On the Most Wanted Folkman Graph

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Abstract

We discuss a branch of Ramsey theory concerning edge Folkman numbers. $F_e(3,3;4)$ involves the smallest parameters for which the problem is open, posing the question "What is the smallest order N of a K_4-free graph, for which any 2-coloring of its edges must contain at least one monochromatic triangle?" This is equivalent to finding the order N of the smallest K_4-free graph which is not a union of two triangle-free graphs. It is known that $16 \leq N$ (an easy bound), and it is known through a probabilistic proof by Spencer that $N \leq 3 \times 10^9$. In this paper, after overviewing related Folkman problems, we prove that $19 \leq N$, and give some evidence for the bound $N \leq 127$.

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1 Scope and notation

We discuss a branch of Ramsey theory concerning mostly edge Folkman numbers. We write $G \to (a_1, \ldots, a_k; p)^e$ if for every edge k-coloring of an undirected simple graph G not containing K_p, a monochromatic K_{a_i} is forced in color i for some $i \in \{1, \ldots, k\}$. The edge Folkman number is defined as $F_e(a_1, \ldots, a_k; p) = \min\{|V(G)| : G \to (a_1, \ldots, a_k; p)^e\}$. In general, much less is known about edge Folkman numbers than the related and more studied vertex Folkman numbers, where we color vertices instead of edges. A brief overview of results concerning vertex numbers is also given throughout the paper.

$F_e(3, 3; 4)$ involves the smallest parameters for which the problem is open, posing the question "What is the smallest order N of a K_4-free graph, for which any 2-coloring of its edges must contain at least one monochromatic triangle?" This is equivalent to finding the order N of the smallest K_4-free graph which is not a union of two triangle-free graphs. It is known that $16 \leq N$ (easy bound), and it is known through a probabilistic proof by Spencer [25] (later updated by Hovey) that $N \leq 3 \times 10^9$. In this paper we overview the area, give a computer-free proof that $18 \leq N$, show with the help of computer algorithms that $19 \leq N$ (see Section 3), and give some evidence that $N \leq 127$ (see Section 7). It is even very likely that $N < 100$.

We consider only simple undirected loopless graphs, and we will use standard graph theory notation: $V(G)$, $E(G)$ for the vertex and edge set of graph G, respectively, K_t for the complete graphs on t vertices, and $\chi(G)$ for the chromatic number of G,

Classical Ramsey numbers $R(s, t)$ (and their multicolor version) play an important role in studying Folkman numbers. $R(s, t)$ is defined as the least positive n such that in any 2-coloring of the edges of K_n there is a monochromatic K_s in the first color or a monochromatic K_t in the second color. A regularly updated survey of the most recent results on the best known bounds on various types of Ramsey numbers is maintained by the first author [22].

A large part of this paper is based on the talk delivered by the first author on October 7, 2005, at the Nineteenth Midwest Conference on Combinatorics, Cryptography and Computing MCCCC'05 in Rochester, NY.

2 Basic concepts

The most important operation in the concept of Folkman numbers is that of arrowing, the same which is used in general Ramsey theory. For graphs F, G, H and positive integers s, t, k, s_i consider the following definitions.

Definition 1

(a) $F \to (s_1, ..., s_k)^e$ if and only if for every k-coloring of the edges of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \leq i \leq k$.

(b) $F \to (s_1, ..., s_k)^v$ if and only if for every k-coloring of the vertices of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \leq i \leq k$.

(c) $F \to (G, H)^e$ if and only if for every red/blue edge-coloring of F, F contains a red copy of G or a blue copy of H.

The definition of Ramsey numbers can be restated in terms of arrowing relation as $R(s,t) = \min\{n \mid K_n \to (s,t)^e\}$, and it also holds for the obvious generalizations of both Ramsey numbers and arrowing to arbitrary graphs G and H, namely we have $R(G,H) = \min\{n \mid K_n \to (G,H)^e\}$.

The complete graph K_6 has the smallest number of vertices among graphs which are not a union of two K_3-free graphs, since $R(3,3) = 6$, or equivalently using arrowing relation we have $K_6 \to (3,3)^e$ and $K_5 \not\to (3,3)^e$. Similarly, the best known bounds $43 \leq R(5,5) \leq 49$ [22] can be written as $K_{49} \to (5,5)^e$ and $K_{42} \not\to (5,5)^e$.

