On Some Multicolor Ramsey Numbers Involving $K_3 + e$ and $K_4 - e$

Daniel S. Shetler
Department of Mathematics
Whitworth University
Spokane, WA 99251
dshetler12@my.whitworth.edu

Michael A. Wurtz
Department of Mathematics
Northwestern University
Evanston, IL 60208
wurtz@u.northwestern.edu

Stanisław P. Radziszowski
Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623
spr@cs.rit.edu

Abstract: The Ramsey number $R(G_1, G_2, G_3)$ is the smallest positive integer n such that for all 3-colorings of the edges of K_n there is a monochromatic G_1 in the first color, G_2 in the second color, or G_3 in the third color. We study the bounds on various 3-color Ramsey numbers $R(G_1, G_2, G_3)$, where $G_i \in \{K_3, K_3 + e, K_4 - e, K_4\}$. The minimal and maximal combinations of G_i's correspond to the classical Ramsey numbers $R_3(K_3)$ and $R_3(K_4)$, respectively, where $R_3(G) = R(G, G, G)$. Here, we focus on the much less studied combinations between these two cases.

Through computational and theoretical means we establish that $R(K_3, K_3, K_4 - e) = 17$, and by construction we raise the lower bounds on $R(K_3, K_4 - e, K_4 - e)$ and $R(K_4, K_4 - e, K_4 - e)$. For some G and H it was known that $R(K_3, G, H) = R(K_3 + e, G, H)$; we prove this is true for several more cases including $R(K_3, K_3, K_4 - e) = R(K_3 + e, K_3 + e, K_4 - e)$.

Ramsey numbers generalize to more colors, such as in the famous 4-color case of $R_4(K_3)$, where monochromatic triangles are avoided. It is known that $51 \leq R_4(K_3) \leq 62$. We prove a surprising theorem stating that if $R_4(K_3) = 51$ then $R_4(K_3 + e) = 52$, otherwise $R_4(K_3 + e) = R_4(K_3)$.

1 Introduction

For undirected simple graphs $G_1, \ldots, G_m$, a $(G_1, \ldots, G_m)$-coloring is a partition of the edges of a complete graph into m colors such that no color i contains G_i as a subgraph. A $(G_1, \ldots, G_m; n)$-coloring is a $(G_1, \ldots, G_m)$-coloring of K_n. Further, $\mathcal{R}(G_1, \ldots, G_m)$ and $\mathcal{R}(G_1, \ldots, G_m; n)$ will

Research supported by the NSF Research Experiences for Undergraduates Program (#1062128) held at the Rochester Institute of Technology during the summer of 2011.

denote the sets of all corresponding colorings. The Ramsey number $R(G_1, \ldots, G_m)$ is defined as the minimum number of vertices n such that no $(G_1, \ldots, G_m; n)$-coloring exists. Note that a standard graph can be considered as a 2-coloring of the edges of a complete graph, where the edges of the graph are those in the first color. As such, we will call a (G_1, G_2)-coloring a (G_1, G_2)-good graph. The known values and bounds for various types of Ramsey numbers are compiled in the dynamic survey *Small Ramsey Numbers* by the third author [24].

We will use the following notation throughout the paper:

$N_c(v)$: neighborhood of vertex v in color c

$G - v$: coloring or graph induced by $V(G) \setminus \{v\}$

$G \setminus \{u, v\}$: G without edge $\{u, v\}$

J_n : $K_n - e$, equal to K_n with one edge deleted

$K_n + e$: K_n connected to an additional vertex by one edge

$R_n(G)$: n-color Ramsey number $R(G, \ldots, G)$

colors : we refer to consecutive colors corresponding to the parameters of Ramsey colorings as red, green, blue, and yellow

We will be using the Ramsey arrowing operator $\rightarrow$. We say that $F \rightarrow (G_1, \ldots, G_m)$ holds iff for all partitions of the edges of F into m colors $F_1, \ldots, F_m$ there exists $G_i \subseteq F_i$ for some $1 \leq i \leq m$. The Ramsey number $R(G_1, \ldots, G_m)$ can also be defined using the arrowing operator as the smallest n such that $K_n \rightarrow (G_1, \ldots, G_m)$.

Observe that if H' is a subgraph of H, then any (G, H')-good graph is also a (G, H)-good graph. Thus $\mathcal{R}(G, H') \subseteq \mathcal{R}(G, H)$, and therefore $R(G, H') \leq R(G, H)$. The complement of a (G, H)-good graph is an (H, G)-good graph, hence $R(G, H) = R(H, G)$. This monotonicity and symmetry of 2-color Ramsey numbers extend to multiple colors.

