# On Halving Line Arrangements


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**Abstract**

Given a set of $n$ points in general position in the plane, where $n$ is even, a *halving line* is a line going through two of the points and cutting the remaining set of $n-2$ points in half. Let $h(n)$ denote the maximum number of halving lines that can be realized by a planar set of $n$ points. The problem naturally generalizes to pseudoconfigurations; denote the maximum number of halving pseudolines over all pseudoconfigurations of size $n$ by $\hat{h}(n)$. We prove that $\hat{h}(12) = 18$ and that the pseudoconfiguration on 12 points with the largest number of halving pseudolines is unique up to isomorphism; this pseudoconfiguration is realizable, implying $h(12) = 18$. We show several structural results that substantially reduce the computational effort needed to obtain the exact value of $\hat{h}(n)$ for larger $n$. Using these techniques, we enumerate all topologically distinct, simple arrangements of 10 pseudolines with a marked cell. We also prove that $h(14) = 22$ using certain properties of degree sequences of halving edges graphs.


## 1 Introduction

Given a set $S$ of $n$ points in general position (no three collinear) in the plane, where $n$ is even, let a *halving line* be a line going through two of the points and cutting the remaining set of $n-2$ points in half. Simmons raised the following question: What is the maximum number $h(n)$ of halving lines that can be realized by a set of $n$ points? Around 1970, Straus described a construction of a set of $n$ points in the plane with $O(n \log n)$ halving lines. This was generalized by Erdős, Lovász, Simmons, and Straus [ELSS73] (and later independently by Edelsbrunner and Welzl [EW85]) to $\Omega(n \log k)$ lower bound on the maximum number of $k$-sets. A subset $S'$ of $k$ points in $S$ is called a *$k$-set of $S$* if it can be cut off $S$ by a straight line going through two points of $S \backslash S'$. A halving line cuts $S$ into two $(n-2)/2$-sets. Erdős et al. [ELSS73] considered several structural properties of geometric *$k$-graphs induced by $S$*, denoted $G_k(S)$. The vertices of $G_k(S)$ are the points of $S$, and the edges are the directed straight line segments $\overrightarrow{pq}$ such that the directed line through $p$ and $q$ has exactly $k$ points of $S$ to its right, i.e. slices a $k$-set from $S$. Obviously, $G_{n-k-2}(S)$ is equal to $G_k(S)$ with the direction of all edges reversed; hence it suffices to consider $k$-graphs only for $k \leq (n-2)/2$. If $n$ is even, $H(S) = G_{(n-2)/2}(S)$ is the graph of *halving edges* of $S$. Clearly, each edge of $H(S)$ occurs in both directions, so the graph can be considered undirected.

Unaware of Erdős' lower bound, Edelsbrunner and Welzl [EW85] gave a construction of a set of $n$ points with $\lfloor \frac{1}{2} n \log_2(2n/3) \rfloor$ halving lines. This lower bound is sharp for all even $n \leq 14$ and it was tempting to conjecture that it is exact for all even $n$ (see Table 1). However, Tóth [Tót00] recently presented a construction of a planar set of $n$ points with $n2^{\Omega(\sqrt{\log n})}$ halving lines, refuting this conjecture. This was the first essential improvement of the lower bound since the problem was first posed.


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| $n$ | $\lfloor \frac{1}{2} n \log_2(2n/3) \rfloor$ | $h(n)$ | source |
|---|---|---|---|
| 2 | 1 | 1 | trivial |
| 4 | 3 | 3 | trivial |
| 6 | 6 | 6 | trivial |
| 8 | 9 | 9 | easily obtainable |
| 10 | 13 | 13 | [Epp92], [Fel97], [Stö84] |
| 12 | 18 | 18 | [AAHP+98] and this paper |
| 14 | 22 | 22 | this paper |
| 16 | 27 | $\geq 27$ | [Epp92] (heuristic computer search) |
| 18 | 32 | $\geq 32$ | [Epp92] (heuristic computer search) |

Table 1: Values of $h(n)$ for $n \leq 18$.

In the dual setting, the problem naturally generalizes to pseudoline arrangements (sacrificing the straightness, but preserving all combinatorial properties). An *arrangement of $n$ pseudolines* is a finite collection of $n$ simple closed curves in the projective plane not all passing through the same point, with the property that every pair intersects exactly once. An arrangement is said to be *simple* if all the $\binom{n}{2}$ intersection points are distinct. Here we are interested in bounding the maximum possible number of intersection points having exactly $(n-2)/2$ pseudolines above them. An unpublished constructive lower bound $n2^{\Omega(\sqrt{\log n})}$ on this quantity is known [KPP82]. The arrangement of pseudolines produced by this construction is not known to be realizable in the plane; however, it matches the lower bound of Tóth for the plane. For an extended treatment of $k$-sets, $k$-levels and related notions, see a survey of Andrzejak and Welzl [AW97] (where the above construction [KPP82] is described).

