Towards the Exact Value of the Ramsey Number $R(3, 3, 4)$

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Abstract

The classical Ramsey number $R(r_1, \ldots, r_k)$ is the least $n > 0$ such that there is no k-coloring of the edges of K_n which does not contain any monochromatic complete subgraph K_{r_i} in color i, for all $1 \leq i \leq k$. In the multicolor case ($k > 2$), the only known nontrivial value is $R(3, 3, 3) = 17$. The only other case whose evaluation does not look hopeless is $R(3, 3, 4)$, which currently is known to be equal to 30 or 31 by an earlier work of the authors. We report on progress towards deciding which of these two is the correct value. Using computer algorithms we show that any critical coloring of K_{30} proving $R(3, 3, 4) = 31$ must satisfy some additional properties, beyond those implied directly by the definitions, further pruning the search space. This progress, though substantial, is not yet sufficient to launch the final attack on the exact value of $R(3, 3, 4)$.

1. Introduction and Notation

An $(r_1, r_2, \ldots, r_k)$ *coloring*, $r_i \geq 1$ for $1 \leq i \leq k$, is an assignment of one of k colors to each edge in a complete graph, such that it does not contain monochromatic complete subgraph K_{r_i} in color i, for $1 \leq i \leq k$. Similarly, an $(r_1, r_2, \ldots, r_k; n)$ *coloring* is an $(r_1, \ldots, r_k)$ coloring of K_n. Let $\mathcal{R}(r_1, \ldots, r_k)$ and $\mathcal{R}(r_1, \ldots, r_k; n)$ denote the set of all $(r_1, \ldots, r_k)$ and $(r_1, \ldots, r_k; n)$ colorings, respectively. The Ramsey number $R(r_1, \ldots, r_k)$ is defined to be the least $n > 0$ such that $\mathcal{R}(r_1, \ldots, r_k; n)$ is empty.

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A coloring using k colors will be called a k-coloring. In this paper we consider k-colorings only for $k = 2$ (equivalent to graphs) or $k = 3$. In the multicolor case ($k > 2$), the only known nontrivial value of the Ramsey number is $R(3,3,3) = 17$ [GG]. The only other case whose evaluation does not look hopeless is $R(3,3,4)$, which currently is known to be equal to 30 or 31 by an earlier work of the authors [PR]. Here we report on progress towards deciding which of these two is the correct value. An electronic dynamic survey of the most recent results on bounds on Ramsey numbers can be found on-line at [Rad].

Two k-colorings are *isomorphic* if there exist a bijection between the vertices of the underlying complete graphs preserving all the colors of edges, and they are *weakly isomorphic* if there exists a bijection between vertices which preserves the relation of two edges having the same color. It is convenient to think of a weak isomorphism as a composition of permutation of colors with an isomorphism. Let C be a k-coloring and G be a simple undirected graph. In the sequel we will use the same notation as in the previous paper [PR].

$\deg_G(x)$	— the degree of vertex x in graph G
$n(G)$, $e(G)$	— the number of vertices and edges in graph G
$V(G)$, $E(G)$	— the vertex and edge sets of graph G
$N_G(x)$	— the neighborhood of vertex x in graph G
$C[i]$	— the graph formed by edges of color i in coloring C
C_x^i	— the coloring induced in C by vertices in $N_{C[i]}(x)$
$C_x^i[j]$	— the graph formed by edges of color j in coloring C_x^i
$\mathcal{R}(a,b,c;\geq n)$	— $\bigcup_{k\geq n} \mathcal{R}(a,b,c;k)$

Section 2 summarizes old and new results, and section 3 describes the algorithms and computations.

2. Old and New Results

Previous results on the Ramsey number $R(3,3,4)$ by Kalbfleisch (lower bound, [Kalb]) and the authors [PR] are summarized in the following theorem.

Theorem 1. $30 \leq R(3,3,4)$, $R(3,3,4) \leq 31$, *and* $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$ *coloring* C *such that every triangle* $T \subset C[3]$ *has a vertex* $x \in T$ *with* $deg_{C[3]}(x) = 13$. *Furthermore,* C *has at least* 14 *vertices* v *such that* $deg_{C[1]}(v) = deg_{C[2]}(v) = 8$ *and* $deg_{C[3]}(v) = 13$.