In what follows we discuss Ramsey numbers for parameters between (K_3, K_3, K_3) and (K_4, K_4, K_4). In this range there are four classical Ramsey numbers $R(K_p, K_q, K_r)$ of which only one exact value $R(K_3, K_3, K_3) = 17$ is known [24]. Arste, Klamroth, and Mengersen [1] studied a variety of 3-color Ramsey numbers $R(G_1, G_2, G_3)$ for G_i's on at most four vertices. Several of the cases still unsolved fall within the (K_3, K_3, K_3) to (K_4, K_4, K_4) range. Figure 1 below is presented as a poset of possible parameters ordered coordinate-wise under inclusion for $G_i \in \{K_3, J_4, K_4\}$. The only two numbers known in this range are $R(K_3, K_3, K_3) = R(K_3, K_3, J_4) = 17$ (Theorem 4). For the open cases the best known bounds are presented. The Ramsey numbers with at least one parameter involving $K_3 + e$ are studied in Section 2.

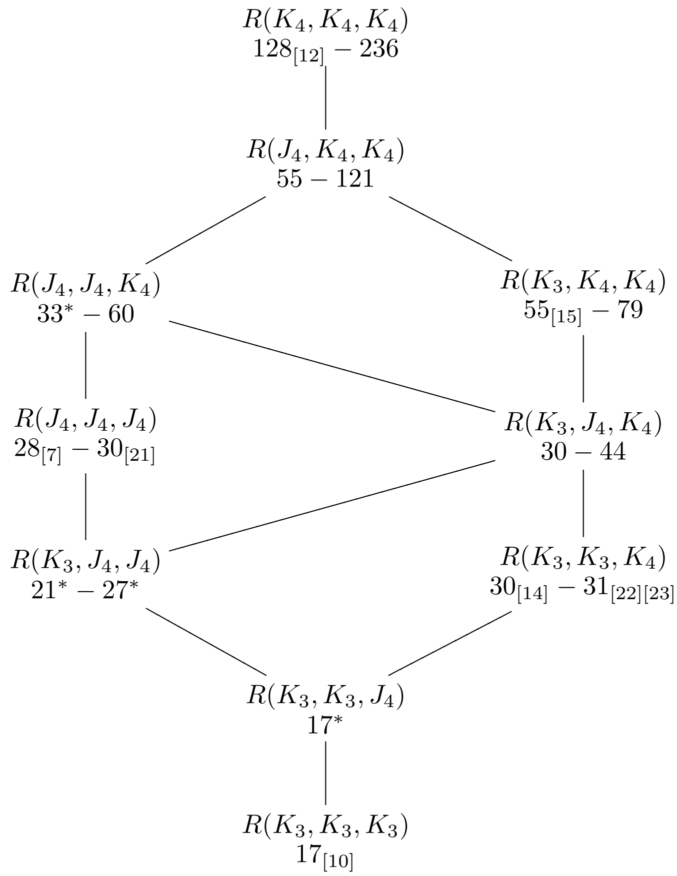

Figure 1: Ramsey numbers for parameters between (K_3, K_3, K_3) and (K_4, K_4, K_4). The results of this paper are marked with a *, and the bounds without references are obtained by monotonicity or by application of the standard upper bound (see 6.1.a in [24]) to the bounds for smaller parameters in this figure or to the results listed in [1].

2 From K_3 to $K_3 + e$

In the case of two colors, Burr, Erdős, Faudree, and Schelp [3] proved that, for $m, n \geq 3$ and $m + n \geq 8$, $R(\widehat{K}_{m,p}, \widehat{K}_{n,q}) = R(K_m, K_n)$, where $p = \lceil m/(n-1) \rceil$, $q = \lceil n/(m-1) \rceil$, and $\widehat{K}_{k,l} = K_{k+1} - K_{1,k-l}$, the graph obtained from a K_k by adding a vertex adjacent to l vertices in K_k.

For more colors, it has been proven that in some cases adding an edge to K_3 leaves Ramsey numbers unchanged, such as the following:

- $R_3(K_3+e) = R_3(K_3) = 17$ [26],
- $R(K_3+e, K_3+e, K_4) = R(K_3, K_3, K_4)$ [1].

Several similar cases are presented in [1]. We give further evidence of such behavior by establishing three new cases. This raises the question of when the parameter K_3 can be extended to $K_3 + e$ without changing the Ramsey number.

In Theorem 4 of the next section we will prove that $R(K_3, K_3, J_4) = 17$. This result will be used in the proof of the following Theorem 1.

Theorem 1 $R(K_3, K_3, J_4) = R(K_3+e, K_3+e, J_4)$ $[= 17]$.

