Diagonal Conjecture for Classical Ramsey Numbers

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, NY, USA

joint work with Meilian Liang and Xiaodong Xu

GGTW, 13 August 2019, Ghent
Ramsey numbers

- $R(G, H) = n$ iff $n$ is minimal such that in any 2-coloring of the edges of $K_n$ there exists a monochromatic $G$ in the first color or a monochromatic $H$ in the second color.

- 2-colorings $\cong$ graphs, $R(k, l) = R(K_k, K_l)$
- Generalizes to $r$ colors, $R(G_1, \cdots, G_r)$

- Theorem (Ramsey 1930): Ramsey numbers exist
Values and bounds on \( R(k, l) \)

two colors, avoiding \( K_k, K_l \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
<td>28</td>
<td>36</td>
<td>40</td>
<td>47</td>
<td>53</td>
<td>60</td>
<td>67</td>
<td>74</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>25</td>
<td>36</td>
<td>41</td>
<td>49</td>
<td>59</td>
<td>73</td>
<td>92</td>
<td>102</td>
<td>128</td>
<td>138</td>
<td>147</td>
<td>155</td>
</tr>
<tr>
<td>5</td>
<td>43</td>
<td>58</td>
<td>80</td>
<td>87</td>
<td>101</td>
<td>143</td>
<td>133</td>
<td>149</td>
<td>183</td>
<td>203</td>
<td>233</td>
<td>267</td>
<td>269</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>115</td>
<td>134</td>
<td>134</td>
<td>183</td>
<td>204</td>
<td>256</td>
<td>294</td>
<td>347</td>
<td>401</td>
<td>401</td>
<td>6911</td>
<td>15263</td>
</tr>
<tr>
<td>7</td>
<td>205</td>
<td>217</td>
<td>252</td>
<td>274</td>
<td>292</td>
<td>405</td>
<td>417</td>
<td>511</td>
<td>10578</td>
<td>15263</td>
<td>22112</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>282</td>
<td>329</td>
<td>343</td>
<td>6090</td>
<td>10630</td>
<td>16944</td>
<td>817</td>
<td>27485</td>
<td>41525</td>
<td>63609</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>565</td>
<td>581</td>
<td>12677</td>
<td>22325</td>
<td>38832</td>
<td>64864</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>798</td>
<td>23556</td>
<td>45881</td>
<td>81123</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[SPR, EJIC survey Small Ramsey Numbers, revision #15, 2017, with updates]
Diagonal Conjecture (DC) motivation

\( R(k, l) \) seem to decrease along \( \rightarrow \) diagonals

<table>
<thead>
<tr>
<th>( k )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
<td>28</td>
<td>36</td>
<td>40</td>
<td>47</td>
<td>53</td>
<td>60</td>
<td>67</td>
<td>74</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>59</td>
<td>73</td>
<td>92</td>
<td>102</td>
<td>128</td>
<td>138</td>
<td>147</td>
<td>155</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>43</td>
<td>58</td>
<td>80</td>
<td>101</td>
<td>133</td>
<td>149</td>
<td>183</td>
<td>203</td>
<td>233</td>
<td>267</td>
<td>269</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>87</td>
<td>143</td>
<td>216</td>
<td>316</td>
<td>442</td>
<td>633</td>
<td>848</td>
<td>1138</td>
<td>1461</td>
<td>1878</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>102</td>
<td>115</td>
<td>134</td>
<td>183</td>
<td>204</td>
<td>256</td>
<td>294</td>
<td>347</td>
<td>401</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>165</td>
<td>298</td>
<td>495</td>
<td>780</td>
<td>1171</td>
<td>1804</td>
<td>2566</td>
<td>3703</td>
<td>5033</td>
<td>6911</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>205</td>
<td>217</td>
<td>252</td>
<td>292</td>
<td>405</td>
<td>417</td>
<td>511</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>540</td>
<td>1031</td>
<td>1713</td>
<td>2826</td>
<td>4553</td>
<td>6954</td>
<td>10578</td>
<td>15263</td>
<td>22112</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1870</td>
<td>3583</td>
<td>6090</td>
<td>10630</td>
<td>16944</td>
<td>27485</td>
<td>41525</td>
<td>63609</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>565</td>
<td>581</td>
<td>12677</td>
<td>22325</td>
<td>38832</td>
<td>64864</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>798</td>
<td>23556</td>
<td>45881</td>
<td>81123</td>
<td>1265</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Best known lower bounds for \( k \leq l \) satisfy

\[ LB(k, l) > LB(k - 1, l + 1), \]

except a mild hick-up at (8,10) vs (7,11).
Diagonal Conjecture (DC)

Two-Color DC:
\[ R(k, l) \geq R(k - 1, l + 1) \] for \( 3 \leq k \leq l \).

As we move away from the diagonal of the table with Ramsey numbers \( R(k, l) \), while preserving \( k + l \), the values decrease.