If C is a $(3,3,4;m)$ coloring, then for every vertex x, C_x^3 is a (3,3,3; $deg_{C[3]}(x)$) coloring, and C_x^i, for $1 \leq i \leq 2$, is a (2,3,4;d) or a (2,3,4;d) coloring, respectively, where $d = deg_{C[i]}(x)$. In the latter cases C_x^i is equivalent to a $(3,4;d)$ graph, since it cannot have edges of color i. Clearly, $deg_{C[1]}(x) + deg_{C[2]}(x) + deg_{C[3]}(x) = m-1$, and furthermore for $m = 30$ the equalities $R(3,4) = 9$ and $R(3,3,3) = 17$ [GG] imply color degree bounds $5 \leq deg_{C[1]}(x)$, $deg_{C[2]}(x) \leq 8$ and $13 \leq deg_{C[3]}(x)) \leq 16$. There are only 48 $(3,4;d)$ graphs (36 for $5 \leq d \leq 8$), and all of them are known since the early work by Kalbfleisch [Kalb].

Our knowledge about possible $(3,3,3;deg_{C[3]}(x))$ subcolorings includes the following. Kalbfleisch and Stanton [KS] proved that there are exactly two nonisomorphic $(3,3,3;16)$ colorings, both vertex transitive. Deleting one point from each leads to two nonisomorphic $(3,3,3;15)$ colorings, and as proved by Heinrich [Hein] there are no others. We found that there are exactly 651 nonisomorphic $(3,3,3;14)$ colorings, which is reduced to only 115 up to weak isomorphism. Further statistics about colorings in $\mathcal{R}(3,3,3;14)$ can be found in [PR]. In summary, we know all of $\mathcal{R}(3,3,3;\geq 14)$, but not $\mathcal{R}(3,3,3;13)$. Exhaustive generation of all $(3,3,3;13)$ colorings is on the edge of feasibility, because of their very large number, and even if their full enumeration were completed, it would not be possible to perform much computations with each coloring.

The results of this work are stated in the next two theorems.

Theorem 2. $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$ *coloring* C *such that every triangle* $T \subseteq C[3]$ *has at least two vertices* $x, y \in T$ *with* $deg_{C[3]}(x) = deg_{C[3]}(y) = 13$.

Proof. The computations described in the next section showed that no coloring in $\mathcal{R}(3,3,4;30)$ can have a triangle T with at most one such vertex. Note that Theorem 1 guarantees at least one. ∎

Theorem 3. $R(3,3,4) = 31$ *if and only if there exists a* $(3,3,4;30)$ *coloring* C *such that every edge in the third color has at least one endpoint* x *with* $deg_{C[3]}(x) = 13$. *Furthermore,* C *has at least* 25 *vertices* v *such that* $deg_{C[1]}(v) = deg_{C[2]}(v) = 8$ *and* $deg_{C[3]}(v) = 13$.

Proof. Assume that C is a $(3,3,4;30)$ coloring. For every vertex x, C_x^3 is a $(3,3,3;\geq 13)$ coloring, in which every vertex is adjacent to at least 2 (and at most $5 = R(3,3)-1$) vertices in each color. Hence, for any edge xy in the third color there must be a vertex z such that xyz forms a triangle in the third color in C. Thus, by Theorem 2, $deg_{C[3]}(x) = 13$ or $deg_{C[3]}(y) = 13$.

Since $R(3,3) = 6$, each subset of vertices $S \subseteq V(C)$ of order higher than 5 must contain an edge in the third color, so one of its endpoints x has $deg_{C[3]}(x) = 13$. ∎

We consider that this work provides additional evidence to support the conjecture that $R(3,3,4) = 30$ [PR]. Further elimination of all vertices with degree at least 14 in the third color is perhaps feasible. Unfortunately, our current approach is not efficient enough to proceed computationally with all $(3,3,3;13)$ colorings, which likely is needed in order to obtain the final answer.

