A Constructive Approach
for the Lower Bounds
on the Ramsey Numbers $R(s,t)$

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Abstract

Graph G is a (k,p)-graph if G does not contain a complete graph on k vertices K_k, nor an independent set of order p. Given a (k,p)-graph G and a (k,q)-graph H, such that G and H contain an induced subgraph isomorphic to some K_{k-1}-free graph M, we construct a $(k,p+q-1)$-graph on $n(G)+n(H)+n(M)$ vertices. This implies that $R(k,p+q-1) \geq R(k,p)+R(k,q)+n(M)-1$, where $R(s,t)$ is the classical two-color Ramsey number. By applying this construction, and some its generalizations, we improve on 22 lower bounds for $R(s,t)$, for various specific values of s and t. In particular, we obtain the following new lower bounds: $R(4,15) \geq 153$, $R(6,7) \geq 111$, $R(6,11) \geq 253$, $R(7,12) \geq 416$, and $R(8,13) \geq 635$. Most of the results did not require any use of computer algorithms.

1. Introduction

We shall only consider graphs without multiple edges or loops, on a nonempty set of vertices. If $G=(V,E)$ is a graph, then by VG we will denote the set of vertices, and by EG the set of edges of G. Let $n(G)=|VG|$. $G[S]$ denotes the subgraph induced in G by a subset of vertices $S \subset VG$. We will use notation $N_G(v)$ for the

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neighborhood of vertex v in G. Also, define the induced subgraphs $G_v^+ = G[N_G(v)]$ and $G_v^- = G[VG - N_G(v) - \{v\}]$.

For positive integers s and t, an (s,t)-graph is a graph F without a subgraph isomorphic to the complete graph on s vertices K_s, and such that F has no independent sets of order t. An $(s,t;n)$-graph is an (s,t)-graph of order n. Let $\mathcal{R}(s,t)$ and $\mathcal{R}(s,t;n)$ denote the set of all (s,t)-graphs and $(s,t;n)$-graphs, respectively. The Ramsey number $R(s,t)$ is defined to be the least $n > 0$ such that there is no $(s,t;n)$-graph. Note that if $G \in \mathcal{R}(s,t)$, then $G_v^+ \in \mathcal{R}(s-1,t)$ and $G_v^- \in \mathcal{R}(s,t-1)$.

Instead of a graph F of order n, one often considers an equivalent concept of the two-coloring of edges of the complete graph K_n, where we identify F with the edges in the first color, and the complement $\overline{F}$ with the edges in the second color. Thus, for example, the Ramsey number $R(s,t)$ can be defined equivalently as the minimal n such that in any two-coloring of the edges of K_n there is a monochromatic K_s in the first color or a monochromatic K_t in the second color.

A regularly updated survey by the third author [5] lists the most recent results on different types of Ramsey numbers. It includes graph constructions implying the best known up to date lower bounds, and, in particular, it covers all cases considered in this paper. By the time this paper appears, [5] will have pointed to this paper as well. Many improvements we are reporting here are with respect to the lower bounds listed in the 2001 revision # 8 of [5].

In Section 2 we present our main construction, which is a generalization of a theorem by Burr et al. [1], and prove some theorems on lower bounds for Ramsey numbers. Based on the latter, Section 3 shows how to construct specific graphs improving several lower bounds for small s and t, including those listed in the abstract.

2. The Main Construction

In 1989, Burr, Erdős, Faudree and Schelp [1] presented a construction, that, for $s,t \geq 2$, given an $(s,t-1;n)$-graph containing K_{s-2} yields an $(s,t;n+2s-3)$-graph. This implies constructively the following lower bound theorem.

Theorem 1. [1] $R(s,t) \geq R(s,t-1) + 2s - 3$ for $s \geq 2$, $t \geq 3$.

We note that theorem 1 does not hold for $t = 2$ and $s > 2$, contrary to the bounds given in [1]. The authors of [1] overlooked the fact that there is no $(s,t-1;n)$-graph, which is needed for this case.