In 1968, Graham [7] asked for which graphs G it still holds that $G \to (3,3)^e$ if we require G to be K_6-free. Graham proved that $G = K_8 - C_5 = K_3 + C_5 \to (3,3)^e$, clearly G has no K_6, and that there are no graphs on less than 8 vertices with these properties. We have just argued that according to the following definition of the edge Folkman numbers we know the values of two of them, namely $F_e(3,3;7) = 6$ and $F_e(3,3;6) = 8$.

Definition 2

(a) The set of edge Folkman graphs $\mathcal{F}_e(s,t;k)$ is defined by $\mathcal{F}_e(s,t;k) = \{G \mid G \rightarrow (s,t)^e \wedge K_k \not\subseteq G\}$.

(b) The edge Folkman number $F_e(s,t;k)$ is defined as the smallest positive n such that there exists an n-vertex graph G in $\mathcal{F}_e(s,t;k)$.

(c) Vertex Folkman graphs and numbers, are defined similarly by 2-coloring vertices instead of edges.

(d) Multicolor vertex/edge graphs/numbers are defined analogously by using more colors.

Theorem 1 (Folkman 1970, [5]) *For all $k > max(s,t)$, edge- and vertex- Folkman numbers $F_e(s,t;k)$, $F_v(s,t;k)$ exist.*

It is easy to see that $k > R(s,t)$ implies $F_e(s,t;k) = R(s,t)$, while for $k \leq R(s,t)$ very little is known in general, and most specific parameter situations seem to be very difficult. The state of knowledge about the cases $F_e(3,3;k)$ is summarized in Table 1 below. The first two cases were discussed above. We comment on the case $k = 5$ in the remainder of this section. The case of $k = 4$ is presented in the following sections.

k	$F_e(3,3;k)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham'68
5	15	659 graphs	[PRU]'99
4	$\leq 3 \times 10^9$	probabilistic	'86,'88,'89

Table 1. Summary of edge Folkman numbers $F_e(3,3;k)$.

Two cases $F_e(3,3;5) = 15$ and $F_v(3,3;4) = 14$ were completed with the help of computer algorithms in [21]. These two cases are connected because of the easy to see implication $H \rightarrow (3,3;4)^v \Rightarrow H+x \rightarrow (3,3;5)^e$, where $H+x$ is obtained from graph H by adding a new vertex x connected to all vertices of H. In [21] it was found that there are exactly 659 graphs on 15 vertices in $\mathcal{F}_e(3,3;5)$, and none on 14. Theorem 5 in [21] states that among these 659 graphs, 153 have a vertex of degree 14, and after deleting it we obtain all nonisomorphic graphs on 14 vertices in $\mathcal{F}_v(3,3;4)$. Note that this is the implication mentioned above in the other direction - it doesn't have to be true in general, but it holds for these specific parameters. There exists exactly one 15-vertex *bicritical* graph for $F_e(3,3;5)$, for which deletion and addition of any edge falsifies the arrowing relation or creates a K_5, respectively. It has a vertex of degree 14, after whose removal we obtain the unique bicritical 14-vertex graph for $F_v(3,3;4)$, which is shown in Figure 1.

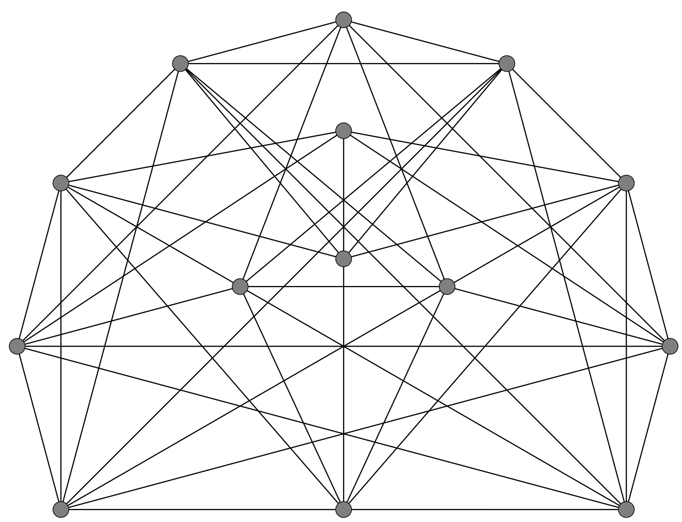

Figure 1. Unique 14-vertex bicritical graph in $\mathcal{F}_v(3,3;4)$ [21]

