Diagonal Conjecture for Classical Ramsey Numbers

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, NY, USA

joint work with Meilian Liang and Xiaodong Xu

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Plan of Talk

- Ramsey numbers: computational perspective
  (16 min)

- Diagonal Conjecture: origins, consequences and evidence
  (32 min)

  LRX, On a Diagonal Conjecture for Classical Ramsey Numbers, *Discrete Applied Mathematics,* 267 (2019) 195-200.

Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.

- $2 - colorings \cong graphs$, $R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist

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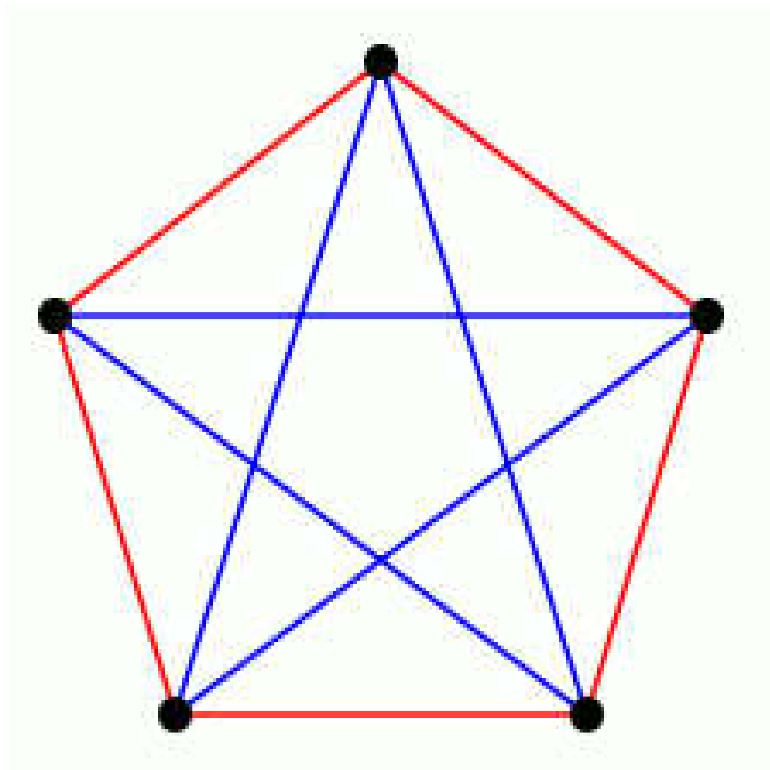
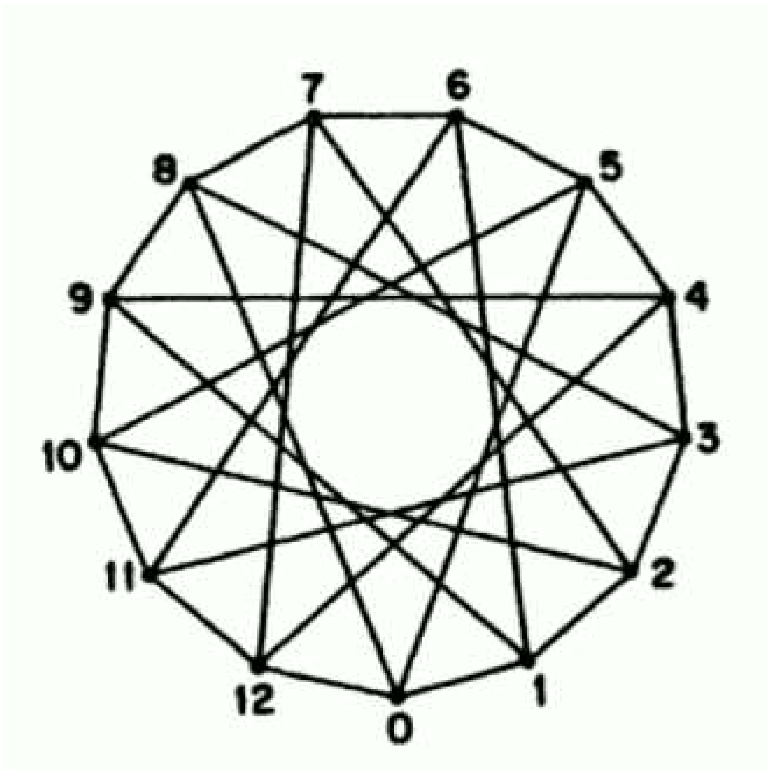
Unavoidable classics



$R(3,3) = 6$



$R(3,5) = 14$ [GRS'90]

Asymptotics

diagonal cases

- **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)

$$\frac{\sqrt{2}}{e}2^{n/2}n < R(n,n) < R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, \$100)
  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.
  If it exists, it is between $\sqrt{2}$ and 4 (\$250 for value).