Suppose that we are given a (k,p)-graph G and a (k,q)-graph H, such that G and H contain an induced subgraph isomorphic to a K_{k-1}-free graph M. Our main construction produces a $(k,p+q-1)$-graph on $n(G)+n(H)+n(M)$ vertices, which implies that $R(k,p+q-1) \geq R(k,p)+R(k,q)+n(M)-1$. This is stronger than, and a generalization of, the method by Burr et al.

Construction 1. We are given a $(k,p;n_1)$-graph G with $VG = \{v_1, v_2, ..., v_{n_1}\}$ and a $(k,q;n_2)$-graph H with $VH = \{u_1, u_2, ..., u_{n_2}\}$, for some $k \geq 3$ and $p,q \geq 2$. Suppose that the induced subgraphs $G[\{v_1, ..., v_m\}]$ and $H[\{u_1, ..., u_m\}]$ are isomorphic to the same K_{k-1}-free graph M with vertices $VM = \{w_1, ..., w_m\}$, $m \leq n_1, n_2$. Assume further that the mappings $\phi(w_i) = v_i$ and $\psi(w_i) = u_i$, for $1 \leq i \leq m$, establish isomorphisms between them. We construct a graph F on n_1+n_2+m vertices, with the vertex set $VF = VG \cup VH \cup VM = \{v_1, ..., v_{n_1}, u_1, u_2, ..., u_{n_2}, w_1, ..., w_m\}$. The set of edges of the graph F is defined by

$$EF = EG \cup EH \cup EM \cup E(G,H) \cup E(G,M) \cup E(H,M),$$

where

$E(G,H) = \{\{v_i, u_i\} \mid 1 \leq i \leq m\}$,
$E(G,M) = \{\{v_i, w_j\} \mid 1 \leq i \leq n_1, 1 \leq j \leq m, \{v_i, v_j\} \in EG\}$, and
$E(H,M) = \{\{u_i, w_j\} \mid 1 \leq i \leq n_2, 1 \leq j \leq m, \{u_i, u_j\} \in EH\}$.

Theorem 2. If the graph F is obtained by construction 1, then

$$F \in \mathcal{R}(k, p+q-1; n_1+n_2+m).$$

Proof. Clearly, F has the number of vertices as claimed. We first prove that the graph F does not contain K_k. Suppose otherwise, and let S be the set of k vertices in F forming K_k. Denote $s_g = |S \cap VG|$, $s_h = |S \cap VH|$ and $s_m = |S \cap VM|$, so that $s_g + s_h + s_m = k$. The structure of $E(G,H)$ implies that if $s_g > 1$ then $s_h = 0$, and symmetrically, if $s_h > 1$ then $s_g = 0$. Since the graph M has no K_{k-1}, we have $s_m \leq k-2$. Thus, there are two cases: (i) one of s_g, s_h is equal to 0, or (ii) $s_g = s_h = 1$ and $s_m = k-2$.

In the case (i) we may assume that $s_h = 0$, because the case $s_g = 0$ is symmetrical. Since $\{v_i, w_i\}$ is not an edge for any i and $S \subset VG \cup VM$, we can write $S = \{t_{i_1}, t_{i_2}, ..., t_{i_k}\}$, where $t_{i_j} = v_{i_j}$ or $t_{i_j} = w_{i_j}$, for $1 \leq j \leq k$. Now, observe that the definition of $E(G,M)$ implies that the set $\{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ induces a K_k in G, which is a contradiction. In the case (ii), let v_{j_1} and u_{j_2} be the unique vertices in S belonging to the sets VG and VH, respectively. Since $\{v_{j_1}, u_{j_2}\} \in E(G,H)$, we must have $j_1 = j_2 \leq m$. From the construction, we easily see that $w_{j_1} \notin S \cap VM$, and observe that $S \cup \{w_{j_1}\} \setminus \{v_{j_1}, u_{j_1}\}$ must induce a complete graph in M. This contradicts the fact that M has no K_{k-1}, and so we can conclude that F does not contain K_k.