Meanwhile, the best known upper bounds are much larger, leaving a fairly big gap. In the very first paper on the subject, Lovász [Lov71] proved that a set of $n$ points in the plane has at most $O(n^{3/2})$ halving lines. Erdős et al. [ELSS73] generalized his result to $O(nk^{1/2})$ for the number of $k$-sets. They also conjectured that this upper bound is far from the true value. Nevertheless, this remained the best known upper bound until 1989, when Pack, Steiger, and Szemerédi [PSS92] (see also a preliminary version [PSS89]) slightly improved it to $O(nk^{1/2}/\log^* n)$. Recently, Dey [Dey98] made the first significant improvement, reducing the upper bound to $O(nk^{1/3})$. His proof uses the notion of "convex chains" [AACS98] to show that the number of pairs of crossing edges of a halving lines graph cannot exceed $O(n^2)$. Using Székely's probabilistic technique [Szé97] (an application of the crossing lemma of Ajtai et al. [ACNS82, Lei83], in disguise) any graph with $n$ vertices and a crossing number $O(n^2)$ has at most $O(n^{4/3})$ edges; thus any halving lines graph has at most $O(n^{4/3})$ edges. The proof of the $O(nk^{1/3})$ upper bound on the number of $k$-sets is lifted from the above upper bound for halving lines using the fact that the number of $(< k)$-sets is at most $n(k-1)$, due to Alon and Győri [AG86]. A different proof of the bound $O(n^{4/3})$ was given by Andrzejak et al. [AAHP+98] by relating the crossing number to the degrees of vertices of $G_k$. Their result implies the exact value of $h(12)$.

Dey's original argument has been extended to pseudoline arrangements [TT97]; thus the current bounds for straight line arrangements match the bounds for pseudoline arrangements. Essentially dual to pseudolines are pseudoconfigurations (or generalized configurations): a *pseudoconfiguration* is a finite set of points in the projective plane, together with a pseudoline joining each pair, the pseudolines forming an arrangement. A pseudoconfiguration is said to be *simple* if the corresponding arrangement is simple. Let $\hat{h}(n)$ be the maximum number of halving pseudolines over all simple pseudoconfigurations of size $n$, for even $n$. Clearly, $\hat{h}(n) \geq h(n)$. It is open whether all pseudoconfigurations maximizing the number of halving pseudolines are realizable as planar point sets. We show that this is true for even $n \leq 12$. (For $n \leq 8$, this claim holds vacuously, because all pseudoconfigurations of size less than 9 are realizable [GP80].)

We obtain the exact value of $\hat{h}(12)$ and show that the pseudoconfiguration maximizing the number of halving pseudolines is *unique* up to isomorphism. Furthermore, it is realizable, giving another proof of $h(12) = 18$. In Section 3 we show some structural properties of halving graphs; these properties dramatically reduce the computational effort needed to compute the exact value of $\hat{h}(n)$ for larger $n$.

We transform the geometric problem into the combinatorial setting of counterclockwise systems (CC-systems) [Knu92]. The *counterclockwise relation* $pqr$ says that points $p$, $q$, $r$ are encountered in this order when the circle through $p$, $q$, $r$ is traversed counterclockwise starting from point $p$. A *CC-system* is a set of ordered triples of points that combinatorially encode (in some precise sense) the orientation properties of a point configuration. Section 5 discusses different equivalence classes and our enumeration results obtained as a byproduct of the search for sets with many halving lines. In particular, we enumerate all topologically distinct, simple arrangements of 10 pseudolines with a marked cell. This implies the enumeration of nonisomorphic CC-systems on 10 points, filling in the last two missing entries in the Knuth's table ([Knu92, page 35], see also [GO97, page 102]) for $n = 10$. Section 4 proves that $h(14) = 22$ using the main identity of Andrzejak et al. [AAHP$^+$98].

## 2 Preliminaries

Let $X$ denote a set of size $n \geq 1$. The elements of $X$ will be referred to as *points.* A *CC system on* $X$, as defined by Knuth [Knu92], is a relation on the set of ordered triples of points from $X$ such that for any three distinct points $p$, $q$, and $r$ the following axioms hold: $pqr \Rightarrow qrp$ (cycle symmetry); $pqr \Rightarrow \neg prq$ (antisymmetry); $pqr \vee prq$ (nondegeneracy); $tqr \wedge ptr \wedge pqt \Rightarrow pqr$ for any point $t \notin \{p, q, r\}$ (interiority); $tsp \wedge tsq \wedge tsr \wedge tpq \wedge tqr \Rightarrow tpr$ for any distinct points $t, s \notin \{p, q, r\}$ (transitivity). (Whenever we quantify over points, we quantify over the points in $X$.)

Let $S$ denote a CC system on $X$. A *halving pair* of $S$ is an ordered pair $pq$ of distinct points such that there are precisely $\lfloor (n-2)/2 \rfloor$ points $t$ such that $ptq$ holds. If $n$ is even, the reverse of any halving pair is also a halving pair; hence the pairs can be considered unordered. Let $H(S)$ denote the set of all halving pairs of $S$. We define the *convex hull* of $S$ to be the set of all ordered pairs $ts$ (called *convex hull edges*) such that $tsp$ holds for all $p \notin \{t, s\}$. A point $t$ is *extreme* if it appears in one of the pairs in the convex hull. An extreme point defines a linear ordering of all the other points in the set; hence it appears exactly twice among the ordered halving pairs, once as the first element and once as the second. Knuth ([Knu92, page 45]) showed that the pairs constituting the convex hull of a CC system always form a unique cycle. A point $p$ is said to *lie in the convex closure of* $S$ if either $p$ is an extreme point of $S$ or $tsp$ holds for every pair $ts$ in the convex hull of $S$.