- **Theorem** (Chung-Grinstead 1983)
  $L = \lim_{k\to\infty} R_k(3)^{1/k}$ exists.

Asymptotics

Ramsey numbers avoiding K_3

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound

$$R(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$$

- Bohman/Keevash 2009/2013, triangle-free process
- Fiz Pontiveros-Griffiths-Morris, lower bound, 2013
  Shearer, upper bound, 1983

$$\left(\frac{1}{4} + o(1)\right) n^2/\log n \leq R(3, n) \leq (1 + o(1))n^2/\log n$$

Test - Hunt - Exhaust

Ramsey numbers

- **Testing:** do it right.
  Graph G is a witness of $R(m, n) > k$ iff
  $|V(G)| = k$, $cl(G) < m$ and $\alpha(G) < n$.

  Lab in a 200-level course.

- **Hunting:** constructions and heuristics.
  Given m and n, find a witness G for k larger than others.
  Challenge projects in advanced courses.

  Master: Geoffrey Exoo 1986–

- **Exhausting:** generation, pruning, isomorphism.
  Prove that for given m, n and k, there does not exist any witness as above. Hard without `nauty/traces`.

  Master: Brendan McKay 1991–

Values and bounds on $R(k, l)$

two colors, avoiding K_k, K_l

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	53 59	60 68	67 77	74 87
4		18	25	36 41	49 61	59 84	73 115	92 149	102 191	128 238	138 291	147 349	155 417
5			43 48	58 87	80 143	101 216	133 316	149 442	183 633	203 848	233 1138	267 1461	269 1878
6				102 165	115 298	134 495	183 780	204 1171	256 1804	294 2566	347 3703	 5033	401 6911
7					205 540	217 1031	252 1713	292 2826	405 4553	417 6954	511 10578	 15263	 22112
8						282 1870	329 3583	343 6090	 10630	 16944	817 27485	 41525	865 63609
9							565 6588	581 12677	 22325	 38832	 64864		
10								798 23556	 45881	 81123			1265

[SPR, EIJC survey *Small Ramsey Numbers*, revision #15, 2017, with updates]

Small $R(k, l)$, references

$R(5, 5) \leq 48$, Angeltveit-McKay 2018.

l / k	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR McZ	Ka2 GR	Ex5 GoR1	Ex20 GoR1	Kol1 Les	Kol1 GoR1	Kol2 GoR1	Kol2 GoR1
4	GG	Ka1 MR4	Ex19 ~~MR5~~	Ex3 ~~Mac~~	ExT ~~Mac~~	Ex16 ~~Mac~~	HaKr1 ~~Mac~~	ExT ~~Spe4~~	SuLL ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~
5		Ex4 AnM	Ex9 ~~HZ1~~	CaET ~~HZ1~~	HaKr1 ~~Spe4~~	Kuz ~~Mac~~	ExT ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	ExT ~~HW+~~
6			Ka2 ~~Mac~~	ExT ~~HZ1~~	ExT ~~Mac~~	Kuz ~~Mac~~	Kuz ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
7				She2 ~~Mac~~	XSR2 ~~HZ1~~	Kuz ~~HZ2~~	Kuz ~~Mac~~	XXER ~~HW+~~	XSR2 ~~HW+~~	XuXR ~~HW+~~	~~HW+~~	~~HW+~~
8					BurR ~~Mac~~	Kuz ~~Eol~~	Kuz ~~HZ2~~	~~HW+~~	~~HW+~~	XXER ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
9						She2 ~~ShZ1~~	XSR2 ~~Eol~~	~~HW+~~	~~HW+~~	~~HW+~~		
10							She2 ~~Shi2~~	~~HW+~~	~~HW+~~			2.3.h

New avalanche of improved upper bounds
after LP attack for higher k and l by Angeltveit-McKay.