In order to complete the proof, we need to show that F cannot have any independent sets of order $p+q-1$. Suppose that I is an independent set in F. Denote by A, B and C the intersection of I with VG, VH and VM, respectively. Consider the cardinalities of parts of I as follows.

$$c = |C|, \quad C = \{w_{i_1}, w_{i_2}, ..., w_{i_c}\},$$

$$a = |A \setminus \{v_{i_1}, v_{i_2}, ..., v_{i_c}\}|,$$
$$b = |B \setminus \{u_{i_1}, u_{i_2}, ..., u_{i_c}\}|.$$

From the construction of F we have

$$a + c \leq p - 1, \qquad b + c \leq q - 1.$$

Consider the set

$$D = I \cap \{v_{i_1}, v_{i_2}, ..., v_{i_c}, u_{i_1}, u_{i_2}, ..., u_{i_c}\}.$$

Because of the edges in $E(G,H)$, we have $|D| \leq c$. On the other hand

$$I = (A \setminus \{v_{i_1}, v_{i_2}, ..., v_{i_c}\}) \cup (B \setminus \{u_{i_1}, u_{i_2}, ..., u_{i_c}\}) \cup C \cup D,$$

and thus

$$|I| \leq a + b + 2c = (a+c) + (b+c) \leq (p-1) + (q-1) < p + q - 1.$$

This completes the proof of theorem 2. ∎

Theorem 3. If $2 \leq p \leq q$ and $3 \leq k$, then

$$R(k, p+q-1) \geq R(k,p) + R(k,q) + \begin{cases} k-3, & \text{if } 2 = p; \\ k-2, & \text{if } 3 \leq p; \\ p-2, & \text{if } 2 = p \text{ or } 3 = k; \\ p-1, & \text{if } 3 \leq p \text{ and } 4 \leq k. \end{cases}$$

Note that the conditions in theorem 3 for the lower bounds on the right hand side are satisfied for some overlapping values, in which case the maximum can be taken.

Proof. Let $G \in \mathcal{R}(k,p;R(k,p)-1)$ and $H \in \mathcal{R}(k,q;R(k,q)-1)$. Both graphs G and H must contain K_{k-1}, since otherwise they wouldn't be critical, i.e. $K_1 + G \in \mathcal{R}(k,p;R(k,p))$ and $K_1 + H \in \mathcal{R}(k,q;R(k,q))$, which is impossible. Let $M_1 = K_{k-2}$. If $p \geq 3$, then observe that G and H can be chosen so that both contain an even larger common induced subgraph $M_2 = K_{k-1} - e$. Similarly, we can argue that graphs G and H must contain independent sets of order $p-1$, and set $M_3 = \overline{K_{p-1}}$. Note finally that, for $p \geq 3$ and $k \geq 4$, they can be chosen so that both contain as an induced subgraph a one edge graph $M_4 = \overline{K_p - e}$. In all cases K_{k-1} is not contained in M_i.

Apply construction 1 to graphs G and H with a common subgraph M_i, for $1 \leq i \leq 4$. Using theorem 2 in each case yields the corresponding lower bound. This completes the proof. ∎

Theorem 3 with $k=3$ and $p=q=2$ gives nicely $R(3,3) \geq 6$, while construction 1 builds the cycle C_5 as the lower bound graph F. Next, for $k=q=3$ and $p=2$, if we use $G=K_2$, $H=C_5$ and $M=K_1$, then the construction produces a $(3,4;8)$-graph, which again is Ramsey-critical (on the largest possible number of vertices) for the number $R(3,4)=9$. We don't expect any further Ramsey number values to be matched this way, but several best known constructions are, as listed in section 3.

We observe that a special case of theorem 3 with $k=s$, $q=t-1$ and $p=2$ gives theorem 1, except for the trivial case $s=2$. Further lower bounds for higher Ramsey numbers can be obtained from the next theorem.

Theorem 4. $R(p,2p-1) \geq \frac{5}{2}R(p,p) - 2$ for $p \geq 3$.