Proof: By Theorem 4 and monotonicity of Ramsey numbers we have that $17 = R(K_3, K_3, J_4) \leq R(K_3+e, K_3+e, J_4)$. Assume towards a contradiction that $R(K_3, K_3, J_4) < R(K_3+e, K_3+e, J_4)$, and let G be a $(K_3+e, K_3+e, J_4; 17)$-coloring. We may assume without loss of generality that there is a red K_3 in G with vertices $\{v_1, v_2, v_3\}$. Let the graph H be the red component of G induced by $V(G) \setminus \{v_1, v_2, v_3\}$. Clearly H contains no K_3+e. Also, H cannot contain a $\overline{K_5}$, since otherwise together with $\{v_1, v_2\}$ it would span a green and blue J_7 in G. By Lemma 2 of the next section, $J_7 \to (K_3+e, J_4)$, which is a contradiction. So H is a $(K_3+e, K_5; 14)$-good graph, which is impossible since $R(K_3+e, K_5) = 14$ [6]. □

In the known non-trivial cases it appears that extending the parameter K_3 to K_3+e does not change Ramsey numbers. Irving [13] stated that for $k > 2$, it seems likely that $R_k(K_3+e) = R_k(K_3)$. The following theorem may add credence to or disprove this statement. It is known that $51 \leq R_4(K_3) \leq 62$ [4][8].

Theorem 2
(a) If $R_4(K_3) = 51$, then $R_4(K_3+e) = R(K_3, K_3, K_3, K_3+e) = 52$, and
(b) If $R_4(K_3) > 51$, then $R_4(K_3+e) = R_4(K_3)$.

Proof: Suppose H is a $(K_3+e, K_3+e, K_3+e, K_3+e; n)$-coloring for some $n \geq R_4(K_3)$ and $n \geq 52$. Then we may assume without loss of generality that H contains a red K_3. Let v be a vertex of this K_3, then we may also assume that $|N_g(v)| \geq \lceil (n-3)/3 \rceil \geq 17$. Note that the green color cannot occur in $N_g(v)$. However, $N_g(v)$ induces a (K_3+e, K_3+e, K_3+e)-coloring, and since $R_3(K_3+e) = 17$ [26], $|N_g(v)| \leq 16$. This gives rise to a contradiction, and thus proves (b) and the upper bound for (a). What remains to be shown is the lower bound in (a).

We construct a $(K_3, K_3, K_3, K_3+e; 51)$-coloring C_{51} by extending the well known Chung $(K_3, K_3, K_3, K_3; 50)$-coloring C_{50} [4]. Partition the set of vertices of C_{50} as $V = R \cup G \cup B \cup \{x, y\}$, where $|R| = |G| = |B| = 16$. The edge $\{x, y\}$ is yellow, edges in $\{\{x, v\}, \{y, v\} : v \in R\}$ are red, edges in $\{\{x, v\}, \{y, v\} : v \in G\}$ are green, and edges in $\{\{x, v\}, \{y, v\} : v \in B\}$

are blue. Each of R, G, and B induces a $(K_3, K_3, K_3; 16)$-coloring, where the first has no red edges, the second no green edges, and the third no blue edges. Chung also described a way to color the edges between R, G, and B without forming a monochromatic K_3. We omit the details as they are irrelevant to our proof. The additional vertex z is connected to R, G, and B in the same way as x and y, and the edges $\{x, z\}$ and $\{y, z\}$ are yellow. This C_{51} on the vertex set $V \cup \{z\}$ has exactly one monochromatic K_3, namely an isolated yellow K_3 on $\{x, y, z\}$. Thus, easily, C_{51} is a $(K_3, K_3, K_3, K_3 + e; 51)$-coloring. By the monotonicity of Ramsey numbers, (a) follows. □

We close this section with a case where a similar but unconditional equality can be proven even when the Ramsey number is unknown, namely for the case $30 \leq R(K_3, K_3, K_4) \leq 31$ [14][22][23]. Our next theorem improves on the old result that $R(K_3, K_3, K_4) = R(K_3 + e, K_3 + e, K_4)$ [26][1]. In the following, P_k will denote a path on k vertices.

Theorem 3 $R(K_3, K_3, K_4) = R(K_3 + e, K_3 + e, K_5 - P_3)$.

Proof: Let $n = R(K_3, K_3, K_4)$, and assume towards a contradiction that G is a $(K_3 + e, K_3 + e, K_5 - P_3; n)$-coloring. By the remarks above we know that $30 \leq n \leq 31$. There is a blue K_4 in G, let its vertices be $\{v_1, v_2, v_3, v_4\}$. If c is red or green and $|N_c(v_i)| > 2$, then $N_c(v_i)$ induces a $(K_3 + e, K_5 - P_3)$-good graph. Since $R(K_3 + e, K_5 - P_3) = 10$ [6], then in both cases $N_c(v_i)$ has order at most 9. Let $N_i = N_b(v_i) \setminus \{v_1, v_2, v_3, v_4\}$, then $|N_i| \geq (n - 4) - 2 \cdot 9 \geq 8$. With four such N_i's covering $n - 4$ vertices, some vertex v must be contained in at least 2 of them. Then $\{v, v_1, v_2, v_3, v_4\}$ forms a blue $K_5 - P_3$ in G. □

3 Ramsey Number $R(K_3, K_3, J_4)$

The smallest open case for complete graphs in Figure 1 is $R(K_3, K_3, K_4)$, of which the current bounds of 30 and 31 have not been improved since 1998 [22]. Obtaining the exact value has continued to remain beyond the reach of computational methods. In this section we prove that the Ramsey number $R(K_3, K_3, K_4 - e)$ is equal to 17, considering it as an intermediate step between $R(K_3, K_3, K_3)$ and the solution to the elusive $R(K_3, K_3, K_4)$. The proof of $R(K_3, K_3, J_4) = 17$ needs some lemmas, which are then used in two different computational approaches.