3 Lower bounds on $F_e(3,3;4)$

The giant gap between the best known lower and upper bounds on $F_e(3,3;4)$ is quite puzzling. Only with very significant effort it was proved in 1988 that $F_e(3,3;4) < 3 \times 10^9$ (see next Section 4 for some details). Further improvements seem to be very hard to obtain, despite that we seem to have at hand a concrete graph which likely is a witness to the upper bound of 127 (see Section 7).

The lower bound $10 \leq F_e(3,3;4)$ was obtained by Lin in 1972. This can be easily improved to 16 by using the results from [21] on $F_e(3,3;5) = 15$. Namely, in that paper all 659 graphs G on 15 vertices such that $G \rightarrow (3,3;5)^e$ were constructed, all of them contain K_4's, and thus $16 \leq F_e(3,3;4)$.

In the remaining part of this section we give two further improvements of the lower bound. First we obtain $18 \leq F_e(3,3;4)$ through a simple proof, and then improve it further to $19 \leq F_e(3,3;4)$ with the help of computations.

Theorem 2 $F_e(3,3;4) \geq 18$.

Proof. Consider the cyclic graph G_{17} on Z_{17} with two vertices connected by an edge iff their circular distance is in the set $\{1,2,4,8\}$. Graph G_{17} is the well known [22] unique critical Ramsey graph for $R(4,4) = 18$. One can easily see that the edges of G_{17} can be split into two cyclic triangle-free parts of distances $\{1,4\}$ and $\{2,8\}$. Thus $G_{17} \not\rightarrow (3,3;4)^e$. Suppose that G is any graph on 17 vertices non-isomorphic to G_{17} such that $G \rightarrow (3,3;4)^e$. Since G has no K_4, it must contain an independent set $I \subset V(G)$ of 4 vertices. Consider the graph H on 13 vertices induced in G by the set $V(G) \setminus I$. Let the graph G' be a supergraph of G with all the edges connecting I to $V(G) \setminus I$ added. We clearly have $G' \rightarrow (3,3;5)^e$, but since all neighborhoods of vertices in I are the same, the arrowing still holds if we drop from G' any 3 vertices from I. This gives a K_5-free graph on 14 vertices contradicting the fact that $F_e(3,3;5) = 15$, which completes the proof. $\diamondsuit$

Theorem 3 $F_e(3,3;4) \geq 19$.

Proof. In order to obtain this bound we follow the main idea of the last proof, this time however computer algorithms are needed to process a large number of possibilities.

For a contradiction, suppose that there exists a K_4-free graph G on 18 vertices such that $G \to (3,3;4)^e$. If G has an independent set of 5 vertices then the same reasoning as in the proof of Theorem 2 leads to a contradiction. Thus, since $R(4,4) = 18$, we may assume that the maximal independent set I in G has order 4, say $I = \{a_1, a_2, a_3, a_4\}$. We first claim that the subgraph H of G induced by the vertices $V(G) \setminus I$ must be isomorphic to one of the 153 graphs in $\mathcal{F}_v(3,3;4)$ on 14 vertices found in [21]. Note that the graph on 15 vertices $H + a_1$ is K_5-free and $H + a_1 \to (3,3)^e$, since any triangle-free split of its edges into two colors would easily extend to such a partition of the edges of G, by assigning any edge $\{a_i, v\} \in E(G)$ the color of $\{a_1, v\} \in E(G)$. Hence, by Theorem 5 in [21] discussed in the last paragraph of Section 2, $H \in \mathcal{F}_v(3,3;4)$.