We introduce the following geometric language that will facilitate the proofs. A pair of points will be identified with a directed *line segment.* The line segments $pq$ and $rs$ *intersect* if and only if $pqr \neq pqs$ and $prs \neq qrs$. A pair of points $pq$ also defines the directed *line* $\overrightarrow{pq}$ that separates $X \setminus \{p, q\}$ into two sets, called *semispaces.* The *right* (respectively, *left*) *semispace* of $\overrightarrow{pq}$ consists of all points $t \notin \{p, q\}$ such that $ptq$ (respectively, $pqt$) holds. A line defined by a halving pair is said to be *halving.*

Following Knuth [Knu92], define the four-point predicate $\Box pqrs = pqr \wedge qrs \wedge rsp \wedge spq$, i.e. $\Box pqrs$ means that points $p$, $q$, $r$, and $s$ are the vertices of a convex quadrilateral, enumerated in counterclockwise order. Whenever $\Box pqsr$ holds, the lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ are said to *meet* if there exists a point $t$ such that both $tqp$ and $trs$ hold.

We say that the lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ *intersect* if one of the following conditions is true:

1. $pqs \neq pqr$ or $srp \neq srq$, i.e. either one of the points $(p, q, r, s)$ is in the convex combination of the other three (in which case exactly one condition is satisfied), or the line segments $pq$ and $rs$ intersect (in which case both conditions are satisfied).
2. $\Box pqsr$ (or the mirror reflection $\Box rsqp$) holds and the lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ meet.
3. $\Box pqrs$ (or the mirror reflection $\Box srqp$) holds and the lines $\overrightarrow{pq}$ and $\overrightarrow{sr}$ meet.

Lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ are said to be *parallel* if they do not intersect. Of course, when a CC-system arises from a set of points in the plane, the above terminology agrees with the standard geometric terminology. (Indeed, it is motivated by the latter.)

## 3 The Symmetry

Given a CC system $S$ on $n$ points, consider all CC systems obtained from $S$ by adding an extreme point. Denote the set of all extensions by $\Gamma = \Gamma(S)$. For any integer $i$, let $\Gamma_i$ be the subset of $\Gamma$ consisting of systems with exactly $i$ halving pairs, and let $e_i = |\Gamma_i|$. Denote the sum of the maximum and the minimum number of halving pairs over all elements of $\Gamma$ by $\delta(\Gamma)$.

**Theorem 3.1** *For all $i$ we have $e_i = e_{\delta(\Gamma)-i}$. Moreover, if $n$ is odd, then $\delta(\Gamma) = |H(S)| + 2$.*

In particular, this theorem implies that if we have the lower bound $\hat{h}(2n) \geq \Delta$, then in order to obtain the exact value of $\hat{h}(2n)$, it suffices to extend only those systems on $2n-1$ points that have at least $(\Delta + n - 2)$ halving pairs.

Before proving Theorem 3.1 we establish several lemmas.

**Lemma 3.2** *Any pair of lines determined by distinct elements of $H(S)$ intersects in the convex closure of $S$.*

*Proof.* Let $\overrightarrow{pq}$ and $\overrightarrow{rs}$ be a pair of distinct halving lines of $S$. If either $pqs \neq pqr$ or $srp \neq srq$ (i.e. either one of the points $(p, q, r, s)$ is in the convex combination of the other three, or the segments $pq$ and $rs$ intersect), then the lemma trivially holds. Otherwise, we have one of the following cases: $\Box pqrs$, $\Box srqp$, $\Box pqsr$, or $\Box rsqp$. We can assume without loss of generality that $\Box pqsr$ is true. Suppose that $\overrightarrow{pq}$ and $\overrightarrow{rs}$ do not intersect. Then there is no point $t \notin \{p, q, r, s\}$, such that $tqp \wedge trs$. Let $(pq)^-$ and $(pq)^+$ be the sets of points to the left and to the right of $pq$, respectively. Similarly define $(rs)^-$ and $(rs)^+$. Then by our assumption, $(pq)^+ \cap (rs)^- = \emptyset$. Since $pq$ and $rs$ are halving, we have $|(rs)^-| = |(pq)^-| = |(pq)^- \cap (rs)^+| + |(rs)^-| + 2$. This implies $|(pq)^- \cap (rs)^+| < 0$, a contradiction. □

For even $n$, a stronger version of Lemma 3.2 will be useful. Let $H_{n/2-2}(S)$ denote the set of ordered pairs $pq$ of distinct points such that there are precisely $n/2 - 2$ points $t \notin \{p, q\}$ such that $ptq$ holds in $S$. Let $\Lambda(S) = H(S) \cup H_{n/2-2}(S)$.