Lemma 1 *Every $(K_3 + e, J_4; 6)$-good graph contains a C_6 or $G = 2K_3$.*

Proof: Suppose that G is a C_6-free $(K_3 + e, J_4; 6)$-good graph with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ that is not equal to $2K_3$. First, we show that G cannot contain a K_3: assume $\{v_1, v_2, v_3\}$ forms a K_3. Then for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, $\{v_i, v_j\}$ cannot be an edge. This means that $\{v_4, v_5, v_6\}$ must induce a K_3 to avoid a $\overline{J_4}$, and the resulting graph is equal to $2K_3$.

We can now assume that G is a (K_3, J_4)-good graph. Next, we consider the cases with respect to the length of the longest path in G as follows.

(P_6) Assume that the longest path is $P_6 = v_1v_2v_3v_4v_5v_6$. To prevent $\{v_1, v_3, v_4, v_6\}$ from forming a $\overline{J_4}$, the graph G must contain either edge $\{v_1, v_4\}$ or $\{v_3, v_6\}$, since any other additional edge would form a C_6 or K_3. Thus, without loss of generality, we can assume that $\{v_3, v_6\} \in E(G)$. Similarly, G must contain an additional edge in $\{v_1, v_2, v_4, v_6\}$, otherwise there is a $\overline{J_4}$. Any such edge forms a K_3 or a C_6, which is a contradiction.

(P_k) Assume that the longest path P in G is of length $k < 6$. By considering one or two forbidden $\overline{J_4}$'s, it can be shown that P together with additional edges would contain P_{k+1}, leading to a contradiction. We leave the details for the reader to verify.

□

Lemma 2 $J_7 \rightarrow (K_3 + e, J_4)$.

Proof: Suppose that there is a coloring C of $J_7 = K_7 \setminus \{x, y\}$ witnessing the contrary. Let G be the graph formed by the edges of the first color of C. Then G contains no $K_3 + e$ and $\overline{G} \setminus \{x, y\}$ contains no J_4. Further, $G - x$ is a $(K_3 + e, J_4; 6)$-good graph, and by Lemma 1 it contains a C_6 or is equal to $2K_3$.

First assume that $G - x$ contains a $C_6 = v_1v_2v_3v_4v_5y$ as shown in Figure 2. Note that $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, y\}$ are independent sets. To avoid J_4 in $\overline{G} \setminus \{x, y\}$ on vertices $\{v_2, v_4, x, y\}$, the graph G must contain at least one of $\{x, v_2\}$ or $\{x, v_4\}$. Without loss of generality assume that $\{x, v_2\}$ is in the graph. Now to avoid $\overline{J_4}$ on the set $S = \{v_1, v_3, v_5, x\}$, G must contain at least two edges with both endpoints in S. However, any two such edges would complete a $K_3 + e$, a contradiction. On the other hand, suppose $G - x = 2K_3$. Any edge from x to $2K_3$ would form a $K_3 + e$, so all 6 edges must be in $\overline{G}$, but this leads to a J_4 in $\overline{G} \setminus \{x, y\}$. □

The above shows that $J_7 \rightarrow (K_3 + e, J_4)$. Note that this also easily implies $R(K_3 + e, J_4) = 7$.

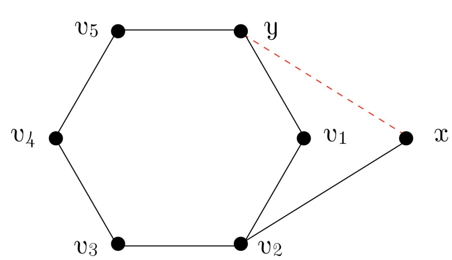

Figure 2: $J_7 \rightarrow (K_3 + e, J_4)$

We will call a graph G *unsplittable* if $G \rightarrow (K_3, J_4)$, otherwise G is *splittable*. Our approach to obtain (K_3, K_3, J_4)-colorings is based on Lemma 2 (here, a weaker arrowing $J_7 \rightarrow (K_3, J_4)$ would suffice), which implies that all such colorings can be produced from a splittable (J_7, K_3)-good graph.