Multicolor DC:
For \( r \geq 3 \), \( a_i \geq 3 \) \((1 \leq i \leq r)\), if \( a_{r-1} \leq a_r \), then
\[ R(a_1, \cdots, a_r) \geq R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1). \]
Diagonal Conjecture cont.

Hints:

▶ Observed long ago ..., probably by many.
▶ Stronger versions of DC with > instead of ≥ are plausible.
▶ Known values and bounds do not contradict either DC.


▶ Wang’s proof is not correct.

A strange alternate definition of Ramsey numbers, followed by unfounded circular arguments between the definitions.
\[ \lim_{r \to \infty} R_r(k)^{1/r} \]

For \( k = a_1 = \cdots = a_r \), let \( R_r(k) = R(a_1, \cdots, a_r) \).

**Theorem** (Chung-Grinstead 1983)
\( L_3 = \lim_{r \to \infty} R_r(3)^{1/r} \) exists, finite or infinite.

- The same argument can be used to show that \( L_k = \lim_{r \to \infty} R_r(k)^{1/r} \) exists for all \( k > 3 \), finite or infinite.
- \( L_3 > 3.1996 \approx 1073^{\frac{1}{6}} \)
- Erdős was inclined to think that \( L_3 = \infty \) (Li,XR).
Consequences of DC
using Abbott’s 1965 construction

Lemma
If DC holds, then for every integer $a \geq 3$ we have

$$R_{2r}(a) > (R_r(a - 1) - 1)(R_r(a + 1) - 1).$$

Proof.
Apply DC $r$ times to $R_{2r}(a)$.
Use a special case of Abbott’s lower bound construction

$$R(a_1, \ldots, a_{2r}) > (R(a_1, \ldots, a_r) - 1)(R(a_{r+1}, \ldots, a_r) - 1).$$
Consequences of DC

main theorem

Theorem
If DC holds and \( \lim_{r \to \infty} R_r(3)^{\frac{1}{r}} \) is finite, then
\( \lim_{r \to \infty} R_r(a)^{\frac{1}{r}} \) is finite for every \( a \geq 3 \).

Proof.
Induction on \( a \) via
\[
\lim_{r \to \infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a-1)^{\frac{1}{r}}} \geq \lim_{r \to \infty} \frac{R_r(a+1)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}}. 
\]
LB vs UB on $R_r(3)$ and $L_3$

<table>
<thead>
<tr>
<th>$r$</th>
<th>lower bound</th>
<th>upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>51</td>
<td>62</td>
</tr>
<tr>
<td>5</td>
<td>162</td>
<td>307</td>
</tr>
<tr>
<td>6</td>
<td>538</td>
<td>1838</td>
</tr>
<tr>
<td>7</td>
<td>1682</td>
<td>12861</td>
</tr>
<tr>
<td>8</td>
<td>5204</td>
<td>102882</td>
</tr>
<tr>
<td>9</td>
<td>16146</td>
<td>925931</td>
</tr>
<tr>
<td>10</td>
<td>51202</td>
<td>9259302</td>
</tr>
</tbody>
</table>

Known bounds on $R_r(3)$ for $r \leq 10$,

$R_r(3) \leq (e - \frac{1}{6})r! + 1 \approx 2.55r!$, based on $R_4(3) \leq 62$.

$L_3 = \lim_{r \to \infty} R_r(3)^{1/r} > 3.1996$
Consequences of DC

another main theorem

**Theorem**

*If DC holds, then for every* $a \geq 3$, *we have*

$$\lim_{r \to \infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a - 1)^{\frac{1}{r}}} > 1.$$  

**Proof.**

First show that

$$\frac{R_r(2a - 1) - 1}{R_r(a) - 1} \geq R_r(3) - 1 \geq 2^r,$$

next that

$$\lim_{r \to \infty} \frac{R_r(2a)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}} \geq 2,$$

then finish by contradiction.
Theorem
If DC holds, then it is true that:

(a) all $L_k$’s are finite or all of them are infinite, and
(b) if $L_3$ is finite then $L_k < L_{k+1}$ for all $k \geq 3$.

$\lim_{k \to \infty} L_k = \infty$, even without assuming validity of DC
(by Abbott 1965, and by the previous theorem).

- If our perspective that the known lower bounds are much closer
to $R_r(k)$ than the upper bounds is correct, it would add weight to
the case that all limits $L_k$ are finite.
Evidence for DC - two colors

$DC(s, t)$ stands for $R(s, t) \geq R(s - 1, t + 1)$, $3 \leq s \leq t$.