**Lemma 3.3** *If $n$ is even, then any pair $\overrightarrow{pq}$ and $\overrightarrow{rs}$ of lines determined by distinct elements of $\Lambda(S)$ intersects in the convex closure of $S$, with the only exception when $pq, rs \in H_{n/2-2}(S)$ and $\Box pqrs$ is true (in which case the lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ are parallel).*

*Proof.* As in the proof of Lemma 3.2, we need only consider the case when points $(p, q, r, s)$ form a convex quadrilateral. Let $(pq)^-$, $(pq)^+$, $(rs)^-$, and $(rs)^+$ be as before. The case when both $pq$ and $rs$ belong to $H(S)$ is covered by Lemma 3.2. First assume that exactly one of $pq$, $rs$ belongs to $H(S)$. Without loss of generality, let this segment be $pq$, and assume that $\Box pqsr$ is true. Then we have $|(rs)^-| = n/2$. On the other hand, $|(pq)^-| = |(pq)^- \cap (rs)^+| + |(rs)^-| + 2 = (n-2)/2$. Together this implies $|(pq)^- \cap (rs)^+| < 0$, which is impossible. Similar contradictions follow in the cases when either $\Box srqp$ or $\Box pqrs$ or $\Box rsqp$ is true.

Now assume that both $pq$ and $rs$ belong to $H_{n/2-2}(S)$. **Case 1**, $\Box pqsr$. We have $|(rs)^-| = n/2$ and $|(pq)^- \cap (rs)^+| + |(rs)^-| + 2 = n/2$, yielding $|(pq)^- \cap (rs)^+| < 0$ as before. **Case 2**, $\Box rsqp$. This is a mirror reflection of Case 1: we have $|(rs)^+| = n/2 - 2$ and $|(pq)^+ \cap (rs)^-| + |(rs)^+| + 2 = n/2 - 2$, and thus $|(pq)^+ \cap (rs)^-| < 0$, which is again impossible. **Case 3**, $\Box srqp$. The same argument gives $|(rs)^-| = n/2$ and $|(pq)^+ \cap (rs)^+| + |(rs)^-| + 2 = n/2 - 2$, implying $|(pq)^+ \cap (rs)^+| < 0$. **Case 4**, $\Box pqrs$ (a mirror reflection of Case 3). We now have $|(pq)^- \cap (rs)^-| + |(rs)^+| + 2 = n/2$ and $|(rs)^+| = n/2 - 2$, implying $(pq)^- \cap (rs)^- = \emptyset$, but not yielding a contradiction. In this case the lines $\overrightarrow{pq}$ and $\overrightarrow{rs}$ are parallel. The smallest example is a convex quadrilateral $S$: all four edges constituting the convex hull belong to $H_{n/2-2}(S)$, and any pair of lines determined by disjoint convex hull edges is parallel. □

Let $D$ be a set of ordered pairs of distinct points of $X$. We say that two extensions in $\Gamma$ are *$D$-equivalent* if and only if they have the same orientation of every triple consisting of the $(n+1)$st point and a pair in $D$.

Let $\Pi(D)$ be the set of $D$-equivalence classes. Clearly, these partition $\Gamma$. For any $Q \in \Gamma$, let $\Pi_Q(D)$ be the equivalence class in $\Pi(D)$ containing $Q$. To simplify notation, let $\Pi$ denote $\Pi(H(S))$.

For the time being, assume that $S$ is realizable as a planar point set in general position. Without loss of generality, we label the points of $S$ by using the numbers $1, \ldots, n$. Let $L$ be a directed line not orthogonal to any direction determined by two points of $S$. Then the orthogonal projection of $S$ on $L$ determines a permutation of $1, \ldots, n$. As $L$ rotates counterclockwise about a fixed point, the permutation changes whenever $L$ passes through the direction orthogonal to that determined by a pair of points in $S$. This defines an infinite sequence of permutations in an obvious way. Following Goodman and Pollack [GP84] we call this sequence the *circular sequence associated to* $S$. Notice that the sequence always has the following properties: it is periodic with period at most $n(n-1)$; the move from each permutation to the next consists of reversing the order of one or more pairs of adjacent numbers; if the points $p$ and $q$ are switched, then every other pair is switched before $p$ and $q$ are switched again. The last property guarantees that each period breaks into two half-periods, with each switch of the first half reversed in the second; hence permutations that are a half-period apart are the reversals of each other. An infinite sequence of permutations of $1, \ldots, n$ satisfying the above properties is called an *allowable sequence* [GP84]. If an allowable sequence is induced by a realizable CC system, then it is said to be *realizable.* An allowable sequence $\Sigma$ associated to $S$ encodes many properties of $S$ that have a sensible geometric interpretation. For example, a point $p$ is an extreme point of $S$ if and only if $p$ is the first (and therefore the last) element in some permutation of $\Sigma$. The line $\overrightarrow{pq}$ is parallel to $\overrightarrow{rs}$ in $S$ if and only if $p$ and $q$ are switched in the same move as $r$ and $s$. The relation $pqr$ holds if and only of $pq$ is reversed before $pr$ (within the half-period of $\Sigma$ containing both ordered pairs). Triples $pqr$ and $pqs$ have different orientation if and only if when $p$ and $q$ switch, $r$ and $s$ are on opposite sides of $pq$ (or $qp$) in the corresponding permutation. Notice that permutations in $\Sigma$ correspond to extensions in $\Gamma$ in an obvious way: the $(n+1)$-st point in any extension is extreme; hence any extension is uniquely determined by a linear ordering (permutation) of the points in $S$. Each move of $\Sigma$ from one permutation to the next consists of one or more switches; a switch $pq$ corresponds to inverting the orientation of the triple containing $p$, $q$, and the $(n+1)$-st point.