Proof. Let $s = R(p,p)-1$, and consider any $(p,p;s)$-graph G. Note that necessarily $\overline{G} \in \mathcal{R}(p,p;s)$. Choose a vertex $v \in VG$ of degree at least $(s-1)/2$. If no such vertex exists, consider $\overline{G}$ instead of G. Let H be an isomorphic copy of G, and define graph M to be an isomorphic copy of the graph G_v^+. Observe that graph M cannot contain K_{p-1}, since otherwise G would contain K_p. Applying construction 1 to graphs G, H and M we obtain graph F, which by theorem 2 is a $(p,2p-1)$-graph on at least $2s+(s-1)/2$ vertices. The theorem follows by substituting $s=R(p,p)-1$. ∎

For $p=3$ and $G=C_5$, the proof of theorem 4 gives $R(3,5) \geq 13$ and it yields a nontrivial $(3,5;12)$-graph, with only one vertex less than the best possible. For $p=4$ we obtain $R(4,7) \geq 43$, while the best known lower bound for this case is $R(4,7) \geq 49$.

3. Computation of Some Lower Bounds on $R(s,t)$

The theorems of the previous section don't seem to lead to any strong asymptotic lower bounds. Nevertheless, they appear to be quite effective in constructing Ramsey (s,t)-graphs for specific small values of s and t. In this section we present several such constructions improving on the currently best known lower bounds. During the last two decades, most of lower bound improvements for $R(s,t)$ were obtained by constructions specific to parameters, very often using circular graphs, or were results of heuristic algorithms like simulated annealing or tabu search. This is in contrast to our method where all improvements are direct or indirect consequences of construction 1.

Theorem 5. The lower bounds on $R(s,t)$ hold as listed in Table I.

The cases listed in Table I are either improving a previously published lower bound reported in the survey [5] (revision #8, 2001, or revision #9, 2002), or are included since they are needed as intermediate constructions in this paper. If an old bound is given without any reference, it is because [5] considered it as an easy result undeserving of credit. We feel that the strongest and most interesting among the new lower bounds are those listed in the abstract. They seem to be more difficult to improve any further, or the previous lower bound was published long ago.

Proof of Theorem 5. We refer to the cases by their position in Table I. If some lower bound on $R(s,t)$ for smaller s,t, or a graph in $\mathcal{R}(s,t;n)$ is needed to complete a given case, either it is a case with a smaller index in Table I, or is referenced to in [5]. The last column specifies the theorem by which new lower bound can be obtained. Theorem 3 subsumes theorem 1, and we point to it only if theorem 1 is not sufficient for the given case. All reasonings using custom constructions via theorem 2 are commented below. In some cases we mention that a graph is circular, and thus it is vertex transitive. Consequently, when we consider G_v^+ or G_v^- in such cases, we don't need to specify the vertex v.

(2) Let $G = G_{127}$ be the circular $(4,12;127)$-graph found in [8]. Let H be any $(4,3;8)$-graph, and set $M = C_5$. Both G and H contain induced C_5. Apply theorem 2.

(3) Take $G = G_{127}$ as in (2). Denote by H the unique circular $(4,4;17)$-graph, and let $M = H_v^+$. Note that $M \in \mathcal{R}(3,4;8)$, and it can be easily verified that G contains induced M. Apply theorem 2.

case no.	s	t	old bound	reference	new bound	theorem/ parameters
1.	4	13	131		133	1
2.	4	14	136		141	2
3.	4	15	145		153	2
4.	4	18	182	[7]	187	1
5.	5	17	284	[3]	285	1
6.	6	7	109	[2]	111	1
7.	6	8	122	[2]	127	2
8.	6	10	167	[2]	177	2
9.	6	11	203		253	4, $p = 6$
10.	6	12	230	[9]	262	1
11.	6	13	242	[11]	278	2
12.	6	14	284	[9]	294	2
13.	7	8			216	1
14.	7	11			322	2
15.	7	12	312	[10]	416	2
16.	7	13			511	4, $p = 7$
17.	7	17	578	[4]	628	2
18.	7	18	618	[4]	722	2
19.	8	9			295	1
20.	8	10			317	3, $p = 3$, $q = 8$
21.	8	13			635	2
22.	8	15	618	[4]	703	4, $p = 8$
23.	8	17	678	[4]	762	2
24.	8	18	740	[9]	871	2
25.	8	19	860	[9]	1054	2
26.	9	10			580	1
27.	9	17			1411	4, $p = 9$
28.	9	21	1278	[10]	1539	3, $p = 5$, $q = 17$
29.	10	16	1052	[4]	1190	2
30.	12	12	1597		1637	2

Table I. New lower bounds on $R(s, t)$.