3. Algorithms and Computations

We describe an algorithm $\mathcal{A}$ constructing all $(3,3,4;30)$ colorings X which have a triangle $T \subseteq X[3]$ with at least two vertices $x, y \in T$ such that $deg_{X[3]}(x)$, $deg_{X[3]}(y) \geq 14$. The starting configurations for algorithm $\mathcal{A}$ are produced by the gluing procedure $\mathcal{GLUE}$, and then the full search is performed using the operations $\mathcal{REDUCE}$, $\mathcal{FILTER}$ and $\mathcal{BRANCH}$.

Procedure $\mathcal{GLUE}$

Given $C, D \in \mathcal{R}(3,3,3;\geq 14)$, $v \in V(C)$ and $w \in V(D)$ such that $\deg_{C[3]}(v) = \deg_{D[3]}(w)$, for each isomorphism $\pi : N_{C[3]}(v) \to N_{D[3]}(w)$ create a new partial coloring $H = Glue(C, D, v, w, \pi)$ by overlapping D with C on the vertices $N_{C[3]}(v)$ using π, so that

(1) for each $x \in N_{C[3]}(v)$ the vertices x and $\pi(x)$ are identified in H,
(2) for each $x \in V(D)$ the edge vx has color 3 in H,
(3) for each $x \in V(C)$ the edge wx has color 3 in H,
(4) for all $x \in V(C) - (N_{C[3]}(v) \cup \{v\})$ and $y \in V(D) - (N_{D[3]}(w) \cup \{w\})$ the edge xy remains uncolored.

The new coloring H has now $n = |V(C)| + |V(D)| - |N_{C[3]}(v)|$ vertices. Finally, extend H to 30 vertices by adding $30 - n$ isolated vertices (with all adjacent edges uncolored).

Let $\mathcal{GLUE}(C, D)$ denote the set of all partial colorings H which can be obtained by $Glue(C, D, v, w, \pi)$ for some $v \in V(C)$, $w \in V(D)$ and some isomorphism π as above. Obviously, any $(3,3,4;30)$ coloring X with an edge xy in color 3 and $\deg_{X[3]}(x)$, $\deg_{X[3]}(y) \geq 14$ contains as a subcoloring an element of $\mathcal{GLUE}(C, D)$, for some colorings $C, D \in \mathcal{R}(3,3,3;\geq 14)$.

Operation $\mathcal{REDUCE}$

In each input partial $(3, 3, 4; 30)$ coloring C, assign three possible colors to each uncolored edge, and iterate the following process. For each edge with more than one possible color, delete colors which lead to a forbidden clique or violate obvious degree restrictions listed in section 2. Terminate with empty output if some edge has no possible colors. If there is no edge for which the deletion of possible colors is enforced, then create the output coloring $\mathcal{REDUCE}(C)$ treating all edges with more than one possible color as uncolored. Clearly, any $(3, 3, 4; 30)$ coloring containing subcoloring C contains $\mathcal{REDUCE}(C)$ as well.

Operation $\mathcal{FILTER}$

Let C be an input partial $(3, 3, 4; 30)$ coloring. Let S be the set of all vertices v such that there is a triangle vxy in color 3 and $\deg_{C[3]}(x) \geq 14$, $\deg_{C[3]}(y) \geq 14$. For each $v \in S$ check whether the subcoloring induced by C_v^1 has a possibility to be extended to one of the three nonisomorphic $(2, 3, 4; 8)$ colorings, and whether the subcoloring induced by C_v^2 has a possibility to be extended to a $(3, 2, 4; 8)$ coloring. If for each $v \in S$ the answer is affirmative in both cases, then we say that the partial coloring C satisfies property F. By Theorem 1, every subcoloring of C must have property F. Operation $\mathcal{FILTER}$ will be used to eliminate colorings failing property F.

Operation $\mathcal{BRANCH}$

Let C be an input partial $(3, 3, 4; 30)$ coloring, and suppose C is a subcoloring of some $D \in \mathcal{R}(3, 3, 4; 30)$. Let S be the set of all vertices v such that there is a triangle vxy in color 3 and $\deg_{C[3]}(x), \deg_{C[3]}(y) \geq 14$. Observe that by Theorem 1 these conditions imply that for $v \in S$ we must have $deg_{D[1]}(v) = deg_{D[2]}(v) = 8$ and $deg_{D[3]}(v) = 13$. We exploit the latter by trying to color some edges in C in a part already almost colored and leading to limited branching.