**Lemma 3.4** $|\Pi| = 2|H(S)|$.

*Proof.* Consider the partition of $\Sigma$ into subsequences obtained by grouping adjacent permutations such that the switches that take us from each one to the next do not involve halving pairs of $S$. We call this sequence of subsequences a *sequence induced by* $H(S)$. According to Lemma 3.2, no two halving lines of $S$ are parallel. Consequently, the period of the induced sequence is precisely $2|H(S)|$. Notice that each element of the induced sequence uniquely corresponds to some equivalence class in $\Pi$ (by associating extensions in $\Gamma$ with permutations of $1, \cdots, n$). Clearly, extensions that belong to the same class (equivalently, permutations that belong to the same subsequence) are $H(S)$-equivalent. The lemma follows. □

If $n$ is odd, let the *degree* of $\Pi_Q$ for $Q \in \Gamma$, denoted by $\deg(\Pi_Q)$, be the number of halving pairs of $S$ that remain halving in $Q$. Let $\overline{Q}$ denote the extension that is identical to $Q$ except that it inverts the orientation of every triple containing the $(n+1)$-st point. We shall call $\overline{Q}$ the *reversal* of $Q$. Notice that the permutation associated with $\overline{Q}$ is the reversal of the permutation associated with $Q$; hence corresponding to each class is the *antipodal* class containing the reversals of all extensions in the class.

**Proposition 3.5** *If $n$ is odd, then for any pair $\Pi_Q$ and $\Pi_{\overline{Q}}$ of antipodal classes in $\Pi$,*

$$\deg(\Pi_Q) + \deg(\Pi_{\overline{Q}}) = |H(S)|.$$

*Proof.* Notice that the $(n+1)$-st point in $Q$ and the $(n+1)$-st point in $\overline{Q}$ lie on opposite sides of any pair in $H(S)$; hence $H(S)$ can be split into two subsets with cardinalities $\deg(\Pi_Q)$ and $\deg(\Pi_{\overline{Q}})$ corresponding to subsets of pairs that remain halving in $Q$ and $\overline{Q}$, respectively. It follows that $\deg(\Pi_Q) + \deg(\Pi_{\overline{Q}}) = |H(S)|$. □

For even $n$, the above definition of degree is not interesting, because all halving pairs of $S$ remain halving in any extension $Q$. However, some elements of $H_{n/2-2}(S)$ may now be halving in $Q$. Therefore, we need to consider $\Lambda(S)$-equivalence of extensions instead of $H(S)$-equivalence. (Recall that $\Lambda(S) = H(S) \cup H_{n/2-2}(S)$.) The *degree* of $\Pi_Q = \Pi_Q(\Lambda(S))$, denoted by $\deg(\Pi_Q)$ as before, is now defined as the number of pairs in $H_{n/2-2}(S)$ that are halving in $Q$. The situation here is slightly more complicated: by Lemma 3.3, a pair of lines determined by elements of $H_{n/2-2}(S)$ may be parallel. Let $pq, rs \in H_{n/2-2}(S)$ be such elements; by Lemma 3.3, $\Box pqrs$ is true, and $\overrightarrow{pq}$, $\overrightarrow{rs}$ are parallel. Consider extensions $Q$ and $\overline{Q}$ that are the same except for the orientation of all triples containing the $(n+1)$-st point and *not* containing $\{p,q\}$ or $\{r,s\}$. We call such pair of reversals *special*. We will also say that $pq$ and $rs$ *define* $(Q,\overline{Q})$. Notice that $Q$ and $\overline{Q}$ do not correspond to permutations in the sequence induced by $\Lambda(S)$; they fall on the switches that simultaneously reverse $(p,q)$ and $(r,s)$.

**Proposition 3.6** *If $n$ is even, then for any pair $\Pi_Q = \Pi_Q(\Lambda(S))$ and $\Pi_{\overline{Q}} = \Pi_{\overline{Q}}(\Lambda(S))$ of antipodal classes in $\Pi(\Lambda)$*

$$\deg(\Pi_Q) + \deg(\Pi_{\overline{Q}}) = |H_{n/2-2}(S)| - \Delta,$$

*where $\Delta = 2$ if the pair $(Q,\overline{Q})$ is special, and $\Delta = 0$ otherwise.*

*Proof.* The $(n+1)$-st point in $Q$ and the $(n+1)$-st point in $\overline{Q}$ lie on opposite sides of any segment in $H_{n/2-2}(S)$, unless the pair $(Q,\overline{Q})$ is special, in which case the points lie on opposite sides of any segment in $H_{n/2-2}(S)$ other than the two segments $pq$ and $rs$ that define $(Q,\overline{Q})$. Furthermore, $\Box pqrs$ holds, and the points lie to the left of both $pq$ and $rs$. Therefore, neither $pq$ nor $rs$ is a halving pair of $Q$ or $\overline{Q}$. The proposition follows. $\Box$

We are now ready to prove Theorem 3.1. In the sequel, let $\chi(\Gamma) = \delta(\Gamma)/2$.