This leads to a simple algorithm which reconstructs all possible graphs G. For each 14-vertex graph $H \in \mathcal{F}_v(3,3;4)$ find the set M of all subsets $A \subset V(H)$ such that A induces in H a maximal triangle-free subgraph (adding any vertex to A creates a triangle in H). The condition of maximality of A greatly reduces the size of M, but clearly still guarantees that some G's will be constructed by the next steps if there are any. Typical H has between 2500 and 3000 induced triangle-free sugbraphs, but in most cases only 50 to 70 maximal ones. Next, recover desired graphs G based on each H by joining the vertices in I to all 4-tuples of sets of vertices in M, then for graphs G with $\chi(G) \geq 6$ (see the first paragraph of section 5) test whether $G \to (3,3)^e$. This was implemented using the algorithms similar to those in [21] and no such G was found, thus there is none at all. Once the programs were written and cross-checked for correctness in several ways, all computations were completed quickly in one afternoon. In total slightly more than $8.6 \cdot 10^7$ candidate graphs G were constructed, but only 68, all of them built from 2 out of 153 H's, satisfied $\chi(G) \geq 6$. Consequently only a very small number of the most expensive tests $G \to (3,3)^e$ were performed. Actually, as we have found later, even these tests were not needed since all 68 constructed graphs have independent sets of order 5 or more, and thus again the reasoning of the proof of Theorem 2 can be applied. $\diamondsuit$

An approach similar to the last proof will not work for $F_e(3,3;4) \geq 20$, since just the number of nonisomorphic graphs on 19 vertices which are K_4-free and have no independent sets of order 5 is estimated to be more than 10^{19} [15]. Any proof or computational technique improving further on the lower bound of 19 very likely will be of significant interest.

4 Upper bounds on $F_e(3,3;4)$

Erdős and Hajnal [4] stated the problem in 1967 by asking if there exists any graph G such that $G \rightarrow (3,3;4)^e$. Its existence follows from a theorem by Folkman [5] proved in 1970, which when instantiated to 2 colors produces a VERY large upper bound for $F_e(3,3;4)$. In 1975, Erdős offered \$100 (or 300 Swiss francs) for deciding if $F_e(3,3;4) < 10^{10}$, which later resulted to be remarkably close to what can be obtained by using probabilistic methods. The first concrete bound was found by Frankl and Rödl [6] in 1986. Spencer [25], in 1988, gave a probabilistic proof of a much better bound $F_e(3,3;4) < 3 \times 10^8$, which despite having a technical error pointed out by Hovey, finally stands at $F_e(3,3;4) < 3 \times 10^9$. This is the best so far and within the limit suggested by Erdős. In section 7 we gather some evidence that this upper bound is still very far from the true value which we conjecture is bounded by 127.

The main idea of probabilistic proofs [6][20][25] is quite simple. Any graph G such that $G \rightarrow (3,3;4)^e$ proves the bound $F_e(3,3;4) \leq |V(G)|$. How to find such a G? First, take randomly a graph F from the set $G(n,p)$ of all graphs on n vertices with edge probability p. Next, remove one edge from every K_4 in F. The resulting graph G is clearly K_4-free and so has some probability of being the graph we need. The very difficult part of probabilistic proofs was analysis showing that this probability is positive for certain values of n and p. For the case of the bound by Spencer this probability is guaranteed to be positive for $n = 3 \times 10^9$ for $p = 6n^{-1/2} \approx 1/9129$.

5 Sample of related facts

In the general case it is true that $G \in \mathcal{F}_e(s,t;k)$ implies $\chi(G) \geq R(s,t)$, so in the previous section we could have used additional restriction that the graph G we were looking for has $\chi(G) \geq 6$. It was not needed since simpler but longer computations were sufficient. For several special cases it was found that $F_e(s,t;k = R(s,t)) = R(s,t) + c$ for some small (2, 4, 5) constant c, and that $F_e(s,t;k < R(s,t)) \geq R(s,t) + 4$ [13]. The following implications are generalizations of the special cases used in Section 3:

$$G \in \mathcal{F}_v(R(s-1,t), R(s,t-1); k-1) \Rightarrow G + x \in \mathcal{F}_e(s,t;k),$$

or equivalently

$$G + x \not\rightarrow (s,t)^e \Rightarrow G \not\rightarrow (R(s-1,t), R(s,t-1))^v.$$

Among the most interesting values and bounds in nontrivial cases for edge Folkman numbers the following are known. $F_e(3,4;9) = 14$ with a critical graph $K_4 + C_5 + C_5$, Nenov 1991 [18], and $F_e(3,4;8) = 16$ with a critical graph $K_4 + C_5 + C_5 + C_5$, Kolev and Nenov, 2006 [12]. It is also known that $F_e(3,5;14) = 16$, $F_e(4,4;18) = 20$, $F_e(3,7;22) \geq 27$, $F_e(3,3,3;17) = 19$ and $F_e(3,3,3;16) = 21$. Observe that the forbidden K_k in these cases satisfies $k = R(s,t)$ or $k = R(s,t) - 1$. We also note that in several cases the critical graphs have the form $K_p + C_q$, $K_p + \overline{C_q} + C_r$, or $K_p - C_q$.

