs)

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Abstract*

We discuss a branch of Ramsey theory concerning edge Folkman numbers and how computer algorithms could help to solve some problems therein. We write $G \rightarrow (a_1, \ldots, a_k; p)^e$ if for every edge k-coloring of an undirected simple graph G not containing K_p, a monochromatic K_{a_i} is forced in color i for some $i \in \{1, \ldots, k\}$. The edge Folkman number is defined as $F_e(a_1, \ldots, a_k; p) = \min\{|V(G)| : G \rightarrow (a_1, \ldots, a_k; p)^e\}$. Folkman showed in 1970 that this number exists for $p > \max(a_1, \ldots, a_k)$.

In general, much less is known about edge Folkman numbers than the related and more studied vertex Folkman numbers, where we color vertices instead of edges. $F_e(3, 3; 4)$ involves the smallest parameters for which the problem is open, namely the question, "What is the smallest order N of a K_4-free graph, for which any edge 2-coloring must contain at least one monochromatic triangle?" This is equivalent to finding the order N of the smallest K_4-free graph which is not a union of two triangle-free graphs. It is known that $19 \le N$, and it is known through a probabilistic proof by Spencer (later updated by Hovey) that $N \le 3 \times 10^9$. We suspect that $N \le 127$.

This talk will present the background, overview some related problems, discuss the difficulties in obtaining better bounds on N, and give some computational evidence why it is very likely that even $N < 100$.

* - slides not shown

Outline

- Arrowing
- Folkman numbers
- Story of $F_e(3, 3; 4)$
- Probabilistic upper bound on $F_e(3, 3; 4)$
- Some general known facts about edge- and vertex- Folkman numbers and bounds for specific small parameters
- Complexity of arrowing
- A very special graph G_{127}
- Can SAT-solvers help?

Graph notation

G - simple undirected loopless graph
$V(G)$ - vertex set of graph G
$E(G)$ - edge set of graph G

$R(s, t)$ - Ramsey number, the least n such that in any 2-coloring of the edges of K_n there is a monochromatic K_s in the first color or a monochromatic K_t in the second color.

$G(n, p)$ - random graph
n vertices, edge probability p

$\chi(G)$ - chromatic number of G

K_n, P_n, C_n - complete graph, path and cycle on n vertices

Arrowing - branch of Ramsey Theory

F, G, H - graphs, s, t, s_i - positive integers

Definitions

$F \rightarrow (s_1, ..., s_k)^e$ iff for every k-coloring of the edges of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \le i \le k$.

$F \rightarrow (s_1, ..., s_k)^v$ iff for every k-coloring of the vertices of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \le i \le k$.

$F \rightarrow (G, H)^e$ iff for every red/blue edge-coloring of F, F contains a red copy of G or a blue copy of H.

Facts

$R(s, t) = \min\{n \mid K_n \rightarrow (s, t)^e\}$
$R(G, H) = \min\{n \mid K_n \rightarrow (G, H)^e\}$

Warming up

$G = K_6$ has the smallest number of vertices among graphs which are not a union of two K_3-free graphs, since $R(3, 3) = 6$.

$$K_6 \rightarrow (K_3, K_3)^e \text{ and } K_5 \not\rightarrow (K_3, K_3)^e$$

and since $43 \le R(5, 5) \le 49$

$$K_{49} \rightarrow (K_5, K_5)^e \text{ and } K_{42} \not\rightarrow (K_5, K_5)^e$$

Warming up

What if we want G to be K_6-free?
Graham (1968) proved that

- $G = K_8 - C_5 = K_3 + C_5 \rightarrow (K_3, K_3)$
clearly, G has no K_6

- $|V(H)| < 8 \wedge K_6 \not\subset H \Rightarrow H \not\rightarrow (K_3, K_3)$

(picture proof of)
$K_3 + C_5 \rightarrow (K_3, K_3)$

Folkman problems

edge Folkman graphs
$\mathcal{F}_e(s, t; k) = \{G \rightarrow (s, t)^e : K_k \not\subseteq G\}$

edge Folkman numbers
$F_e(s, t; k)$ = the smallest n such that there exists an n-vertex graph G in $\mathcal{F}_e(s, t; k)$

vertex Folkman graphs/numbers
2-coloring vertices instead of edges

Theorem 1. (Folkman 1970) For all $k > max(s, t)$, edge- and vertex- Folkman numbers $F_e(s, t; k)$, $F_v(s, t; k)$ exist.