*Proof of Theorem 3.1.*

**Case 1**, $n$ is odd. Consider an extension $Q \in \Gamma$. The $(n+1)$-st point of $Q$ is extreme, and thus defines a linear ordering of the points in $S$, giving rise to exactly one new halving pair. All the other halving pairs of $Q$ belong to $H(S)$; hence $|H(Q)| = \deg(\Pi_Q) + 1$. By Proposition 3.5, $\overline{Q}$ preserves $|H(S)| - \deg(\Pi_Q)$ halving pairs of $S$. Hence $Q$ contributes one to $e_{\deg(\Pi_Q)+1}$, while its reversal $\overline{Q}$ contributes one to $e_{|H(S)|-\deg(\Pi_Q)+1}$. This results in a symmetry centered at $\chi(\Gamma) = |H(S)|/2 + 1$.

**Case 2**, $n$ is even. Let $Q$ be an extension of $S$ as before. The $(n+1)$-st point of $Q$ is extreme, and since $(n+1)$ is odd, it gives rise to exactly two new halving pairs. As remarked earlier, all halving pairs of $S$ remain halving in $Q$; hence we only need to count the pairs in $H_{n/2-2}(S)$, which may be halving in $Q$. These observations imply that $Q$ contributes one to $e_{|H(S)|+\deg(\Pi_Q)+2}$, while By Proposition 3.6, $\overline{Q}$ contributes one to $e_{|H(S)|+|H_{n/2-2}(S)|-\deg(\Pi_Q)+2}$ if the pair $(Q,\overline{Q})$ is not special; otherwise $\overline{Q}$ contributes one to $e_{|H(S)|+|H_{n/2-2}(S)|-\deg(\Pi_Q)}$. This leads to a symmetry similar to that of Case 1. However, since pairs of lines determined by elements of $H_{n/2-2}(S)$ may be parallel, the sequence of permutations induced by $\Lambda(S)$ may have more than one switch within the same move. Therefore, the period of this induced sequence is not necessarily fixed, and the center of the symmetry does not depend only on the number of halving pairs in $S$. $\Box$

Theorem 3.1 gives a considerable restriction on the extension process sufficient to obtain the value of $\hat{h}(n)$ for small $n$. Let an $(n,k)$-*configuration* be a CC system on $n$ points with $k$ halving pairs, and let $f(n,k)$ denote the largest integer such that any $(n, \geq k)$-configuration is an extension of an $(n-1, \geq f(n,k))$-configuration. If $n$ is even, the center of symmetry $\chi(\Gamma)$ is $\frac{f(n,k)}{2} + 1$. On the other hand, we certainly have $\chi(\Gamma) \geq \frac{n/2+k}{2}$. Thus $f(n,k) \geq n/2 + k - 2$. In particular, in order to obtain all $(12, \geq 18)$-configurations, it is sufficient to extend only $(11, \geq 22)$-configurations. (The maximum number of halving pairs over all systems of 11 points is 24, while the minimum is 11.)

We can further bound the extension process by considering two-point *restrictions* of $S$.

| Number of halving lines | Total number of extensions | Number of nonisomorphic extensions | Ratio |
|---|---|---|---|
| 5 | 2,247,826 | 517,423 | 0.23 |
| 6 | 10,596,609 | 2,584,235 | 0.24 |
| 7 | 19,204,602 | 4,865,400 | 0.25 |
| 8 | 16,482,171 | 4,290,426 | 0.26 |
| 9 | 6,578,464 | 1,757,011 | 0.27 |
| 10 | 1,021,892 | 283,580 | 0.28 |
| 11 | 73,972 | 21,389 | 0.29 |
| 12 | 2,326 | 713 | 0.31 |
| 13 | 14 | 5 | 0.36 |

Table 2: Statistics on the number of CC systems on 10 points (obtained by extending CC systems on 9 points) according to the number of halving pairs. The inverse of the ratio indicates the average number of extreme points that systems with a corresponding number of halving pairs tend to have; thus systems with at least 12 halving pairs tend to have three extreme points on average.

**Lemma 3.7** *Any $(2n, k)$-configuration with $m$ extreme points is a two-point extension of a $(2n-2, \geq \lceil 2\,(k/m - 2/(m-1)) \rceil)$-configuration.*

*Proof.* Let $S$ be a $(2n, k)$-configuration with $m$ extreme points. Consider the family of $\binom{m}{2}$ restrictions obtained from $S$ by removing a pair of extreme points. Call a pair $pq \in H(S)$ *good* for a restriction $T$ if it remains halving in $T$. In this case $T$ is said to *preserve* $pq$. Clearly, $pq$ has at least one extreme point on each of its sides, since any induced subsystem of a CC system (and in particular the one induced by points in a semispace of $pq$ together with $p$ and $q$) is also a CC system; and any CC system has at least three extreme points.

If a halving pair $pq$ does not involve an extreme point, it will be good for at least $m-1$ restrictions obtained by removing a pair of extreme points that lie on opposite sides of $pq$. Notice that there are at least $k-m$ such $pq$. Similarly, a halving pair one (or both) of whose elements are extreme, will be good for at least $m-3$ restrictions. By the averaging argument, there must exist a restriction preserving at least

$$\left\lceil \frac{(k-m)(m-1)+m(m-3)}{\binom{m}{2}} \right\rceil = \left\lceil 2\left(\frac{k}{m} - \frac{2}{m-1}\right) \right\rceil$$

halving pairs. □

In the special case when $S$ has three extreme points, any halving pair can contain at most one of them. Evidently, any halving pair that does contain an extreme point, will be good for precisely one restriction. Any other halving pair will be good for precisely two restrictions. Therefore, there exists a restriction of $S$ preserving at least $\lceil 2k/3 \rceil - 1$ halving pairs.