(a) $DC(3, t)$ is true for all $t \geq 3$.
(b) $DC(4, t)$ is true for all $t \geq 4$.
(c) $DC(5, 5)$, $DC(5, 6)$ and $DC(5, 7)$ are true.
(d) The above establishes the validity of $DC(s, t)$ for all $s < 5$, and all cases with $s + t \leq 12$, except $DC(6, 6)$.
(e) The further we go from the diagonal of the DC conjecture, the easier it seems to corroborate it. We anticipate DC to be the hardest on the diagonal itself, i.e. proving that $R(t, t) \geq R(t - 1, t + 1)$ for any $t \geq 6$.
(f) Little bump at $DC(8, 10)$.
Evidence for DC - more colors
relying on lower bounds

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$LB_1$</th>
<th>$LB_2$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,3,5</td>
<td>45</td>
<td>55</td>
<td>3,4,4</td>
</tr>
<tr>
<td>3,3,6</td>
<td>61</td>
<td>89</td>
<td>3,4,5</td>
</tr>
<tr>
<td>3,3,7</td>
<td>85</td>
<td>117</td>
<td>3,4,6</td>
</tr>
<tr>
<td>3,3,8</td>
<td>103</td>
<td>152</td>
<td>3,4,7</td>
</tr>
<tr>
<td>3,3,9</td>
<td>129</td>
<td>193</td>
<td>3,4,8</td>
</tr>
<tr>
<td>3,3,10</td>
<td>150</td>
<td>242</td>
<td>3,4,9</td>
</tr>
<tr>
<td>3,4,6</td>
<td>117</td>
<td>139</td>
<td>3,5,5</td>
</tr>
<tr>
<td>3,4,7</td>
<td>152</td>
<td>181</td>
<td>3,5,6</td>
</tr>
<tr>
<td>3,4,8</td>
<td>193</td>
<td>241</td>
<td>3,5,7</td>
</tr>
<tr>
<td>4,3,5</td>
<td>89</td>
<td>128</td>
<td>4,4,4</td>
</tr>
<tr>
<td>3,3,3,5</td>
<td>162</td>
<td>171</td>
<td>3,3,4,4</td>
</tr>
</tbody>
</table>

Known lower bounds $LB_1$ and $LB_2$ on $R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1)$ and $R(a_1, \cdots, a_r)$ for some DC-adjacent pairs of parameters $A_1$ and $A_2$. 

14/20  Evidence for DC
Shannon capacity $c(G)$ and limits $L_k$

\[ \alpha(G^r) = \text{independence of the strong } r\text{-th power of graph } G \]
\[ c(G) = \text{Shannon capacity of a noisy channel modeled by } G \]
\[ c(G) = \lim_{r \to \infty} \alpha(G^r)^{\frac{1}{r}} \]

We proved (XR 2013):

- For any fixed $k \geq 3$, $L_k = \lim_{r \to \infty} R_r(k)^{1/r}$ is equal to the supremum of the Shannon capacity $c(G)$ over all graphs $G$ with $\alpha(G) = k - 1$, but this supremum cannot be achieved by any finite graph power, $G^{r_0}$. \\

15/20 Evidence for DC
Papers to look at


▶ Meilian Liang, SPR, Xiaodong Xu
On a Diagonal Conjecture for Classical Ramsey Numbers
*arXiv* 1810.11386, *October 2018*.


777+ papers by many authors ...
Thanks for listening!
Asymptotics for 2 colors

diagonal cases

▶ **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)

\[
\sqrt{2} \frac{2^{n/2} n}{e} < R(n, n) < \frac{R(n+1, n+1)}{n} \leq \left(\frac{2n}{n}\right) n^{-c \log \log n}
\]

▶ **Conjecture** (Erdős 1947, $100)

\[
\lim_{n \to \infty} R(n, n)^{1/n} \text{ exists.}
\]

If it exists, it is between \(\sqrt{2}\) and 4 ($250 for the value).

▶ **Theorem** (Chung-Grinstead 1983)

\[
L = \lim_{k \to \infty} R_k(3)^{1/k} \text{ exists.}
\]

\[
3.199 < L, \quad (\text{Fredricksen-Sweet 2000, X-Xie-Exoo-R 2004})
\]
New avalanche of improved upper bounds after LP attack for higher $k$ and $l$ by Angeltveit-McKay.

$$R(5, 5) \leq 48,$$ Angeltveit-McKay 2018.
Two problems beyond DC

▶ Generalizing DC.

For connected graphs $G_i$ with $s \leq t$, is it true that

$$R(G_1, G_2, \cdots, K_{s-1}, K_{t+1}) \leq R(G_1, G_2, \cdots, K_s, K_t)?$$

We think ‘YES’, but make no more conjectures.

▶ Let $r \geq 3$, $a_i \geq 3$, $a_{r-1} \leq a_r$, and $C$ be a coloring witnessing

$$n < R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1).$$

Let $G = \text{all edges of } C \text{ in colors } r - 1 \text{ and } r$, $|V(G)| = n$.

Is it true that $G \notightarrow (a_{r-1}, a_r)^e$?

i.e. that there exists a 2-coloring of $E(G)$ without any $K_{a_{r-1}}$ in the first color and without $K_{a_r}$ in the second color?

We think ‘weaker YES’.