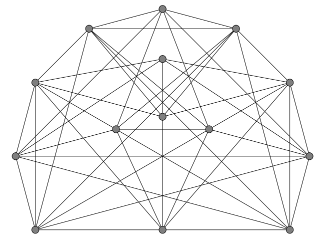
Known values/bounds for $F_e(3,3;k)$

Our goal $F_e(3,3;4)$

k	$F_e(3,3;k)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham'68
5	15	659 graphs	[PRU]'99
4	$\leq 3 \times 10^9$	probabilistic	'86,'88,'89

$k > R(s,t) \Rightarrow F_e(s,t;k) = R(s,t)$
$k \leq R(s,t)$, very little known in general

$F_e(3,3;5) = 15$ and $F_v(3,3;4) = 14$



unique 14-vertex bicritical graph G [PRU'99]

$H \rightarrow (3,3;4)^v$ implies
$H + x \rightarrow (3,3;5)^e$

History of upper bounds on $F_e(3,3;4)$

- 1967 - Erdős, Hajnal state the problem
- 1970 - Folkman proves his theorem for 2 colors. VERY large bound for $F_e(3,3;4)$.
- 1975 - Erdős offers \$100 (or 300 Swiss francs) for deciding if $F_e(3,3;4) < 10^{10}$
- 1988 - Spencer gives a probabilistic proof of $F_e(3,3;4) < 3 \times 10^8$
- 1989 - Hovey finds an error in Spencer's proof, bound up to $F_e(3,3;4) < 3 \times 10^9$
- 2007 - nothing better so far ...
- 2013 - "$F_e(3,3;4) < 100$" is decided (?)

History of lower bounds on $F_e(3,3;4)$

$10 \leq F_e(3,3;4)$ Lin (1972)

$16 \leq F_e(3,3;4)$ (PRU 1999)
since $F_e(3,3;5) = 15$, all graphs in $\mathcal{F}_e(3,3;5)$ on 15 vertices are known, and all of them contain K_4's

$19 \leq F_e(3,3;4)$ (RX 2006)
$18 \leq F_e(3,3;4)$ - proof "by hand"
$19 \leq F_e(3,3;4)$ - computations

ANY proof technique improving on 19 very likely will be of interest

Lower Bound

Proof "by hand" that $18 \le F_e(3,3;4)$

- G_{17} critical for $R(4,4) = 18$, check that $G_{17} \not\to (3,3;4)^e$.
- $G_{17} \not\approx G \to (3,3;4)^e$, $|V(G)| = 17$, G must have indset I on 4 vertices.
- $G\prime = I + G[V(G) \setminus I] \to (3,3;5)^e$.
- Dropping any three vertives from I, gives K_5-free graph on 14 vertices.
- Contradiction with $F_e(3,3;5) = 15$.

Computing $19 \le F_e(3,3;4)$

Quite similar, but much more work,
Use all 153 graph $H \in \mathcal{F}_v(3,3;4)$.

Probabilistic construction

Frankl, Rődl, Spencer, Hovey
used graph G^* constructed as follows:

Construction

1: input an integer n, and probability p
2: $G \leftarrow G(n,p)$
3: remove random edge from each K_4 in G
4: output G^*, the result of step 3

Sometimes $G^* \to (3,3)^e$

Frankl, Rődl:
very difficult probabilistic graph theory
$n = 7 \times 10^{11}$

Spencer/Hovey:
difficult probabilistic graph theory
$n = 3 \times 10^9$, $p = 6n^{-1/2} \approx 1/9129$

Probabilistic construction*

main proof steps

Let

$$U(G) = \{(x, xyz) \mid \triangle xyz \ \ in \ \ G\}$$

$$U^* = U(G^*)$$

For each $x \in V(G)$, define (maximum over all partitions $N(x) = T \cup B, \ T \cap B = \emptyset$)

$$A(x) = \max |\{yz \in E(G) \mid y \in T \wedge z \in B\}|$$

Theorem 2. (Spencer)

$$\sum_{x \in V(G)} A(x) < \frac{2}{3}|U^*|$$

holds with positive probability for $n = 3 \times 10^9$, $p \approx 0.00011$, and $|E(G)| \approx 4 \times 10^{14}$.

Probabilistic construction*

main counting trick

Theorem 3.
If

$$\sum_{x \in V(G)} A(x) < \frac{2}{3}|U^*|$$

then

$$G^* \in \mathcal{F}_e(3,3;4).$$

Proof.
G has no K_4 by construction.
Suppose f colors $E(G^*)$ in $\triangle$-free way.

