$28 \leq R(C_4, C_4, C_3, C_3) \leq 36$

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Abstract

Using four colors we construct a coloring of the edges of K_{27} which has no monochromatic quadrilaterals in the first two colors and no monochromatic triangles in the other two colors. This gives a new lower bound of 28 on the Ramsey number $R(C_4, C_4, C_3, C_3)$. We also prove an upper bound of 36 for the same number using an estimate of the maximum number of edges in C_4-free graphs.

1 Background

For graphs $G_1, \cdots, G_k$, the generalized multicolor graph Ramsey number $R(G_1, \cdots, G_k)$ is the least natural number n such that in any edge coloring with k colors of K_n, there exists monochromatic G_i in some color i, $1 \leq i \leq k$. A regularly updated survey of the most recent results on the best known bounds on various types of Ramsey numbers is maintained by the second author [6].

The best known lower bound of 27 for the graphs C_4, C_4, C_3, C_3 was established in 2005 by Engström [4]. In the next section we construct a 4-coloring $\mathcal{C}_2$ of the edges of K_{27} which has no monochromatic quadrilaterals in the first two colors and no monochromatic triangles in the other two

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colors. This improves the best known lower bound on the Ramsey number $R(C_4, C_4, C_3, C_3)$ from 27 to 28. We also show that the known bound on the maximum number of edges in any C_4-free graph on 36 vertices implies that $R(C_4, C_4, C_3, C_3) \leq 36$. In the following we use notation similar to that of our earlier paper [7], in which we presented a few similar constructions avoiding complete graphs.

2 The Construction

The construction of the coloring $\mathcal{C}_2$ of the edges of K_{27} using 4 colors is accomplished in three stages. The matrix of the final coloring is presented in Figure 1.

Stage 1. Start with two 2-colorings G and H of K_5 with vertex sets $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$, respectively, where the edges of $C_5 = u_1u_2u_3u_4u_5u_1$ in G have color 1, other 5 edges of G have color 2, the edges of $C_5 = v_1v_2v_3v_4v_5v_1$ in H have color 3, and the other 5 edges in H have color 4. Clearly, there is no monochromatic C_3 or C_4 in G or H. Next, construct the product coloring $\mathcal{C}_0$ of the edges of K_{25} on the vertex set $U \times V$ by defining:

$$\mathcal{C}_0((u_{p_1}, v_{q_1}), (u_{p_2}, v_{q_2})) = \begin{cases} G(u_{p_1}, u_{p_2}) & \text{if } q_1 = q_2, \\ H(v_{q_1}, v_{q_2}) & \text{if } q_1 \neq q_2, \end{cases}$$

for $1 \leq p_1, p_2, q_1, q_2 \leq 5$, and $p_1 \neq p_2$ or $q_1 \neq q_2$.

One can see the coloring $\mathcal{C}_0$ as five copies of G interconnected by the edges of colors 3 and 4 according to H. There are no monochromatic triangles in any color, but there is a large number of C_4's in colors 3 and 4.

Stage 2. Obtain $\mathcal{C}_1$ from $\mathcal{C}_0$ by recoloring the pentagon $(u_1, v_1)(u_1, v_2)(u_1, v_3)(u_1, v_4)(u_1, v_5)(u_1, v_1)$ from color 3 to color 2, and two pentagons $(u_2, v_1)(u_2, v_3)(u_2, v_5)(u_2, v_2)(u_2, v_4)(u_2, v_1)$ and $(u_5, v_1)(u_5, v_3)(u_5, v_5)(u_5, v_2)(u_5, v_4)(u_5, v_1)$ from color 4 to color 1. Observe that no C_4's in colors 1 and 2 where introduced by this step.

Stage 3. The final coloring $\mathcal{C}_2$ of the edges of K_{27} on the vertex set $U \times V \cup \{s, t\}$ is defined as an extension of $\mathcal{C}_1$. For $1 \leq p, q \leq 5$, the edges adjacent to $\{s, t\}$ are colored as follows:

$$\mathcal{C}_2(s, (u_p, v_q)) = \begin{cases} 1 & \text{if } p = 3, \\ 2 & \text{if } p = 2 \text{ or } p = 4, \\ 3 & \text{if } p = 1, \\ 4 & \text{if } p = 5, \end{cases}$$

