An Improvement to Mathon's Cyclotomic Ramsey Colorings

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Abstract

In this note we show how to extend Mathon's cyclotomic colorings of the edges of some complete graphs without increasing the maximum order of monochromatic complete subgraphs. This improves the well known lower bound construction for multicolor Ramsy numbers, in particular we obtain $R_3(7) \geq 3214$.

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1 Introduction and Notation

A $(k_1, k_2, \ldots, k_m)$-*coloring*, for integers $m, k_i \geq 1$, is an assignment of one of m colors to each edge in a complete graph, such that it does not contain any monochromatic complete subgraph K_{k_i} in color i, for $1 \leq i \leq m$. Similarly, a $(k_1, k_2, \ldots, k_m; n)$-*coloring* is a $(k_1, \ldots, k_m)$-coloring of the complete graph on n vertices K_n. Let $\mathcal{R}(k_1, \ldots, k_m)$ and $\mathcal{R}(k_1, \ldots, k_m; n)$ denote the set of all $(k_1, \ldots, k_m)$- and $(k_1, \ldots, k_m; n)$-colorings, respectively. The Ramsey number $R(k_1, \ldots, k_m)$ is defined to be the least $n > 0$ such that $\mathcal{R}(k_1, \ldots, k_m; n)$ is empty. In the diagonal case, where $k_1 = \ldots = k_m = k$, we will use simpler notation $\mathcal{R}_m(k)$ and $\mathcal{R}_m(k; n)$ for sets of colorings, and $R_m(k)$ for the Ramsey numbers.

In the case of 2 colors ($m = 2$) we deal with classical *graph* Ramsey numbers, which have been studied extensively for 50 years. Much less has been done for multicolor numbers ($m \geq 3$). A related area of interest has been the study of generalized Ramsey colorings, wherein the forbidden monochromatic subgraphs are not restricted to complete graphs. The second author maintains a regularly updated survey [2] of the most recent results on the best known bounds on various types of Ramsey numbers.

The next section shows how to improve on the well known construction by Mathon [1] for establishing lower bounds for $R_m(k)$.

2 Extending Mathon's Construction

In 1987, Mathon [1] gave a very elegant algebraical construction of certain m-class cyclotomic association schemes over finite field $\mathcal{F}_p$, which when interpreted as m-colorings of the edges of K_p and $K_{m(p+1)}$ give constructive lower bounds for the corresponding classical diagonal Ramsey numbers. Specifically, for a prime power $p = mt + 1$ with even t, one considers the basic m-th residue graph H_p^m with vertices in $\mathcal{F}_p$ and $\{x, y\}$ an edge if for some $0 \neq z \in \mathcal{F}_p$, $x - y = z^m$, Then, if α is the order of the maximum clique in H_p^m, the construction gives m-colorings of the edges of K_p and $K_{m(p+1)}$ with the orders of the maximum monochromatic cliques equal to α and $\alpha + 1$, respectively.

In the case of quadratic ($m = 2$) cyclotomic relations Mathon's construction is equivalent to the "doubling" of Paley graph described independently by Shearer [3], which, directly and indirectly, led to several best known lower bounds for Ramsey numbers (cf. [4]).

Higher order ($m \geq 3$) cyclotomic relations beyond the basic H_p^m so far seem to be not much exploited in the context of Ramsey constructions. Here, our main interest is in the Mathon's cubic association scheme (also pointed to, but not analyzed, by Shearer [3]). We show how to improve on Mathon's scheme in the case of cubic residues in $\mathcal{Z}_p$ for K_{3p+3}, though as remarked at the end, a similar improvement holds for all $\mathcal{F}_p$ and $m \geq 2$. In the following we show how to include three additional vertices and obtain a 3-coloring of the edges of K_{3p+6} without increasing the order of monochromatic complete subgraphs.

We begin with a description of Mathon's construction instantiated for 3 colors over $\mathcal{Z}_p$. Let p be a prime of the form $p = 3t + 1$ with even t, and let β be a primitive element (generator) of $\mathcal{Z}_p^*$. The condition $p \equiv 1 \pmod 6$ implies that $-1 \equiv (\beta^{q/2})^3 \pmod p$ is a cubic residue, which is needed for the associated coloring to be well defined. Consider 3-coloring H_p^3 with the vertex set $\mathcal{Z}_p$, where the edge $\{x, y\}$ has color of the cubic character of $x - y$ in $\mathcal{Z}_p^*$, i.e. $\{x, y\}$ has color $i \in \{0, 1, 2\}$ if and only if $x - y \equiv \beta^{3s+i} \pmod p$, for some integer s. It is well known that the subgraphs induced in H_p^3 by the three colors are isomorphic to each other [1]. Let α_p denote the order of the largest monochromatic clique in H_p^3.