Count marked triangles (x, xyz) such that $f(xz) \neq f(xy)$. It is $2|U^*|/3$, but also bounded by $\sum_{x \in V(G)} A(x)$. Contradiction. ∎

General facts on $F_e(s,t;k)$

- $G \in \mathcal{F}_e(s,t;k) \Rightarrow \chi(G) \geq R(s,t)$
  no k in the bound!, easy

- $\mathcal{F}_e(s,t;\ k > R(s,t)) = R(s,t)$
- $\mathcal{F}_e(s,t; k = R(s,t)) = R(s,t) + c$
  in most cases c is small (2, 4, 5)
- $\mathcal{F}_e(s,t; k < R(s,t)) \geq R(s,t) + 4$

Special cases (other than $F_e(3,3;4)$)*

$F_e(3,4;\geq 10) = 9$, K_9 since $R(3,4) = 9$
$F_e(3,4;9) = 14$, $K_4 + C_5 + C_5$, Nenov (1991)
$F_e(3,4;8) = 16$, Kolev/Nenov (2006)
$F_e(3,4;7) =?$

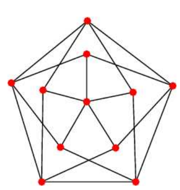
$F_e(3,5;14) = 16$
$F_e(4,4;18) = 20$
$F_e(3,7;22) \geq 27$
$F_e(3,3,3;17) = 19$
$F_e(3,3,3;16) = 21$

forbidden K_k in the above items has
$k = R(s,t)$ or $k = R(s,t) - 1$

several critical graphs have the form
$K_p + C_q$, $K_p + \overline{C_q} + C_r$, or $K_p - C_q$

Vertex Folkman numbers pearls

$F_v(2,2,2;3) = 11$
the smallest 4-chromatic triangle-free graph



Grőtzsch graph [mathworld.wolfram.com]

$F_v(2,2,2,2;4) = 11$
the smallest 5-chromatic K_4-free graph has
11 vertices, Nenov (1984), also 1993

$F_v(2,2,2,2;3) = 22$
the smallest 5-chromatic triangle-free graph
has 22 vertices, Jensen/Royle (1995)

Vertex Folkman numbers pearls

Theorem 4. (ancient folklore)
$F_v(\underbrace{2,\cdots,2}_{r};r) = r + 5$, for $r \geq 5$.

Theorem 5. (Nenov 2003)
$F_v(\underbrace{3,\cdots,3}_{r};2r) = 2r + 7$, for $r \geq 3$.

For $r = 2$, a small but hard case,
$F_v(3,3;4) = 14$ (PRU 1999)

Complexity of arrowing

- Testing whether $F \rightarrow (3,3)^e$ is **coNP**-complete (Burr 1976).
- Determining if $R(G,H) < m$ is **NP**-hard (Burr 1984).
- For any fixed 3-connected graphs G and H, testing whether $F \not\rightarrow (G,H)^e$ is **NP**-complete (Burr 1990).
- Testing whether $F \rightarrow (G,H)^e$ is $\mathbf{\Pi_2^p}$-complete (Schaefer 2001).

Testing whether $F \rightarrow (K_2, K_n)^e$ is the same as checking $K_n \subset F$, so it is NP-hard.

Complexity of (edge) arrowing*

Compendium of arrowing complexity including contributions by Cook (1971), Burr (1976, 1984, 1990), Rutenburg (1986) and Schaefer (2001)

Problem	Fixed	Complexity
$F \rightarrow (G,H)$		Π_2^p-complete
$F \rightarrow (G,H)$	G, H	in coNP
$F \rightarrow (K_2,H)$		NP-complete
$F \rightarrow (K_2,H)$	H	NP-complete
$F \rightarrow (T,K_n)$	T, $e(T) \geq 2$	Π_2^P-complete
$F \rightarrow (G,H)$	$G, H \in \Gamma_3$	coNP-complete
$F \rightarrow (P_4,P_4)$		coNP-complete
$F \rightarrow (kK_2,H)$	k, H	P
$F \rightarrow (K_{1,n},K_{1,m})$		P
$K_n \rightarrow (G,H)$		NP-hard

Tools in complexity of arrowing*

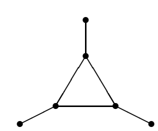
(G,H)-enforcers, -signal senders, -cleavers, -determiners are the tools (gadgets) used in reductions (Burr, Schaefer).