**Corollary.** Any $(12, \geq 18)$-configuration is an extension of a $(10, \geq 11)$-configuration. Therefore, only about *one tenth of a percent* of all CC systems on 10 points need to be extended (see Table 2).

# 4 $h(14) = 22$

Let $S$ be a set of $n$ points in general position in the plane, where $n$ is even, and let $H(S)$ be a geometric graph of halving edges of $S$. We say that $S$ is an *$(n, k)$-set* if it contains $k$ halving edges, i.e. an $(n, k)$-set is an $(n, k)$-configuration realizable as an actual set of points in the plane.

The set of halving segments of a planar point set is completely characterized by the Lovász crossing lemma. It was introduced by Lovász in [Lov71] and has been a major technique for proving upper bounds for the related problems ever since. The lemma appears in different disguised forms, as in [Dey98, AW98, ELSS73, Lov71].

**Lemma 4.1** (Lovász) *Take any point $p$ in $S$ and a line $l$ going through $p$ and missing every other point of $S$. Call the side of $l$ that contains more (less) points the* larger *(*smaller*)* side *of $l$. These are uniquely defined, since there are $n-1$ points on both sides of $l$.*

1. *There is exactly one more halving edge emanating from $p$ into the larger side of $l$ than into the smaller side of $l$.*
2. *For any pair of halving edges emanating from $p$ into one side of $l$, there must exist a halving edge emanating from $p$ into the other side of $l$ with an intermediate slope. Thus, $p$ is adjacent to an odd number of halving edges.*

Andrzejak et al. [AAHP+98] showed the following identity.

**Lemma 4.2** (Andrzejak et al.)

$$C + \sum_{p \in S} \binom{(\deg p + 1)/2}{2} = \binom{n/2}{2},$$

*where* $\deg p$ *is the number of halving edges incident to $p$ and $C$ is the number of crossing pairs of halving edges.*

Let $(d_1, d_2, \ldots, d_n)$ denote the non-decreasing sequence of degrees of vertices of $H(S)$, and let $n_i$ denote the number of vertices of degree $i$ in $H(S)$. Note that $i$ must be odd. The following inequality is immediately implied by Lemma 4.2:

$$\sum_i \binom{(i+1)/2}{2} n_i \leq \binom{n/2}{2}.$$

We also have $\sum_i i\, n_i = 2h(n)$, because every edge contributes 2 to the sum of all degrees; and just counting vertices gives $\sum_i n_i = n$. Since there are at least three extreme points and every extreme point has degree one, we also have $n_1 \geq 3$.

If we want to show the upper bound $h(n) < \Delta$ for some $n$ and $\Delta$, we must prove the nonexistence of an $(n, \geq \Delta)$-set. The properties of the coefficients in the sum $\sum_i \binom{(i+1)/2}{2} n_i$ imply that if there does not exist an $(n, \geq \Delta)$-set with $n_1$ extreme points, then there does not exist an $(n, \geq \Delta)$-set with more than $n_1$ extreme points. It is easy to see that the sequence minimizing the sum $\sum_i \binom{(i+1)/2}{2} n_i$ for given $n$ and $\Delta$, is

$$(1, 1, 1, \underbrace{q, \cdots, q}_{n_q}, \underbrace{p, \cdots, p}_{n_p}),$$

where $n_p = n_{2\lceil d/(n-3) \rceil + 1} = d \bmod (n-3)$, $n_q = n_{2\lfloor d/(n-3) \rfloor + 1} = n - n_p - 3$, and $d = (2h - n)/2$. These observations imply the following upper bounds for small values of $n$:

- $h(12) \leq 18$; In conjunction with the known lower bound ([Epp92]), this implies the equality $h(12) = 18$. The only degree sequence that a $(12, 18)$-set can have is $(1, 1, 1, 3, 3, 3, 3, 3, 3, 5, 5, 5)$.
- $h(14) \leq 23$; Together with the known lower bound ([Epp92]), we have $22 \leq h(14) \leq 23$. Moreover, $h(14) = 23$ if and only if there exists a planar set of 14 points with the degree sequence

$$(1, 1, 1, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5).$$

- $27 \leq h(16) \leq 28$; Furthermore, $h(16) = 28$ if and only if there exists a planar set of 16 points with one of three degree sequences:

$$(1, 1, 1, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5),$$
$$(1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 7),$$
$$(1, 1, 1, 1, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5).$$

Throughout the rest of the section, let $S$ be a hypothetical $(14, 23)$-set. From the identity in Lemma 4.2, the crossing number of $H(S)$ must be 0. In other words, if $S$ exists, then $H(S)$ is planar. The degree sequence of $H(S)$ implies that $S$ must have exactly three extreme points.