Most of this paper is focused on edge Folkman numbers and graphs, but still we wish to close this section with some results on vertex Folkman numbers, often associated with graph coloring problems not necessarily in the context of arrowing.

$F_v(2,2,2;3) = 11$, or the smallest 4-chromatic triangle-free graph has 11 vertices. It is the Grőtzsch graph presented in Figure 2.

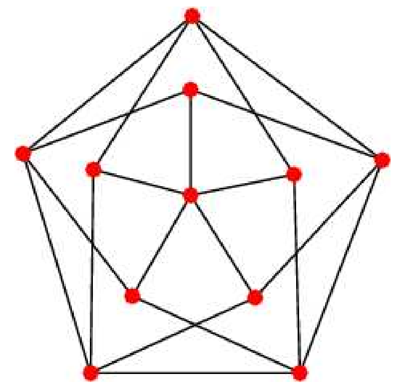

Figure 2. Grőtzsch graph [mathworld.wolfram.com]

It is known that $F_v(2,2,2,2;4) = 11$, or the smallest 5-chromatic K_4-free graph has 11 vertices [17], Nenov (1984), and $F_v(2,2,2,2;3) = 22$, or the smallest 5-chromatic triangle-free graph has 22 vertices [11], Jensen and Royle (1995). Also in terms of graph coloring, the next Theorem 4 states that the smallest $(r+1)$-chromatic K_r-free graph has $r+5$ vertices, for $r \geq 5$. The last Theorem 5 of this section was much harder to prove, but clearly deals with a similar property.

Theorem 4 (folklore)

$$F_v(\underbrace{2,\cdots,2}_{r};r) = r+5, \quad \text{for} \quad r \geq 5.$$

Sketch of proof. For the upper bound consider as the critical graph $K_{r-5} + C_5 + C_5$, for the lower bound take any K_r-free graph G on $r+4$ vertices, then assemble matchings in $\overline{G}$ to show $\chi(G) \leq r$. $\diamondsuit$

Theorem 5 (Nenov 2003, [19])

$$F_v(\underbrace{3,\cdots,3}_{r};2r) = 2r+7, \quad \text{for} \quad r \geq 3.$$

For $r = 2$ we have $F_v(3,3;4) = 14$ [21]. More complete overview of other general and special cases can be found in [14][17][18][19].

6 Complexity of edge arrowing

The problem of deciding whether $F \rightarrow (G, H)$ for general graphs plays a very special role in computational complexity theory. In 1976, Burr proved that testing whether $F \rightarrow (3,3)^e$ is **coNP**-complete [1]. Also, Burr in 1984 showed that determining if $R(G, H) < m$ is **NP**-hard [2], though he already suspected that this problem in general might be harder. In 1990, he proved that for any fixed 3-connected graphs G and H, testing whether $F \not\rightarrow (G, H)^e$ is **NP**-complete [3]. Finally, in 2001, in a breakthrough paper by Schaefer [23] it was proved that testing whether $F \rightarrow (G, H)^e$ is $\mathbf{\Pi_2^p}$-complete.

The variety of questions one can pose within this formalism by restricting or fixing some of the three arguments to arrowing is quite rich. Let us see a few examples. Note that testing whether $F \rightarrow (K_2, K_n)^e$ is the same as checking $K_n \subset F$, so it is clearly NP-hard. Schaefer [23] showed that $F \rightarrow (G, H)^e$ remains $\mathbf{\Pi_2^p}$-complete for G fixed to any tree of order at least 3. On the other hand some arrowing instances are apparently easy. For example, the results by Burr [3] include polynomial time algorithms for deciding $F \rightarrow (K_{1,n}, K_{1,m})$, and $F \rightarrow (kK_2, H)$ for any fixed $k \geq 1$ and graph H.