(7) Let $G = G_{101}$ be the only known $(6,6;101)$-graph. This graph is circular and self-complementary. All graphs in $\mathcal{R}(6,3;17)$ and $\mathcal{R}(5,3;8)$ are well known (cf. [5]). Using simple computations, we found graphs $H \in \mathcal{R}(6,3;17)$ and $M \in \mathcal{R}(5,3;8)$, such that both G and H contain M as an induced graph. Apply theorem 2.

(8) Take $G = G_{101}$ as in (7), and let $H \in \mathcal{R}(6,5;50)$ be isomorphic to G_v^-. H has a vertex w of degree 25, and we take $M \in \mathcal{R}(5,5;25)$ to be isomorphic to H_w^+. Apply theorem 2 to obtain a $(6,10;176)$-graph.

(11) Let $G \in \mathcal{R}(6,8;126)$ be the graph obtained in (7), and set $H = G_{101}$. Note that H is isomorphic to a subgraph of G, and thus both G and H contain induced $M = H_v^- \in \mathcal{R}(5,6;50)$. Apply theorem 2 to obtain a $(6,13;277)$-graph.

(12) Take $G \in \mathcal{R}(6,11;252)$ as obtained in (9). Let $H \in \mathcal{R}(6,4;34)$ be the complement of a $(4,6)$-graph found by Exoo (cf. [5]). We have found a 7-vertex graph M without K_5, which is an induced subgraph of both G and H. Apply theorem 2 to obtain a $(6,14;293)$-graph.

(14) Given $(k,p;n)$-graph, first construct a $(k+1,p;n+2p-3)$-graph G using theorem 2 (or 1). Next, for $H = G$, and setting M to be the original $(k,p;n)$-graph, construction 1 yields a $(k+1,2p-1;3n+4p-6)$-graph F. Starting with G_{101}, for $k = p = 6$, this gives construction of a $(7,11;321)$-graph.

(15) Let $G = G_{204} \in \mathcal{R}(7,7;204)$. This, and other diagonal Ramsey constructions were described by Shearer [6]. Graph G_{204} contains $M = G_{101}$ as an induced subgraph. Take $H \in \mathcal{R}(7,6;110)$ to be the complement of $(7,6;110)$-graph obtained in (6). Since G_{101} is self-complementary, it is also an induced subgraph of H. Apply theorem 2 to obtain a $(7,12;415)$-graph.

(17) Take $G \in \mathcal{R}(7,11;321)$ as obtained in (14), $H = G_{204}$, and set $M = K_1 \cup G_{101} \in \mathcal{R}(6,7;102)$. Following construction in (14) we see that graph G contains an induced subgraph isomorphic to M. Graph H defined in [6] contains M. Apply theorem 2 to obtain a $(7,17;627)$-graph.

(18) Take $G \in \mathcal{R}(7,12;415)$ as obtained in (15), $H = G_{204}$, and set $M = K_1 \cup G_{101} \in \mathcal{R}(6,7;102)$. Graph G contains G_{204}, which in turn contains an induced subgraph isomorphic to M. Apply theorem 2 to obtain a $(7,18;721)$-graph.

(21) Use the same construction as in (14), but starting with G_{204}, for $k = p = 7$. This gives construction of an $(8,13;634)$-graph.

(23) Take $G \in \mathcal{R}(8,13;634)$ as obtained in (21). We define graph $H \in \mathcal{R}(8,5;70)$ as the complement of the result of construction 1 applied to the unique $(5,3;13)$-graph, $G_{101}^+ \in \mathcal{R}(5,6;50)$, and a common 7-vertex induced graph M (the same

as in (12)). With some work it can be traced that G and H were build from some shared subgraphs, resulting in a common induced $(7, 5; 57)$-graph. Apply theorem 2 to obtain a $(8, 17; 761)$-graph.

