Diagonal Conjecture for Classical Ramsey Numbers

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joint work with Meilian Liang and Xiaodong Xu

DisCoMathS
9 October 2019, Henrietta
Plan of Talk

▶ Ramsey numbers: computational perspective
   (16 min)

▶ Diagonal Conjecture: origins, consequences and evidence
   (32 min)

Ramsey Numbers

- $R(G, H) = n$ iff minimal $n$ such that in any 2-coloring of the edges of $K_n$ there is a monochromatic $G$ in the first color or a monochromatic $H$ in the second color.

- 2-colorings $\cong$ graphs, $R(m, n) = R(K_m, K_n)$

- Generalizes to $k$ colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist
Unavoidable classics

\[ R(3, 3) = 6 \]

\[ R(3, 5) = 14 \] [GRS'90]
Asymptotics

diagonal cases

- **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)
  \[
  \frac{\sqrt{2}}{e} 2^{n/2} n < R(n, n) < R(n + 1, n + 1) \leq \left( \frac{2n}{n} \right) n^{-c \frac{\log n}{\log \log n}}
  \]

- **Conjecture** (Erdős 1947, $100)
  \[
  \lim_{n \to \infty} R(n, n)^{1/n} \text{ exists.}
  \]
  If it exists, it is between $\sqrt{2}$ and 4 ($250$ for value).

- **Theorem** (Chung-Grinstead 1983)
  \[
  L = \lim_{k \to \infty} R_k(3)^{1/k} \text{ exists.}
  \]
Asymptotics
Ramsey numbers avoiding $K_3$

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound
  \[ R(3, n) = \Theta \left( \frac{n^2}{\log n} \right) \]

- Bohman/Keevash 2009/2013, triangle-free process
- Fiz Pontiveros-Griffiths-Morris, lower bound, 2013
  Shearer, upper bound, 1983
  \[ \left( \frac{1}{4} + o(1) \right) \frac{n^2}{\log n} \leq R(3, n) \leq (1 + o(1)) \frac{n^2}{\log n} \]
Test - Hunt - Exhaust
Ramsey numbers

- **Testing**: do it right.
  Graph $G$ is a witness of $R(m, n) > k$ iff $|V(G)| = k$, $cl(G) < m$ and $\alpha(G) < n$.
  Lab in a 200-level course.

- **Hunting**: constructions and heuristics.
  Given $m$ and $n$, find a witness $G$ for $k$ larger than others.
  Challenge projects in advanced courses.
  Master: Geoffrey Exoo 1986–

- **Exhausting**: generation, pruning, isomorphism.
  Prove that for given $m, n$ and $k$, there does not exist any witness as above. Hard without nauty/traces.
  Master: Brendan McKay 1991–
Values and bounds on $R(k, l)$

two colors, avoiding $K_k, K_l$

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[SPR, EJJC survey Small Ramsey Numbers, revision #15, 2017, with updates]
Small $R(k, l)$, references

$R(5, 5) \leq 48$, Angeltveit-McKay 2018.

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New avalanche of improved upper bounds after LP attack for higher $k$ and $l$ by Angeltveit-McKay.
Diagonal Conjecture (DC) motivation

\( R(k, l) \) seem to decrease along \( \uparrow \) diagonals

Best known lower bounds for \( k \leq l \) satisfy

\[
LB(k, l) > LB(k - 1, l + 1),
\]

except a mild hick-up at (8,10) vs (7,11).
Diagonal Conjecture (DC)

**Two-Color DC:**
\[ R(k, l) \geq R(k - 1, l + 1) \text{ for } 3 \leq k \leq l. \]

As we move away from the diagonal of the table with Ramsey numbers \( R(k, l) \), while preserving \( k + l \), the values decrease.

**Multicolor DC:**
For \( r \geq 3 \), \( a_i \geq 3 \) \( (1 \leq i \leq r) \), if \( a_{r-1} \leq a_r \), then
\[ R(a_1, \cdots, a_r) \geq R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1). \]
Diagonal Conjecture

cont.

Hints:

▶ Observed long ago ..., probably by many.
▶ Stronger versions of DC with $>$ instead of $\geq$ are plausible.
▶ Known values and bounds do not contradict either DC.


▶ Wang’s proof is not correct.

A strange alternate definition of Ramsey numbers, followed by unfounded circular arguments between the definitions.
\[
\lim_{r \to \infty} R_r(k)^{1/r}
\]

For \( k = a_1 = \cdots = a_r \), let \( R_r(k) = R(a_1, \cdots, a_r) \).

**Theorem** (Chung-Grinstead 1983)

\( L_3 = \lim_{r \to \infty} R_r(3)^{1/r} \) exists, finite or infinite.

- The same argument can be used to show that \( L_k = \lim_{r \to \infty} R_r(k)^{1/r} \) exists for all \( k > 3 \), finite or infinite.
- \( L_3 > 3.1996 \approx 1073^{1/6} \)
  
- Erdős was inclined to think that \( L_3 = \infty \) (Li,XR).
Lemma

*If DC holds, then for every integer* \( a \geq 3 \) *we have*

\[
R_{2r}(a) > (R_r(a - 1) - 1)(R_r(a + 1) - 1).
\]

Proof.