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u1 v1        1221 23333 44444 44444 23333 33
u2 v1       1 122 33333 41444 41444 33333 24
u3 v1       21 12 33333 44444 44444 33333 12
u4 v1       221 1 33333 44444 44444 33333 21
u5 v1       1221  33333 44441 44441 33333 42

u1 v2       23333  1221 23333 44444 44444 33
u2 v2       33333 1 122 33333 41444 41444 24
u3 v2       33333 21 12 33333 44444 44444 12
u4 v2       33333 221 1 33333 44444 44444 21
u5 v2       33333 1221  33333 44441 44441 42

u1 v3       44444 23333  1221 23333 44444 33
u2 v3       41444 33333 1 122 33333 41444 24
u3 v3       44444 33333 21 12 33333 44444 12
u4 v3       44444 33333 221 1 33333 44444 21
u5 v3       44441 33333 1221  33333 44441 42

u1 v4       44444 44444 23333  1221 23333 33
u2 v4       41444 41444 33333 1 122 33333 24
u3 v4       44444 44444 33333 21 12 33333 12
u4 v4       44444 44444 33333 221 1 33333 21
u5 v4       44441 44441 33333 1221  33333 42

u1 v5       23333 44444 44444 23333  1221 33
u2 v5       33333 41444 41444 33333 1 122 24
u3 v5       33333 44444 44444 33333 21 12 12
u4 v5       33333 44444 44444 33333 221 1 21
u5 v5       33333 44441 44441 33333 1221  42

  s         32124 32124 32124 32124 32124  4
  t         34212 34212 34212 34212 34212 4
```

Figure 1. (C_4, C_4, C_3, C_3)-good 4-coloring $\mathcal{C}_2$ of the edges of K_{27}

$$\mathcal{C}_2(t,(u_p,v_q)) = \begin{cases} 1 & \text{if } p=4, \\ 2 & \text{if } p=3 \text{ or } p=5, \\ 3 & \text{if } p=1, \\ 4 & \text{if } p=2, \end{cases}$$

and, finally, let $\mathcal{C}_2(s,t)=4$. Still a straightforward but little less obvious check shows that $\mathcal{C}_2$ avoids C_4 and C_3 as required, i.e. it is (C_4,C_4,C_3,C_3)-good. The matrix form of coloring $\mathcal{C}_2$ is presented in Figure 1, where s and t correspond to the last two rows and columns. Note that the colors of the edges added at this stage depend only on p, and not on q.

By the construction of $\mathcal{C}_2$ we have:

Theorem 1. $R(C_4,C_4,C_3,C_3) \geq 28$.

Observe that since $K_3=C_3$ and Ramsey numbers are preserved under permutations of arguments, the above theorem is equivalent to $R(K_3,K_3,C_4,C_4) \geq 28$.

As far as we are aware, no upper bound for $R(C_4,C_4,C_3,C_3)$ has been published. One can easily derive an upper bound of 59 by using the known facts, $R(K_3,C_4,C_4)=12$ and $R(K_3,K_{12}) \leq 59$ (cf. [6]). However, we can do much better by employing bounds on the function $\mathrm{ex}(n;C_4)$, which is the maximum number of edges in any C_4-free graph on n vertices. Let $t_n = \mathrm{ex}(n;C_4)$.

Theorem 2. $R(C_4,C_4,C_3,C_3) \leq 36$.

Proof. Suppose $\mathcal{C}$ is a (C_4,C_4,C_3,C_3)-good coloring of the edges of K_{36}. Since $R(C_4,C_4,K_3)=12$ (cf. [6]), each vertex in $\mathcal{C}$ has degree at most 11 in color 3 and in color 4, and thus $\mathcal{C}$ has at most 198 edges in each of the colors 3 and 4. Irving [5] proved (see also Bollobás [1]) that $t_n < n(1+\sqrt{4n-3})/4$, which used with $n=36$ implies that $\mathcal{C}$ has at most 115 edges in each of the colors 1 and 2. Hence, we can color at most $2(198+115)=626$ edges, while $\binom{36}{2}=630$ need to be colored. This is a contradiction. $\diamondsuit$

We note that a similar reasoning does not disprove the existence of a (C_4,C_4,C_3,C_3)-good coloring of K_{35} (in this case the same bound by Irving gives $t_{35} \leq 111$). We conclude with some easy bounds on a similar Ramsey number $R(C_4,C_4,C_4,C_3)$, which so far was not listed in the survey [6]. A (C_4,C_4,C_4,C_3)-good coloring of K_{20} can be obtained by "doubling" Clapham construction: take two disjoint copies of the Clapham (C_4,C_4,C_4)-good coloring on 10 vertices [2], and color all edges in-between in color 4. The upper bound of 27 follows from a reasoning similar to that in the proof of Theorem 2, using the known value $t_{27} = 71$ [8] (the exact values of t_n are known up to $n = 31$ [3, 8]). Hence, we have

$$21 \leq R(C_4, C_4, C_4, C_3) \leq 27.$$

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