Next, we "triple" the coloring H_p^3 to the coloring M_p on the vertex set $X = U \cup V$ of $3p + 3$ vertices, where $U, V \subset \mathcal{Z}_p \times \mathcal{Z}_p$, $|U| = 3$, $|V| = 3p$, and U and V are defined by

$$U = \{u_0, u_1, u_2\} = \{(0, 1), (0, \beta), (0, \beta^2)\},$$
$$V = V_0 \cup V_1 \cup V_2, \text{ where } V_i = \{(\beta^i, a) | a \in \mathcal{Z}_p\}, \text{ for } i \in \{0, 1, 2\}.$$

Each edge $e = \{(x, y), (s, t)\}$ in M_p is colored according to the cubic character of $xt - ys$ in $\mathcal{Z}_p$. If $xt - ys = 0$ then e has the special color $c \notin \{0, 1, 2\}$ (which later will be recolored), otherwise e has color $i \in \{0, 1, 2\}$ if and only if $xt - ys \equiv \beta^{3s+i} \pmod p$, for some integer s.

The main result related to this construction obtained by Mathon [1] is that the order of any monochromatic clique in M_p is at most $\alpha_p + 1$. In addition, the coloring M_p satisfies the following properties:

A. Color c induces $p+1$ vertex disjoint triangles, U is one of them. For each $i, j \in \{0,1,2\}$, u_i's neighborhood in color j, $N_j(u_i)$, is $V_{j+1 \pmod 3}$.

B. M_p is vertex transitive, and colors $\{0,1,2\}$ induce isomorphic colorings. Thus, each vertex $x \in X$ has degree p in each color $i \in \{0,1,2\}$, and color i neighborhood of x, $N_i(x)$, induces a coloring isomorphic to H_p^3.

C. If the edge $\{x,y\}$ has color c, then $N_i(x) \cap N_i(y) = \emptyset$ for all i in $\{0,1,2\}$. Consequently, after an arbitrary recoloring of the edges from color c to colors $\{0,1,2\}$, any monochromatic clique in M_p contains at most one vertex from U if it contains any vertices not in U.

Theorem 1 *For prime $p \equiv 1 \pmod 6$, let α_p denote the order of the largest monochromatic clique in the cubic residues coloring H_p^3. If $k = \alpha_p + 2 \geq 4$ then $R(k,k,k) > 3(p+2)$.*

```
           U   V   W
          012 012 012

u0         01 120 012
u1        0 2 201 201
u2        12  012 120

V0        120 xxx 102
V1        201 xxx 021
V2        012 xxx 210

w0        021 102  01
w1        102 021 0 2
w2        210 210 12
```

Figure 1. Extending coloring M_p by vertices $\{w_0, w_1, w_2\}$ to M'_p.

Proof. We will extend the coloring M_p described above (isomorphic to the construction by Mathon [1] for $m = 3$) by three additional vertices to M'_p, without incrementing the order of the largest monochromatic complete subgraph. We define 3-coloring M'_p of the edges of the complete graph on the vertex set $X \cup W = U \cup V \cup W$, where $W = \{w_0, w_1, w_2\}$, and X, U, V are as before. Figure 1 gives the colors of the edges.

The middle $3p \times 3p$ section of the matrix with $\mathtt{x}$'s is defined by the starting coloring M_p, while other entries in the rows/columns corresponding to V_i mean that all the edges adjacent to this set have the same color as indicated in the matrix.

Let S be the maximum order clique in M'_p in color i. If $S \cap W = \emptyset$ then S is restricted to original M_p, so $|S| \leq \alpha_p + 1$. One can easily check in Fig. 1 that every monochromatic triangle can have at most one vertex in the set $U \cup W$, so we can assume that $S \cap U = \emptyset$ but there exists $x \in S \cap W$. By properties **B** and **C** of M_p we have $S \setminus \{x\} \subset V_j$, so again $|S| \leq \alpha_p + 1$. $\diamondsuit$

The next corollary improves on the old bound 3211 by Mathon [1]. The new bound was not published, though it was already cited as an unpublished result in the survey [2].

Corollary $R(7,7,7) \geq 3214$.

Proof. For prime $p = 1069$, it is known that $\alpha_p < 6$ [1]. The bound 3214 follows from the Theorem. $\diamondsuit$

One can similarly improve Shearer/Mathon's construction on K_{mp+m} for other values of m, by producing m-colorings of $K_{mp+m!}$. Note that for $m = 2$ there is no improvement, and the case $m = 3$ is that of the Theorem. For general m with $\alpha_p + 1 \geq m$, after chosing the set U (now of m vertices) we add a new set of vertices W so that $|U \cup W| = m!$. With each $x \in Y = U \cup W$ we associate a permutation $(i_0, \cdots, i_{m-1})$ and color all the edges from x to V_{i_j} in color j. For $x, y \in Y$ the edge $\{x, y\}$ has color equal to the minimal index of the position at which the corresponding permutations are different. We omit the details since we don't know any specific parameters with $m > 3$ for which this would improve on a best known published lower bound as in the Corollary.

References

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