Such gadgets permit to construct F for which we are in control of whether $F \rightarrow (G,H)$.

Definition (Grossman 1983)
F is a (G,G)-cleaver iff there exists unique coloring of F witnessing $F \not\rightarrow (G,G)$.

Cleavers*

P_4 cleaved graph F, $F \not\rightarrow (P_4,P_4)$, but there is only one witness coloring.



graph F

Known K_3-cleaved graphs contain K_4.
K_5 is not C_5-cleaved, P_3 cleaves C_{2n}.

$G_{127} \rightarrow (3,3)^e$?

Exoo suggested to look at the well known Ramsey graph (Hill, Irving 1968), defined by:

$G_{127} = (\mathcal{Z}_{127}, E)$
$E = \{(x,y) | x - y = \alpha^3 \pmod{127}\}$

- 127 vertices, 2667 edges, 9779 triangles
- regular of degree 42
- independence number 11, no K_4's !
- vertex- and edge-transitive
- 5334 $(= 127 * 42)$ automorphisms
- $(127, 42, 11, \{14, 16\})$ - regularity, almost strongly regular graph
- K_{127} can be partitioned into three G_{127}'s

When to expect $G \rightarrow (3,3)^e$?

- G has a large number of triangles
- G has many small dense subgraphs
- Spencer's proof is far from useful for G_{127}

Conjecture: $G_{127} \rightarrow (3,3)^e$

If $G_{127} \rightarrow (3,3)^e$ then it gives 23,622,047-fold improvement over Spencer/Hovey bound.

Proving $G \rightarrow (3,3)^e$

First, solve a simpler task: find a small subgraph H, embedded in G in many places, such that there is a small number of colorings witnessing $H \not\rightarrow (3,3)^e$

Second, try to extend all (not many) colorings for $H \not\rightarrow (3,3)^e$ to whole G,

or, if this is too expensive ...

go via SAT ...

Reducing $\{G \mid G \not\rightarrow (3,3)^e\}$ to 3-SAT

edges in $G \longmapsto$ variables of ϕ_G
each (edge)-triangle xyz in $G \longmapsto$ add to ϕ_G

$$(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z})$$

Clearly,

$$G \not\rightarrow (3,3)^e \iff \phi_G \quad \text{is satisfiable}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 3-clauses, 2 for each of the 9779 triangles.

Note: By taking only the positive clauses, we obtain a reduction to ϕ'_G in NAE-3-SAT with half of the clauses.

Algorithms for 3-SAT*

Randomized algorithms finding a satisfying assignment to n-variable 3-SAT in expected time

$$O(c^n)$$

Between 1997 and 2004, c was sliding down from 1.782 to 1.324 (Iwama, Tamaki - 2004) in a dozen of papers.

8-authors TCS 2002 paper presenting a deterministic algorithm for k-SAT running in time

$$\left(2 - \frac{2}{k+1}\right)^n$$

SAT-solvers - enhanced/tuned Davis-Putnam Algorithm

zChaff
Well known solver since 2001, winner of competitions. EE Princeton group: Fu, Mahajan, Zhao, Zhang, Malik, joined by Madigan (MIT), Moskewicz (UC Berkeley).

Satzoo → MiniSat → SatELite
New contender since 2003, strong for combinatorial/handmade instances, 4 gold medals in 2005, Eén and Sörensen (Chalmers U., Sweden)

SAT-solvers

SAT 2005 Competition
3 medals in each of 9 categories

(random, crafted, industrial)
× (SAT, UNSAT, ALL)

SatELite - winner of 2005 competition in the category (crafted, UNSAT)

March_eq, Vallst, Adaptnovelty, Kcnfs, Jerusat
other recent less known leading SAT-solvers

GRASP'99, SATO'97, POSIT'95
other older more known SAT-solvers

zChaff experiments on $\phi_{G_{127}}$

- Pick $H = G_{127}[S]$ on $m = |S|$ vertices. Use zChaff to split H:
  - $m \leq 80$, H easily splittable
  - $m \approx 83$, phase transition ?
  - $m \geq 86$, splitting H is very difficult