Choose a vertex $v \in S$ and color $k \in \{1, 2\}$ maximizing the number of colored edges in C_v^k, ignoring v and k giving complete subcolorings of C_v^k (necessarily on 8 vertices). If there is no such pair the process terminates with unchanged C on output. Otherwise, each of the three nonisomorphic $(2, 3, 4; 8)$ colorings ($k = 1$) or each of the three nonisomorphic $(3, 2, 4; 8)$ colorings ($k = 2$) is embedded into C in all possible ways, so that C_v^k becomes completely colored when the vertices of $(2, 3, 4; 8)$ (or $(3, 2, 4; 8)$) are identified with the corresponding vertices of $V(C)$, and the colors of already colored edges in C are preserved. Denote the set of all such extensions by $\mathcal{BRANCH}(C)$. Note that such branching is correct in the sense that if input C is a subcoloring of a $(3,3,4;30)$ coloring D, then at least one of the elements of $\mathcal{BRANCH}(C)$ is also a subcoloring of D.

Algorithm $\mathcal{A}$

Step 1. For each pair of colorings $C, D \in \mathcal{R}(3,3,3;\geq 14)$ create the set of starting partial colorings $\mathcal{M} = \mathcal{GLUE}(C, D)$.

Step 2. For each partial coloring $C \in \mathcal{M}$ replace C with $\mathcal{REDUCE}(C)$ (remove C from $\mathcal{M}$ if $\mathcal{REDUCE}(C)$ has empty output). Using $\mathcal{FILTER}$, delete from $\mathcal{M}$ all partial colorings which fail property F. Remove from $\mathcal{M}$ isomorphic copies of colorings.

Step 3. Iterate until $\mathcal{M}'$ after step (3.3) is the same as $\mathcal{M}$ at step (3.1).

(1) Assign $\mathcal{M}' = \emptyset$.

(2) For each partial coloring $C \in \mathcal{M}$ add $\mathcal{BRANCH}(C)$ to $\mathcal{M}'$.

(3) For each partial coloring $C \in \mathcal{M}'$ replace C with $\mathcal{REDUCE}(C)$. Using $\mathcal{FILTER}$, delete from $\mathcal{M}'$ all partial coloring C which fail property F. Remove from $\mathcal{M}'$ isomorphic copies of colorings.

(4) Assign $\mathcal{M} = \mathcal{M}'$.

Proof. (of Theorem 2) Suppose there exists a $(3,3,4;30)$ coloring X which has a triangle $T \subseteq X[3]$ with at least two vertices $x, y \in T$ such that $deg_{X[3]}(x)$, $deg_{X[3]}(y) \geq 14$. It is easy to see that the set of starters generated by $\mathcal{GLUE}$ in Step 1 of algorithm $\mathcal{A}$ must produce a subcoloring of X. The other three operations $\mathcal{REDUCE}, \mathcal{FILTER}$ and $\mathcal{BRANCH}$ used in algorithm $\mathcal{A}$ produce at least one coloring if performed on subcolorings of X. Consequently, the set $\mathcal{M}$ must contain a subcoloring of X at the termination of $\mathcal{A}$, or if $\mathcal{M} = \emptyset$ then no such X exists.

The algorithm $\mathcal{A}$ was run for all starting pairs of colorings $C, D \in \mathcal{R}(3,3,3;\geq 14)$. In each case it produced an empty set $\mathcal{M}$ after a small number of iterations (in most cases 4, 5 or 6) of Step 3. This completes the proof of Theorem 2. ∎

Computations

The set $\mathcal{M}$ after Step 1 in algorithm $\mathcal{A}$ was split into 11135 smaller parts, and the computations were performed separately for each of them. Significant parts of computations were repeated with an independent implementation by the second author, and the two implementations agreed on all partial colorings produced in sample cases, up to isomorphism.

Two general public domain programs written by Brendan McKay were used in this work: *nauty* [McK1] for testing isomorphism of edge colorings, and *autoson* [McK2] for distributing jobs over a local network. The total time required for all computations was about 4 CPU years, mostly on Sun Ultra 5 and 10 systems. This was achieved in a reasonable amount of time by employing a large number of computers simultaneously.

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