Lemma 3 *If m is the largest order of all splittable (J_7, K_3)-good graphs, then $R(K_3, K_3, J_4) = m + 1$.*

Proof: By Lemma 2, the complement of the red subgraph (union of green and blue subgraphs) of any $(K_3, K_3, J_4; n)$-coloring is a splittable $(J_7, K_3; n)$-good graph. This shows $R(K_3, K_3, J_4) \leq m + 1$. The edges of the complement of a splittable graph G of order m give the red part of a $(K_3, K_3, J_4; m)$-coloring, while any witness to the splittability of G defines the other two colors. This shows $R(K_3, K_3, J_4) \geq m + 1$. □

Using the argument in the proof of Lemma 3 we can construct all $(K_3, K_3, J_4; n)-$colorings by splitting every $(J_7, K_3; n)$-good graph. The full set $\mathcal{R}(K_3, J_7)$ has been enumerated [9][18], and $R(K_3, J_7) = 21$ [11]. We independently computed $\mathcal{R}(K_3, J_7)$ using a simple vertex by vertex extension algorithm that generates $\mathcal{R}(K_3, J_7; n + 1)$ from $\mathcal{R}(K_3, J_7; n)$, and utilizes the program *nauty* [16][17] to eliminate graph isomorphs. Our results agreed exactly with previously reported data shown in Table 1.

n	$\lvert\mathcal{R}(K_3, J_7; n)\rvert$	#edges
1	1	0
2	2	0-1
3	3	0-2
4	7	0-4
5	14	0-6
6	38	0-9
7	105	2-12
8	392	3-16
9	1697	4-20
10	9430	5-25
11	58522	8-30
12	348038	11-36
13	1323836	15-36
14	2447170	19-40
15	1358974	24-45
16	158459	30-48
17	4853	37-50
18	225	43-51
19	1	54
20	1	60

Table 1: Statistics of $\mathcal{R}(K_3, J_7)$

None of the complements of graphs in $\mathcal{R}(K_3, J_7; n)$ for $n \geq 17$ could be split into a (K_3, K_3, J_4)-coloring, which implies the following theorem.

Theorem 4 $R(K_3, K_3, J_4) = 17$.

Proof: We determined all splittable (J_7, K_3)-good graphs of maximal order via two independent computational methods. First, for each $(J_7, K_3; n)$-good graph G we created a conjunctive normal form (CNF) Boolean formula $\phi(G)$ which is satisfiable iff $G \not\rightarrow (K_3, J_4)$. The satisfiability of $\phi(G)$ was tested using a standard SAT-solver. In the second method we implemented our own computer algorithm which exhaustively searched through all relevant edge colorings.

Neither of the two methods found any splittable $(J_7, K_3; n)$-good graphs for $n \geq 17$, and both found the same 11813 splittable $(J_7, K_3; 16)$-good graphs. So, by Lemma 3, $R(K_3, K_3, J_4) = 17$. Below we give further details about each method.

Splittability via Satisfiability:
If G is a (J_7, K_3)-good graph, we wish to see if $G \rightarrow (K_3, J_4)$. We consider each edge of G to be a Boolean variable, and our colors as F and T. We define the clauses of $\phi(G)$ as follows:

- For each K_3 with edges $\{e_1, e_2, e_3\}$ include the clause $(e_1 \vee e_2 \vee e_3)$. This forces at least one edge to have color T, so no K_3 will be formed in color F.
- For each J_4 with edges $\{e_1, e_2, e_3, e_4, e_5\}$ include the clause $(\overline{e_1} \vee \overline{e_2} \vee \overline{e_3} \vee \overline{e_4} \vee \overline{e_5})$. This forces at least one edge to have color F, so no J_4 will be formed in color T.

Clearly, the resulting $\phi(G)$ is satisfiable if and only if G is splittable. We used the SAT-solver PicoSAT [2], the gold medal winner of the 2007 International SAT Competition in the industrial category, and found that no $(J_7, K_3; 17)$-good graphs were splittable.

Recursive Coloring:
We implemented the function $f(\mathit{uncolored}, \mathit{green}, \mathit{blue})$ that takes three graphs as input. It attempts to take an uncolored edge and add it to the current set of green edges or blue edges, and recurse. If either recursion is successful, **True** is returned. In this way, $f(E(G), \emptyset, \emptyset)$ returns **True** if G can be split into a (K_3, J_4)-good graph using the following algorithm.

$f(\mathit{uncolored}, \mathit{green}, \mathit{blue}) =$
False if *green* contains a K_3 or *blue* contains a J_4
True if *uncolored* is empty
Else let $\{i, j\}$ be an edge in *uncolored*,
return $f(\mathit{uncolored} \setminus \{i, j\}, \mathit{green} \cup \{i, j\}, \mathit{blue})$
$\vee f(\mathit{uncolored} \setminus \{i, j\}, \mathit{green}, \mathit{blue} \cup \{i, j\})$

□

It could be tempting to obtain Theorem 4 by a simpler approach of splitting (K_7, K_3)-good graphs. However, the number of such graphs is much larger than (J_7, K_3)-good graphs, and it seems infeasible even just to enumerate the set $\mathcal{R}(K_7, K_3)$.