*Apply DC* \( r \) *times to* \( R_{2r}(a) \).

*Use a special case of Abbott’s lower bound construction*

\[
R(a_1, \ldots, a_{2r}) > (R(a_1, \ldots, a_r) - 1)(R(a_{r+1}, \ldots, a_r) - 1).
\]
**Theorem**

*If DC holds and \( \lim_{r \to \infty} R_r(3)^{\frac{1}{r}} \) is finite, then \( \lim_{r \to \infty} R_r(a)^{\frac{1}{r}} \) is finite for every \( a \geq 3 \).*

**Proof.**

Induction on \( a \) via

\[
\lim_{r \to \infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a - 1)^{\frac{1}{r}}} \geq \lim_{r \to \infty} \frac{R_r(a + 1)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}}.
\]
LB vs UB on $R_r(3)$ and $L_3$

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Known bounds on $R_r(3)$ for $r \leq 10$,

$R_r(3) \leq (e - \frac{1}{6})r! + 1 \approx 2.55r!$, based on $R_4(3) \leq 62$.

$$L_3 = \lim_{r \to \infty} R_r(3)^{1/r} > 3.1996$$
Consequences of DC

another main theorem

Theorem

If DC holds, then for every $a \geq 3$, we have

$$\lim_{r \to \infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a-1)^{\frac{1}{r}}} > 1.$$ 

Proof.

First show that

$$\frac{R_r(2a-1) - 1}{R_r(a) - 1} \geq R_r(3) - 1 \geq 2^r,$$

next that

$$\lim_{r \to \infty} \frac{R_r(2a)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}} \geq 2,$$

then finish by contradiction.
Consequences of DC

summary

Theorem

If DC holds, then it is true that:

(a) all $L_k$’s are finite or all of them are infinite, and
(b) if $L_3$ is finite then $L_k < L_{k+1}$ for all $k \geq 3$.

▶ $\lim_{k \to \infty} L_k = \infty$, even without assuming validity of DC
(by Abbott 1965, and by the previous theorem).

▶ If our perspective that the known lower bounds are much closer
   to $R_r(k)$ than the upper bounds is correct, it would add weight to
   the case that all limits $L_k$ are finite.
Evidence for DC - two colors

$DC(s, t)$ stands for $R(s, t) \geq R(s - 1, t + 1)$, $3 \leq s \leq t$.

(a) $DC(3, t)$ is true for all $t \geq 3$.
(b) $DC(4, t)$ is true for all $t \geq 4$.
(c) $DC(5, 5)$, $DC(5, 6)$ and $DC(5, 7)$ are true.
(d) The above establishes the validity of $DC(s, t)$ for all $s < 5$, and all cases with $s + t \leq 12$, except $DC(6, 6)$.
(e) The further we go from the diagonal of the DC conjecture, the easier it seems to corroborate it. We anticipate DC to be the hardest on the diagonal itself, i.e. proving that $R(t, t) \geq R(t - 1, t + 1)$ for any $t \geq 6$.
(f) Little bump at $DC(8, 10)$.
Evidence for DC - more colors
relying on lower bounds

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Known lower bounds $LB_1$ and $LB_2$ on $R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1)$ and $R(a_1, \cdots, a_r)$ for some DC-adjacent pairs of parameters $A_1$ and $A_2$. 
Shannon capacity $c(G)$ and limits $L_k$

$\alpha(G^r) =$ independence of the strong $r$-th power of graph $G$
$c(G) =$ Shannon capacity of a noisy channel modeled by $G$

$$c(G) = \lim_{r \to \infty} \alpha(G^r)^{\frac{1}{r}}$$

We proved (XR 2013):

- For any fixed $k \geq 3$, $L_k = \lim_{r \to \infty} R_r(k)^{1/r}$ is equal to the supremum of the Shannon capacity $c(G)$ over all graphs $G$ with $\alpha(G) = k - 1$, but this supremum cannot be achieved by any finite graph power, $G^{r_0}$. 
Two problems beyond DC

▶ Generalizing DC.

For connected graphs $G_i$ with $s \leq t$, is it true that

$$R(G_1, G_2, \ldots, K_{s-1}, K_{t+1}) \leq R(G_1, G_2, \ldots, K_s, K_t)?$$

We think ‘YES’, but make no more conjectures.

▶ Let $r \geq 3$, $a_i \geq 3$, $a_{r-1} \leq a_r$, and $C$ be a coloring witnessing

$$n < R(a_1, \ldots, a_{r-2}, a_{r-1} - 1, a_r + 1).$$

Let $G =$ all edges of $C$ in colors $r - 1$ and $r$, $|V(G)| = n$.

Is it true that $G \not\rightarrow (a_{r-1}, a_r)^e$?

i.e. that there exists a 2-coloring of $E(G)$ without
any $K_{a_{r-1}}$ in the first color and without $K_{a_r}$ in the second color?

We think ‘weaker YES’.
Papers to look at


▶ Meilian Liang, SPR, Xiaodong Xu
On a Diagonal Conjecture for Classical Ramsey Numbers

▶ SPR, revision #15 of the dynamic survey paper,
*Small Ramsey Numbers*,

777+ papers by many authors ...
Thanks for listening!