Diagonal Conjecture (DC) motivation

$R(k, l)$ seem to decrease along $\nearrow$ diagonals

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	53 59	60 68	67 77	74 87
4		18	25	36 41	49 61	59 84	73 115	92 149	102 191	128 238	138 291	147 349	155 417
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Best known lower bounds for $k \leq l$ satisfy

$$LB(k, l) > LB(k-1, l+1),$$

except a mild hick-up at (8,10) vs (7,11).

Diagonal Conjecture (DC)

Two-Color DC:
$R(k, l) \geq R(k-1, l+1)$ for $3 \leq k \leq l$.

As we move away from the diagonal of the table with Ramsey numbers $R(k, l)$, while preserving $k + l$, the values decrease.

Multicolor DC:
For $r \geq 3$, $a_i \geq 3$ $(1 \leq i \leq r)$, if $a_{r-1} \leq a_r$, then

$$R(a_1, \cdots, a_r) \geq R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1).$$

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Diagonal Conjecture

cont.

Hints:

- Observed long ago ..., probably by many.
- Stronger versions of DC with $>$ instead of $\geq$ are plausible.
- Known values and bounds do not contradict either DC.

- Wang Rui (2008) published a theorem implying two-color DC, and its extensions to multicolor cases (without proof).

  Wang Rui, Another definition for Ramsey numbers, IEEE *Int. Symp. Information Science and Engineering*, 2 (2008) 405–409.

- Wang's proof is not correct.

  A strange alternate definition of Ramsey numbers, followed by unfounded circular arguments between the definitions.

$\lim_{r\to\infty} R_r(k)^{1/r}$

For $k = a_1 = \cdots = a_r$, let $R_r(k) = R(a_1, \cdots, a_r)$.

Theorem (Chung-Grinstead 1983)
$L_3 = \lim_{r\to\infty} R_r(3)^{1/r}$ exists, finite or infinite.

- The same argument can be used to show that $L_k = \lim_{r\to\infty} R_r(k)^{1/r}$ exists for all $k > 3$, finite or infinite.
- $L_3 > 3.1996 \approx 1073^{\frac{1}{6}}$ (Fredricksen-Sweet 2000, X-Xie-Exoo-R 2004).
- Erdős was inclined to think that $L_3 = \infty$ (Li,XR).

Consequences of DC

using Abbott's 1965 construction

Lemma

If DC holds, then for every integer $a \geq 3$ we have

$$R_{2r}(a) > (R_r(a-1)-1)(R_r(a+1)-1).$$

Proof.

Apply DC r times to $R_{2r}(a)$.

Use a special case of Abbott's lower bound construction

$$R(a_1, \ldots, a_{2r}) > (R(a_1, \ldots, a_r) - 1)(R(a_{r+1}, \ldots, a_r) - 1).$$

$\square$

Consequences of DC

main theorem

Theorem

If DC holds and $\lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ is finite, then $\lim_{r\to\infty} R_r(a)^{\frac{1}{r}}$ is finite for every $a \geq 3$.

Proof.

Induction on a via

$$\lim_{r\to\infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a-1)^{\frac{1}{r}}} \geq \lim_{r\to\infty} \frac{R_r(a+1)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}}.$$

$\square$

LB vs UB on $R_r(3)$ and L_3

r	lower bound	upper bound
2	**6**	6
3	**17**	17
4	**51**	**62**
5	162	307
6	538	1838
7	1682	12861
8	5204	102882
9	16146	925931
10	51202	9259302

Known bounds on $R_r(3)$ for $r \leq 10$,

$R_r(3) \leq (e - \frac{1}{6})r! + 1 \approx 2.55r!$, based on $R_4(3) \leq 62$.

$$L_3 = \lim_{r \to \infty} R_r(3)^{1/r} > 3.1996$$

Consequences of DC

another main theorem

Theorem

If DC holds, then for every $a \geq 3$, we have

$$\lim_{r\to\infty} \frac{R_r(a)^{\frac{1}{r}}}{R_r(a-1)^{\frac{1}{r}}} > 1.$$

Proof.