In another attempt to construct $\mathcal{R}(K_3, K_3, J_4; n)$ for $n \geq 17$ we tried to enumerate $\mathcal{R}(K_6, J_4)$, since $K_6 \rightarrow (K_3, K_3)$ and thus splitting

(K_6, J_4)-good graphs leads to all (K_3, K_3, J_4)-colorings. For more than 12 vertices the number of (K_6, J_4)-good graphs became too large to handle. The attempt was continued by extending only suitably selected $(K_6, J_4; 12)$-good graphs. Eventually, all 6817238 $(K_6, J_4; 19)$- and 24976 $(K_6, J_4; 20)$-good graphs were constructed, and none were found on 21 vertices, confirming the previously unpublished results by McNamara that $R(K_6, J_4) = 21$ [19]. No $(K_6, J_4; 19)$-good graphs could be split into a (K_3, K_3, J_4)-coloring, proving $R(K_3, K_3, J_4) \leq 19$. However, the attempt to enumerate $\mathcal{R}(K_6, J_4; 18)$ was computationally infeasible.

4 More Bounds

Theorem 5 $21 \leq R(K_3, J_4, J_4) \leq 27$.

Proof: The lower bound is established by a $(K_3, J_4, J_4; 20)$-coloring presented in Figure 3. It was obtained by splitting the unique $(J_7, K_3; 20)$-good graph.

```
0 2 2 3 3 2 2 3 2 3 3 3 2 2 1 1 1 1 1 1
2 0 3 2 3 1 2 3 3 2 1 2 3 1 2 1 1 1 3 2
2 3 0 3 2 1 3 2 2 1 2 3 1 3 1 2 1 1 3 2
3 2 3 0 2 1 3 2 1 2 3 1 3 2 1 1 2 1 2 3
3 3 2 2 0 1 1 1 3 3 2 2 2 3 1 1 1 2 2 3
2 1 1 1 1 0 3 2 3 2 2 2 3 3 2 2 3 3 1 1
2 2 3 3 1 3 0 2 3 3 2 1 1 1 3 2 2 1 2 1
3 3 2 2 1 2 2 0 1 1 1 3 3 2 2 3 3 1 1 2
2 3 2 1 3 3 3 1 0 2 3 2 1 1 2 3 1 2 2 1
3 2 1 2 3 2 3 1 2 0 2 1 2 1 3 1 3 2 3 1
3 1 2 3 2 2 2 1 3 2 0 1 1 2 1 3 2 3 3 1
3 2 3 1 2 2 1 3 2 1 1 0 2 3 3 2 1 3 1 2
2 3 1 3 2 3 1 3 1 2 1 2 0 2 2 1 2 3 1 3
2 1 3 2 3 3 1 2 1 1 2 3 2 0 1 2 3 2 1 3
1 2 1 1 1 2 3 2 2 3 1 3 2 1 0 3 2 3 2 3
1 1 2 1 1 2 2 3 3 1 3 2 1 2 3 0 3 2 2 3
1 1 1 2 1 3 2 3 1 3 2 1 2 3 2 3 0 2 3 2
1 1 1 1 2 3 1 1 2 2 3 3 3 2 3 2 2 0 3 2
1 3 3 2 2 1 2 1 2 3 3 1 1 1 2 2 3 3 0 2
1 2 2 3 3 1 1 2 1 1 1 2 3 3 3 3 2 2 2 0
```

Figure 3: A $(K_3, J_4, J_4; 20)$-coloring

For the upper bound, consider the graph G formed by the green edges of any (K_3, J_4, J_4)-coloring. By Lemma 2, G must be a (J_4, J_7)-good graph. $R(J_4, J_7) = 28$, and it is known that there exists a unique $(J_4, J_7; 27)$-good graph [20]. This is the well known strongly 10-regular Schläfli graph [25]. Reducing graph splittability to Boolean satisfiability as in Section 3, we determined that the complement of the Schläfli graph is unsplittable, and thus $R(K_3, J_4, J_4) \leq 27$. □

We note that, interestingly, the same Schläfli graph can be split into two J_4-free graphs, which establishes the bound $R_3(J_4) \geq 28$ [7].

Theorem 6 $33 \leq R(J_4, J_4, K_4)$.

