NOTE

Computation of the Ramsey Number $R(W_5, K_5)$

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Abstract

We determine the value of the Ramsey number $R(W_5, K_5)$ to be 27, where $W_5 = K_1 + C_4$ is the 4-spoked wheel of order 5. This solves one of the four remaining open cases in the tables given in 1989 by George R. T. Hendry, which included the Ramsey numbers $R(G, H)$ for all pairs of graphs G and H having five vertices, except seven entries. In addition, we show that there exists a unique up to isomorphism critical Ramsey graph for W_5 versus K_5. Our results are based on computer algorithms.

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1. Overview

This note is a continuation of the work reported in [1], which contained the result $R(B_3, K_5) = 20$. It has similar origins, and also the scenario of work arrangements was similar. The main result of this note, the equality $R(W_5, K_5) = 27$, was obtained with the help of computer algorithms that were a part of an MS thesis work by Josh Stinehour, under supervision of Stanisław Radziszowski, and verified with independently written programs by Kung-Kuen Tse. We will use the same definitions and notation as in [1], which appeared in this Bulletin [Vol. 41 (2004) 71-76].

In 1989, George R. T. Hendry [2] presented a table of Ramsey numbers $R(G,H)$ for all pairs of graphs G and H having five vertices, with the exception of seven cases: $R(C_5+e,K_5)$, $R(W_5,K_5-e)$, $R(B_3,K_5)$, $R(W_5,K_5)$, $R(K_5-P_3,K_5)$, $R(K_5-e,K_5)$ and $R(K_5,K_5)$. Until now, only three of these open cases have been solved: $R(C_5+e,K_5)=17$, $R(W_5,K_5-e)=17$ and $R(B_3,K_5)=20$. A regularly updated survey by the first author [5] reports on old and the most recent results on various types of Ramsey numbers, including those of the form $R(G,H)$. In particular, [5] lists the developments related to all seven cases missing in Hendry's table, and gives references to papers discussing them.

In this work, we eliminate one of these open cases by computing $R(W_5,K_5)=27$. This result improves the bounds $27 \le R(W_5,K_5) \le 29$ given in [2]. In addition, we show that there exists a unique up to isomorphism critical graph, i.e. $|\mathcal{R}(W_5,K_5;26)|=1$.

Thus, the remaining open cases of two-color Ramsey numbers for general graphs on at most five vertices are: $25 \le R(K_5-P_3,K_5) \le 28$, $30 \le R(K_5-e,K_5) \le 34$, and $43 \le R(K_5,K_5) \le 49$ (see [5] for references to all bounds). The expected difficulty of these cases is discussed in [1].

2. Enumerations and Results

It is known that $R(C_4,K_5)=14$ and $R(W_5,K_4)=17$ (see [5]). The set of all 1888 graphs in $\mathcal{R}(C_4,K_5)$ was enumerated in [6], and for this project fairly simple algorithms were sufficient to generate all 3071561 graphs in $\mathcal{R}(W_5,K_4)$. The statistics of both families by the number of graphs with fixed number of vertices are given in Table I.

For a graph G, if $v \in VG$, then $N_G(v)=\{w \in VG \,|\, vw \in EG\}$. The subgraph of G induced by W will be denoted by $G[W]$. Also, for $v \in VG$, define the induced subgraphs $G_v^+ = G[N_G(v)]$ and $G_v^- = G[VG - N_G(v) - \{v\}]$. Note that if $G \in \mathcal{R}(W_5,K_5;n)$ and $v \in VG$, then necessarily $G_v^+ \in \mathcal{R}(C_4,K_5;d)$, where $d=\deg_G(v)$, and $G_v^- \in \mathcal{R}(W_5,K_4;n-d-1)$.

For all cases, the construction of $\mathcal{R}(W_5,K_5;n)$ proceeds by using the results in Table I and applying the gluing algorithm to $G_v^+ \in \mathcal{R}(C_4,K_5;s)$ and $G_v^- \in \mathcal{R}(W_5,K_4;t)$, for all possible s and t satisfying $s+t+1=n$. The gluing algorithm used in this work was similar to that described in [1, 4, 6], except for some modifications which were needed in order to avoid the graph W_5 instead of B_3, K_4 or C_4.

s	$\lvert\mathcal{R}(C_4, K_5; s)\rvert$	$\lvert\mathcal{R}(W_5, K_4; s)\rvert$
1	1	1
2	2	2
3	4	4
4	8	10
5	17	26
6	38	94
7	85	401
8	190	2307
9	385	15452
10	574	104314
11	457	531892
12	126	1437877
13	1	865055
14		111153
15		2891
16		82
total	1888	3071561

Table I. Statistics for $\mathcal{R}(C_4, K_5)$ and $\mathcal{R}(W_5, K_4)$.

All $(W_5, K_5; 26)$-graphs were obtained by performing gluing of graphs G_v^+ to G_v^- for $s \in \{9, 10, 11, 12, 13\}$ and $t = 25 - s$. Table II presents the statistics of the gluings that were completed. The computations led to the unique $(W_5, K_5; 26)$-graph, which is cyclic and regular of degree 9, with the edges connecting pairs of vertices belonging to $\mathcal{Z}_{26}$ in circular distances 1, 5, 8, 12 and 13.

s	$\lvert\mathcal{R}(C_4, K_5; s)\rvert$	$\lvert\mathcal{R}(W_5, K_4; 25 - s)\rvert$	no. of generated $(W_5, K_5; 26)$-graphs
9	385	82	1
10	574	2891	0
11	457	111153	0
12	126	865055	0
13	1	1437877	0

Table II. Statistics for computation of $(W_5, K_5; 26)$-graphs

All $(W_5, K_5; 27)$-graphs were obtained in two ways: by performing gluing as above for $s \in \{10, 11, 12, 13\}$, $t = 26 - s$, and independently by constructing and (W_5, K_5)-filtering all one-vertex extensions of the unique $(W_5, K_5; 26)$-graph. Both paths led to no graphs, and thus $R(W_5, K_5) = 27$.

Theorem. $R(W_5, K_5) = 27$.

Proof. The computations and results described above prove that no $(W_5, K_5; 27)$-graph exists, so $R(W_5, K_5) \leq 27$. It is easy to verify that a cyclic graph with the edges joining vertices belonging to $\mathcal{Z}_{26}$ which are in circular distance 1, 5, 8, 12 or 13, has no W_5 and no $\overline{K}_5$. This implies the lower bound. ∎

Two separate implementations of the algorithms were prepared and their results compared. In order to corroborate the correctness of both implementations, we have performed a number of gluings yielding large output. Table III lists some special cases of gluing instances producing a large number of graphs in $\mathcal{R}(W_5, K_5)$ on which the two implementations agreed exactly. The computational effort of this project was moderate — all computations can now be repeated overnight on a small local departmental network.

s	$\lvert\mathcal{R}(C_4, K_5; s)\rvert$	t	$\lvert\mathcal{R}(W_5, K_4; t)\rvert$	no. of generated graphs G with $\delta(G) = s$ in $\mathcal{R}(W_5, K_5; s + t + 1)$
6	38	16	82	869853
7	85	16	82	17421
8	190	15	2891	1768

Table III. Further data on generated (W_5, K_5)-graphs

A general utility program for graph isomorph rejection, *nauty* [3], together with other graph manipulation tools, written by Brendan McKay, was used extensively. All graphs whose statistics were given in this paper are available from the authors.

References

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