First show that

$$\frac{R_r(2a-1)-1}{R_r(a)-1} \geq R_r(3)-1 \geq 2^r,$$

next that

$$\lim_{r\to\infty} \frac{R_r(2a)^{\frac{1}{r}}}{R_r(a)^{\frac{1}{r}}} \geq 2,$$

then finish by contradiction. $\square$

Consequences of DC

summary

Theorem

If DC holds, then it is true that:

(a) all L_k's are finite or all of them are infinite, and
(b) if L_3 is finite then $L_k < L_{k+1}$ for all $k \geq 3$.

- $\lim_{k\to\infty} L_k = \infty$, even without assuming validity of DC (by Abbott 1965, and by the previous theorem).
- If our perspective that the known lower bounds are much closer to $R_r(k)$ than the upper bounds is correct, it would add weight to the case that all limits L_k are finite.

Evidence for DC - two colors

$DC(s, t)$ stands for $R(s, t) \geq R(s-1, t+1)$, $3 \leq s \leq t$.

(a) $DC(3, t)$ is true for all $t \geq 3$.
(b) $DC(4, t)$ is true for all $t \geq 4$.
(c) $DC(5, 5)$, $DC(5, 6)$ and $DC(5, 7)$ are true.
(d) The above establishes the validity of $DC(s, t)$ for all $s < 5$, and all cases with $s + t \leq 12$, except $DC(6, 6)$.
(e) The further we go from the diagonal of the DC conjecture, the easier it seems to corroborate it. We anticipate DC to be the hardest on the diagonal itself, i.e. proving that $R(t, t) \geq R(t-1, t+1)$ for any $t \geq 6$.

(f) Little bump at $DC(8, 10)$.

Evidence for DC - more colors

relying on lower bounds

A_1	LB_1	LB_2	A_2
3,3,5	45	55	3,4,4
3,3,6	61	89	3,4,5
3,3,7	85	117	3,4,6
3,3,8	103	152	3,4,7
3,3,9	129	193	3,4,8
3,3,10	150	242	3,4,9
3,4,6	117	139	3,5,5
3,4,7	152	181	3,5,6
3,4,8	193	241	3,5,7
4,3,5	89	128	4,4,4
3,3,3,5	162	171	3,3,4,4

Known lower bounds LB_1 and LB_2 on $R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1)$ and $R(a_1, \cdots, a_r)$ for some DC-adjacent pairs of parameters A_1 and A_2.

Shannon capacity $c(G)$ and limits L_k

$\alpha(G^r) =$ independence of the strong r-th power of graph G
$c(G) =$ Shannon capacity of a noisy channel modeled by G

$$c(G) = \lim_{r\to\infty} \alpha(G^r)^{\frac{1}{r}}$$

We proved (XR 2013):

- For any fixed $k \geq 3$, $L_k = \lim_{r\to\infty} R_r(k)^{1/r}$ is equal to the supremum of the Shannon capacity $c(G)$ over all graphs G with $\alpha(G) = k - 1$, but this supremum cannot be achieved by any finite graph power, G^{r_0}.

Two problems beyond DC

- Generalizing DC.

  For connected graphs G_i with $s \leq t$, is it true that

  $$R(G_1, G_2, \cdots, K_{s-1}, K_{t+1}) \leq R(G_1, G_2, \cdots, K_s, K_t)?$$

  We think 'YES', but make no more conjectures.

- Let $r \geq 3$, $a_i \geq 3$, $a_{r-1} \leq a_r$, and C be a coloring witnessing

  $$n < R(a_1, \cdots, a_{r-2}, a_{r-1} - 1, a_r + 1).$$

  Let G = all edges of C in colors $r - 1$ and r, $|V(G)| = n$.

  $$\text{Is it true that } G \not\rightarrow (a_{r-1}, a_r)^e?$$

  i.e. that there exists a 2-coloring of $E(G)$ without
  any $K_{a_{r-1}}$ in the first color and without K_{a_r} in the second color?

  We think 'weaker YES'.

Papers to look at

- Wang Rui, Another definition for Ramsey numbers,
  IEEE *International Symposium on Information Science and Engineering*, 2 (2008) 405–409.

- Meilian Liang, SPR, Xiaodong Xu
  On a Diagonal Conjecture for Classical Ramsey Numbers
  Discrete Applied Mathematics, 267 (2019) 195-200.

- SPR, revision #15 of the dynamic survey paper,
  Small Ramsey Numbers,
  Electronic Journal of Combinatorics, DS1, March 2017.

  777+ papers by many authors ...

Thanks for listening!