(24) Take $G \in \mathcal{R}(8, 13; 634)$ as obtained in (21). Let $H \in \mathcal{R}(8, 6; 126)$ be the complement of the graph constructed in (7). By following the details of constructions of G and H one can easily see that both contain an induced $M \in \mathcal{R}(7, 6; 110)$. Apply theorem 2 to obtain an $(8, 18; 870)$-graph.

(25) Take $G \in \mathcal{R}(8, 13; 634)$ as obtained in (21). Let $H \in \mathcal{R}(8, 7; 215)$ be the complement of the graph constructed in (13). Both G and H contain as an induced graph $M = G_{204}$. Apply theorem 2 to obtain an $(8, 19; 1053)$-graph.

(29) Take $G \in \mathcal{R}(10, 9; 579)$ and $H \in \mathcal{R}(10, 8; 316)$ to be the complements of graphs obtained in (26) and (20), respectively. By following the details of constructions of G and H one can easily see that both contain an induced $M \in \mathcal{R}(9, 8; 294)$. Apply theorem 2 to obtain a $(10, 16; 1189)$-graph.

(30) First, apply theorem 3 with $k = 11$, $p = 2$, $q = 11$, so $R(11, 12) \geq 1616$. Next, use theorem 3 with $k = 12$, $p = 2$, $q = 11$. ∎

This work did not require any intensive computations. We have used computer algorithms mainly for verification of induced subgraphs. It is quite likely that in many cases better lower bounds can be obtained by finding larger common induced subgraph M. This could be approached by using heuristic algorithms, or even by performing exhaustive searches within special graphs G and H.

References

[1] S. A. Burr, P. Erdős, R. J. Faudree and R. H. Schelp, On the Difference between Consecutive Ramsey Numbers, *Utilitas Mathematica*, **35** (1989) 115–118.

[2] G. Exoo, Some New Ramsey Colorings, *The Electronic Journal of Combinatorics*, `http://www.combinatorics.org/`, #R29, **5** (1998), 5 pages. The constructions are available at `http://isu.indstate.edu/ge/RAMSEY`.

[3] G. Exoo, Some Applications of *pq*-groups in Graph Theory, *preprint*, (2002). The constructions are available at `http://isu.indstate.edu/ge/RAMSEY`.

[4] Luo Haipeng, Su Wenlong, Zhang Zhengyou and Li Guiqing, New Lower Bounds for Twelve Classical 2-Color Ramsey Numbers $R(k,l)$ (in Chinese), *Guangxi Sciences*, **7**, 2 (2000) 120–121.

[5] S. P. Radziszowski, Small Ramsey Numbers, *Electronic Journal of Combinatorics*, Dynamic Survey 1, revision #9, July 2002, `http://www.combinatorics.org`.

[6] J. B. Shearer, Lower Bounds for Small Diagonal Ramsey Numbers, *Journal of Combinatorial Theory*, Series A, **42** (1986) 302–304.

[7] Su Wenlong and Luo Haipeng, Prime Order Cyclic Graphs and New Lower Bounds for Three Classical Ramsey Numbers $R(4,n)$ (in Chinese), *Journal of Mathematical Study*, **31**, 4 (1998) 442–446.

[8] Su Wenlong, Luo Haipeng and Li Qiao, New Lower Bounds of Classical Ramsey Numbers $R(4,12)$, $R(5,11)$ and $R(5,12)$, *Chinese Science Bulletin*, **43**, 6 (1998) 528.

[9] Su Wenlong, Luo Haipeng and Li Qiao, A Method for Obtaining Lower Bounds for Some Ramsey Numbers (in Chinese), *Journal of Guangxi Academy of Sciences*, **15**, 4 (1999) 145–147.

[10] Su Wenlong, Luo Haipeng and Li Qiao, New Lower Bounds for Seven Classical Ramsey Numbers $R(k,l)$ (in Chinese), *Journal of Systems Science and Mathematical Sciences*, **20**, 1 (2000) 55–57.

[11] Su Wenlong, Luo Haipeng and Zhang Zhengyou, Five New Prime Order Cyclic Graphs (in Chinese), *Guangxi Sciences*, **5**, 1 (1998) 4–5.