Typical tools in studying the complexity of arrowing are (G, H)-enforcers, (G, H)-signal senders, or determiners, which are special graphs which do not necessarily arrow the pair (G, H), but in any coloring avoiding the pair (G, H) some strong additional properties of such colorings are guaranteed. Such gadgets permit the construction of graphs F for which we are in tight control of whether $F \rightarrow (G, H)$. This type of components was used by Burr [1][2][3] and Schaefer [23] in building complexity theory reductions. On another path Grossman [8] studied (G, G)-cleavers, which are graphs F such that there exists unique coloring of F witnessing $F \not\rightarrow (G, G)$. Related complexity results are surveyed in [24].

7 Is $F_e(3, 3; 4)$ bounded by 127 ?

Geoffrey Exoo suggested to look at the well known Ramsey coloring of K_{127} defined by Hill and Irving [9] in 1982 in order to establish the bound $128 \leq R(4,4,4)$. Consider the graph G_{127} on 127 vertices defined by

$$G_{127} = (\mathcal{Z}_{127}, E), \quad E = \{(x,y) | x - y = \alpha^3 \pmod{127}\}.$$

The goal of this section is to gather some evidence that $G_{127} \rightarrow (3,3)^e$. With some routine work, one can check that G_{127} has the following properties: it has 2667 edges and 9779 triangles, it is regular of degree 42, it has independence number 11, has no K_4's (so it is a feasible candidate to be in the set $\mathcal{F}_e(3,3;4)$), it is vertex- and edge-transitive, has 5334 ($= 127 * 42$) automorphisms, has regularity type $(127, 42, 11, \{14, 16\})$ (it is almost a strongly regular graph), and the edges of K_{127} can be partitioned into three isomorphic copies of G_{127}.

In general, a graph G can be expected to satisfy $G \rightarrow (3,3)^e$ if G has a large number of triangles and has many other small dense subgraphs. Beyond this we really don't know much. There are some deep results characterizing such G's, though they deal mostly with asymptotics and probabilities which seem to give little insight into how to proceed in the case of a specific graph. Still, we venture into the following conjecture.

Conjecture. $G_{127} \rightarrow (3,3)^e$.

Let's remark that if this conjecture is true it would imply that $F_e(3,3;4) \leq 127$, and thus give a 23,622,047-fold improvement over Spencer/Hovey bound. As far as we can see Spencer's proof logic of the upper bound is not useful in the case of G_{127}.

When proving $G \rightarrow (3,3)^e$ in general, one could proceed as follows. First, solve a simpler task: find a small subgraph H embedded in G in many places such that there is a small number of colorings witnessing $H \not\rightarrow (3,3)^e$. Second, try to extend all colorings for $H \not\rightarrow (3,3)^e$ to the whole G so that monochromatic triangles are avoided. However, this algorithm seems far too expensive for G_{127}.

The most promising approach we found for verifying the conjecture is by reducing $\{G \mid G \not\rightarrow (3,3)^e\}$ to the problem 3-SAT of satisfiability of boolean formulas in conjunctive normal form with exactly three literals per each clause. We map the edges $E(G)$ to the variables of $\phi_G \in$ 3-SAT, and for each (edge)-triangle xyz in $E(G)$ we add to ϕ_G two clauses

$$(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z}).$$

One can easily see that

$$G \not\rightarrow (3,3)^e \Longleftrightarrow \phi_G \text{ is satisfiable.}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 3-clauses, two for each of the 9779 triangles. Note that by taking only the positive clauses, we can obtain a reduction to ϕ'_G in NAE-3-SAT (not all equal) with half of the clauses. Many hard 3-SAT instances of similar or larger sizes can be solved by one of rapidly improving SAT-solvers. We tried to decide ϕ_G for $G = G_{127}$ with two well known SAT-solvers, namely *zChaff* [16] from EE Princeton group, the winner of SAT-solver competitions since 2001, and more recent *March_eq* [10], winner in one of the categories in 2004 competition. Unfortunately, these solvers seem to be far from being able to decide ϕ_G. Various experiments indicate that subformulas of ϕ_G corresponding to subgraphs of G induced by no more than 80 vertices are almost always easily satisfiable, while those corresponding to more than 86 vertices are very difficult to satisfy. None corresponding to 90 or more vertices was satisfied. Our conjecture above is equivalent to the unsatisfiability of ϕ_G.

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