Proof: The lower bound is established by a $(J_4, J_4, K_4; 32)$-coloring presented in Figure 4. This coloring was found using a standard simulated annealing algorithm.

```
0 1 3 3 2 3 3 2 2 1 3 2 2 1 2 3 3 1 3 2 1 1 1 2 2 1 1 1 3 2 3 3
1 0 3 1 2 1 3 2 1 3 1 2 1 3 3 1 3 3 3 1 2 3 2 1 1 2 2 3 3 2 1 2
3 3 0 2 2 2 1 1 3 3 3 2 3 2 2 1 1 3 3 2 3 2 2 1 1 3 2 1 1 3 1 1
3 1 2 0 3 3 2 3 2 3 3 2 2 3 3 3 1 1 2 3 3 1 1 2 3 1 1 2 1 2 2 1
2 2 2 3 0 1 3 3 1 2 2 3 2 3 1 1 2 1 3 1 1 3 1 1 3 3 1 2 3 3 2 3
3 1 2 3 1 0 1 1 3 3 2 3 2 3 2 2 1 2 3 3 3 3 1 2 2 1 3 1 1 2 3 1
3 3 1 2 3 1 0 3 1 1 2 3 1 1 2 2 2 1 3 3 3 1 3 3 2 1 2 3 3 2 1 2
2 2 1 3 3 1 3 0 3 1 3 1 1 2 3 2 3 2 2 1 1 3 3 2 1 2 1 3 2 3 3 3
2 1 3 2 1 3 1 3 0 1 2 1 3 2 2 2 3 3 3 3 1 2 2 3 1 2 2 1 3 1 3 1
1 3 3 3 2 3 1 1 1 0 2 2 3 2 1 1 1 2 1 1 2 3 1 1 3 2 2 3 2 3 3 3
3 1 3 3 2 2 2 3 2 2 0 1 3 3 1 1 1 1 2 2 1 3 1 2 3 3 3 1 3 3 3 1
2 2 2 2 3 3 3 1 1 2 1 0 3 3 3 2 3 2 1 3 3 1 1 1 2 1 1 3 1 3 3 3
2 1 3 2 2 2 1 1 3 3 3 3 0 2 1 3 1 2 2 3 3 3 1 1 3 2 2 1 1 1 3 2
1 3 2 3 3 3 1 2 2 2 3 3 2 0 1 3 1 3 1 1 2 3 3 1 2 3 1 3 1 2 2 2
2 3 2 3 1 2 2 3 2 1 1 3 1 1 0 3 2 3 2 3 2 1 3 3 1 1 3 2 3 3 1 1
3 1 1 3 1 2 2 2 2 1 1 2 3 3 3 0 3 3 1 2 2 1 3 3 3 1 1 3 1 1 2 2
3 3 1 1 2 1 2 3 3 1 1 3 1 1 2 3 0 3 3 2 1 1 2 2 3 2 3 3 3 1 2 3
1 3 3 1 1 2 1 2 3 2 1 2 2 3 3 3 3 0 1 2 3 2 2 1 1 3 3 2 1 1 2 3
3 3 3 2 3 3 3 2 3 1 2 1 2 1 2 1 3 1 0 2 1 2 2 3 2 3 3 1 2 3 1 1
2 1 2 3 1 3 3 1 3 1 2 3 3 1 3 2 2 2 2 0 2 3 3 3 3 2 2 1 1 2 1 1
1 2 3 3 1 3 3 1 1 2 1 3 3 2 2 2 1 3 1 2 0 2 3 2 3 1 3 2 1 3 1 3
1 3 2 1 3 3 1 3 2 3 3 1 3 3 1 1 1 2 2 3 2 0 3 2 1 2 3 1 3 2 2 2
1 2 2 1 1 1 3 3 2 1 1 1 1 3 3 3 2 2 2 3 3 3 0 2 1 3 3 2 3 2 1 3
2 1 1 2 1 2 3 2 3 1 2 1 1 1 3 3 2 1 3 3 2 2 2 0 3 1 2 3 3 3 3 1
2 1 1 3 3 2 2 1 1 3 3 2 3 2 1 3 3 1 2 3 3 1 1 3 0 2 2 2 3 3 2 3
1 2 3 1 3 1 1 2 2 2 3 1 2 3 1 1 2 3 3 2 1 2 3 1 2 0 3 2 3 1 3 3
1 2 2 1 1 3 2 1 2 2 3 1 2 1 3 1 3 3 3 2 3 3 3 2 2 3 0 3 2 3 1 1
1 3 1 2 2 1 3 3 1 3 1 3 1 3 2 3 3 2 1 1 2 1 2 3 2 2 3 0 2 3 3 2
3 3 1 1 3 1 3 2 3 2 3 1 1 1 3 1 3 1 2 1 1 3 3 3 3 3 2 2 0 2 2 2
2 2 3 2 3 2 2 3 1 3 3 3 1 2 3 1 1 1 3 2 3 2 2 3 3 1 3 3 2 0 1 1
3 1 1 2 2 3 1 3 3 3 3 3 3 2 1 2 2 2 1 1 1 2 1 3 2 3 1 3 2 1 0 3
3 2 1 1 3 1 2 3 1 3 1 3 2 2 1 2 3 3 1 1 3 2 3 1 3 3 1 2 2 1 3 0
```

Figure 4: A $(J_4, J_4, K_4; 32)$-coloring

□

5 Future Work

Our work answers some of the open questions of Arste, Klamroth, and Mengersen [1], while others remain open and should be studied more. In particular, we think that further progress on the known bounds for

$R(K_3, J_4, J_4)$ and $R(J_4, J_4, J_4)$ is feasible, definitely more so than for $R(K_3, K_3, K_4)$. Another interesting project would be to study 3-color Ramsey numbers with the parameters as in this paper, but in addition with at least one color avoiding C_4.