- $\#(clauses)/\#(variables) = 7.483$ for G_{127}, far above conjectured phase transition ratio $r \approx 4.2$ for 3-SAT. It is known that

$$3.52 \leq r \leq 4.596$$

References*

S. A. Burr, 1976 result mentioned in the Garey and Johnson 1979 NP-book.

S. A. Burr, Determining generalized Ramsey numbers is NP-hard, *Ars Combinatoria* **17** (1984), 21–25.

S. A. Burr, On the Computational Complexity of Ramsey-Type Problems, in Mathematics of Ramsey theory, *Algorithms Combin.* **5** (1990), 46–52,

E. Dantsin, A. Goerdt, E. A. Hirsch, R. Kannan, J. Kleinberg, C. Papadimitriou, P. Raghavan and U. Schőning, A Deterministic $(2-\frac{2}{k+1})^n$ Algorithm for k-SAT Based on Local Search, *Theoretical Computer Science* **289** (2002), 69–83.

P. Erdős, A. Hajnal, Research problem 2-5, *J. Combin. Theory* **2** (1967), 104.

J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.* **18** (1970), 19–24.

P. Frankl, V. Rödl, Large triangle-free subgraphs in graphs without K_4, *Graphs and Combinatorics* **2** (1986), 135–144.

R. L. Graham, On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon, *J. Combin. Theory* **4** (1968), 300.

J. W. Grossman, Graphs with Unique Ramsey Colorings, *J. Graph Theory* **7** (1983), 85–90.

R. Hill and R. W. Irving, On Group Partitions Associated with Lower Bounds for Symmetric Ramsey Numbers, *European Journal of Combinatorics* **3** (1982), 35–50.

References cont.*

K. Iwama and S. Tamaki, Improved upper bounds for 3-SAT, *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, New Orleans, Louisiana (2004), 328–329.

T. Jensen, D. Royle, Small graphs with chromatic number 5: a computer search, *J. Graph Theory* **19** (1995), 107–116.

S. Lin, On Ramsey numbers and K_r-coloring of graphs, *J. Combin. Theory Ser. B* **12** (1972), 82–92.

T. Łuczak, A. Ruciński, S. Urbański, On minimal Folkman graphs, *Discrete Math.* **236** (2002), 245–262.

N. Kolev and N. Nenov, The Folkman number $F_e(3,4;8)$ is equal to 16, *Compt. R. Acad. Bulgare Sci.* **59** (2006), 25–30.

N. Nenov, The chromatic number of any 10-vertex graph without 4-cliques is at most 4 (in Russian), *C.R. Acad. Bulgare Sci.* **37** (1984), 301–304.

N. Nenov, On $(3,4)$ Ramsey graphs without 9-cliques (in Russian), *Annuaire Univ. Sofia Fac. Math. Inform.* **85** (1991), no. 1-2, 71-81.

N. Nenov, On the triangle vertex Folkman numbers, *Discrete Math.* **271** (2003), 327–334.

J. Nešetřil and V. Rödl, The Ramsey Property for Graphs with Forbidden Complete Subgraphs, *Journal of Combinatorial Theory, Series B*, **20** (1976), 243–249.

K. Piwakowski, S. Radziszowski, S. Urbański, Computation of the Folkman number $F_e(3,3;5)$, *J. Graph Theory* **32** (1999), 41–49.

M. Schaefer, Graph Ramsey theory and the polynomial hierarchy, *J. Comput. System Sci.* **62** (2001), 290–322.

J. Spencer, Three hundred million points suffice, *J. Combin. Theory Ser. A* **49** (1988), 210–217. Also see erratum in Vol. **50**, p. 323.

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SAT solvers*

ZCHAFF
M. Moskewicz and C. Madigan and Y. Zhao and L. Zhang and S. Malik, Chaff: Engineering an Efficient SAT Solver, *Proceedings of the 39th Design Automation Conference, Las Vegas*, June, 2001. Available at `http://www.princeton.edu/~chaff` (2004).

MARCH_EQ
Marijn Heule and Hans van Maaren, March_eq SAT-solver, 2004. Available at `http://www.isa.ewi.tudelft.nl/sat/march_eq.htm`.

Links to other SAT-solvers can be easily found on the web.

Revisions*

Revision #1, October 28, 2004
presented at MCCCC'04, Rochester NY

Revision #2, February 7, 2005
presented at the University of Rochester, Rochester NY

Revision #3, October 7, 2005
presented at MCCCC'05, Rochester NY

Revision #4, November 23, 2006
presented at the Technical University of Gdańsk, Poland

Revision #5, March 25, 2007

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Revision #n, June 7, 2013
presenting solution to the G_{127} problem, Playa Azul, Cozumel QR