Acknowledgement. We would like to thank the anonymous reviewers whose suggestions led to improved presentation of this work.

References

[1] J. Arste, K. Klamroth and I. Mengersen, Three Color Ramsey Numbers for Small Graphs, *Utilitas Mathematica*, **49** (1996) 85–96.

[2] A. Biere, PicoSAT Essentials, *Journal on Satisfiability, Boolean Modeling and Computation (JSAT)*, **4** (2008) 75–97.

[3] S.A. Burr, P. Erdős, R.J. Faudree and R.H. Schelp, On the Difference between Consecutive Ramsey Numbers, *Utilitas Mathematica*, **35** (1989) 115–118.

[4] F.R.K. Chung, On the Ramsey Numbers $N(3, 3, ..., 3; 2)$, *Discrete Mathematics*, **5** (1973) 317–321.

[5] V. Chvátal and F. Harary, Generalized Ramsey Theory for Graphs, III. Small Off-Diagonal Numbers, *Pacific Journal of Mathematics,* **41** (1972) 335–345.

[6] M. Clancy, Some Small Ramsey Numbers, *Journal of Graph Theory*, **1** (1977) 89–91.

[7] G. Exoo, Three Color Ramsey Number of $K_4 - e$, *Discrete Mathematics*, **89** (1991) 301–305.

[8] S. Fettes, R.L. Kramer and S.P. Radziszowski, An Upper Bound of 62 on the Classical Ramsey Number $R(3, 3, 3, 3)$, *Ars Combinatoria*, **72** (2004) 41–63.

[9] R. Fidytek, Ramsey Graphs $R(K_n, K_m - e)$, `http://fidytek.inf.ug.edu.pl/ramsey`.

[10] R.E. Greenwood and A.M. Gleason, Combinatorial Relations and Chromatic Graphs, *Canadian Journal of Mathematics*, **7** (1955) 1–7.

[11] U. Grenda and H. Harborth, The Ramsey Number $r(K_3, K_7 - e)$, *Journal of Combinatorics, Information & System Sciences,* **7** (1982) 166–169.

[12] R. Hill and R.W. Irving, On Group Partitions Associated with Lower Bounds for Symmetric Ramsey Numbers, *European Journal of Combinatorics*, **3** (1982) 35–50.

[13] R.W. Irving, Generalised Ramsey Numbers for Small Graphs, *Discrete Mathematics*, **9** (1974) 251–264.

[14] J.G. Kalbfleisch, Chromatic Graphs and Ramsey's Theorem, *Ph.D. thesis*, University of Waterloo, January 1966.

[15] D.L. Kreher, Li Wei and S.P. Radziszowski, Lower Bounds for Multi-Colored Ramsey Numbers From Group Orbits, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **4** (1988) 87–95.

[16] B.D. McKay, Practical Graph Isomorphism, *Congressus Numerantium*, **30** (1981) 45–87.

[17] B.D. McKay, nauty User's Guide (Version 2.4), Technical Report TR-CS-90-02, Department of Computer Science, Australian National University, (1990). The latest version of the software is available at `http://cs.anu.edu.au/~bdm/nauty`.

[18] B.D. McKay, K. Piwakowski and S.P. Radziszowski, Ramsey Numbers for Triangles versus Almost-Complete Graphs, *Ars Combinatoria*, **73** (2004) 205–214.

[19] J. McNamara, SUNY Brockport, personal communication (1995).

[20] J. McNamara and S.P. Radziszowski, The Ramsey Numbers $R(K_4 - e, K_6 - e)$ and $R(K_4 - e, K_7 - e)$, *Congressus Numerantium*, **81** (1991) 89–96.

[21] K. Piwakowski, A New Upper Bound for $R_3(K_4 - e)$, *Congressus Numerantium*, **128** (1997) 135–141.

[22] K. Piwakowski and S.P. Radziszowski, $30 \leq R(3,3,4) \leq 31$, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **27** (1998) 135–141.

[23] K. Piwakowski and S.P. Radziszowski, Towards the Exact Value of the Ramsey Number $R(3,3,4)$, *Congressus Numerantium*, **148** (2001) 161–167.

[24] S.P. Radziszowski, Small Ramsey Numbers, *Electronic Journal of Combinatorics*, DS1, revision #13, August 2011, 84 pages, `http://www.combinatorics.org/Surveys`.

[25] J.J. Seidel, Strongly Regular Graphs, in *Surveys in Combinatorics* (2nd ed.), edited by B. Bollobás, London Math. Soc. Lecture Note Series **38**, Cambridge U.P. (1979), 157–180.

[26] Yang Yuansheng and P. Rowlinson, The Third Ramsey Numbers for Graphs with at Most Four Edges, *Discrete Mathematics*, **125** (1994) 399–406.