**Definition 4.3** *Let $p$ be an extreme point of $S$, and let $q$ be the only neighbor of $p$ in $H(S)$. Denote the other two neighbors of $q$ by $r$ and $s$ in such a way that $qrs$ holds; hence $r$ is to the right and $s$ is to the left of $\overrightarrow{pq}$. Call the region bounded by rays $\overrightarrow{qr}$ and $\overrightarrow{qs}$ (i.e. the region consisting of all points $t$ such that $tsq$ and $tqr$ holds) the* wedge *of $q$, denoted $\angle sqr$, and $s$ and $r$ the* points defining *it.*

**Proposition 4.4** *The wedge of any extreme point of $S$ is empty (i.e. does not contain any points of $S$).*

*Proof.* Let $p$ be an extreme point of $S$, and let $q$, $r$, $s$ be as in Definition 4.3. The ray $\overrightarrow{pq}$ splits $\angle sqr$ into a *left wedge* and a *right wedge*. The number of points that lie to the left of $\overrightarrow{pq}$ and $\overrightarrow{qs}$ is the same; hence the left wedge of $q$ is empty. By a symmetric argument, the right wedge is also empty. □

**Proposition 4.5** *All six points defining the wedges of $S$ are distinct.*

*Proof.* Let $p$ and $w$ be two of the three extreme points of $S$, and let $q$ and $x$ be their respective neighbors in $H(S)$.

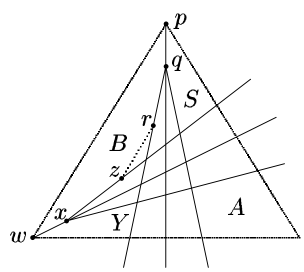


Figure 1.

Let $r$ and $z$ be the points defining the right wedge of $q$ and the left wedge of $x$, respectively; see Figure 1. The segments $qr$ and $xz$ cannot cross by the assumption of planarity. We will show that $r$ and $z$ cannot coincide. Assume the contrary. There are nine unlabeled points remaining. Since $pq$ is halving, region $Y$ must contain exactly three points; similarly, since $wx$ is halving, region $S$ must contain exactly three points. Hence the remaining three points must lie in region $A$, which contradicts the fact that the segment containing $x$ and the point defining the right wedge of $x$ is halving. (Notice that region $B$ bounded by points $p$, $q$, $r$, $z$, $x$, $w$, cannot contain any points of $S$. Suppose that it does, and let $i$ be the point with the shortest orthogonal segment connecting it to $pq$, among all the points in $B$. Consider the line through $i$ parallel to $pq$ (we may assume that it misses all the other points of $S$); evidently, there can be no halving segment incident to $i$ emanating into the smaller side of this this line. The Lovász crossing lemma applied to $i$ says that $i$ must have degree 1, which is impossible. Therefore $i$ cannot belong to $B$, but then the same argument can be applied to the point with the shortest orthogonal segment among the points remaining in $B$ when $i$ is removed, and so on, until we show that $B$ is empty.) □

**Proposition 4.6** *The edges defining a wedge of $S$ cannot cross any other wedges of $S$.*

*Proof.* Let $a$, $p$, $w$ be the extreme points of $S$, and let $b$, $q$, $x$ be their respective neighbors in $H(S)$. Denote the six points defining the wedges by $c$, $d$, $r$, $s$, $y$, $z$, as in Figure 2. Let $i$ and $j$ be the last two "free" points, and label the six non-empty regions in Figure 2 with $C$, $D$, $R$, $S$, $Y$, and $Z$, depending on which of the six wedge points are on the boundary. We shall assume that the edge defining the right wedge of $x$ crosses at least one of $\angle dbc$, $\angle sqr$, and try to obtain a contradiction.

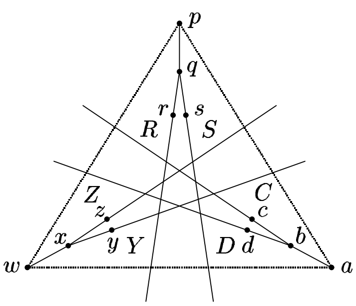


Figure 2.

Call a segment defining a wedge *bad*, if it crosses any other wedge of $S$. **Case 1**, $y$ lies in region $C$, so that $xy$ crosses both $\angle dbc$ and $\angle sqr$ (hence $xy$ is bad). Then none of the other segments defining the wedges is bad, and the free points $i$ and $j$ lie in regions $Y$ and $Z$, respectively. By the Lovász crossing lemma, $d$ must have a halving neighbor to the left of $\overrightarrow{bd}$. The only possibility is a point $i$ in $Y$, but then consider the line through $i$ parallel to $qr$ (and without loss of generality missing all the other points of $S$): there is no point to the left of this line that can be incident to $i$ without contradicting the planarity of $H(S)$; hence by the Lovász crossing lemma, $i$ must have degree 1, which is impossible, because $H(S)$ has only three points with degree 1. **Case 2**, $y$ lies in region $D$ (i.e. $xy$ crosses only $\angle sqr$). Neither $qr$ nor $qs$ is bad; otherwise they would cross $xy$. The segments $bc$ and $bd$ cannot both be bad, because this would imply that $wx$ has seven points to its left, contradicting the fact that it is halving. Assume that only $bc$ is bad; then the remaining points $i$ and $j$ must be in regions $Y$ and $C$, but then the Lovász crossing lemma would say that $i$ and $j$ have degree 1, which is impossible. Using the same argument, we can easily verify that $bd$ is not bad. Then $i$ and $j$ must lie in regions $Y$ and $R$, respectively. Consider the line going through $i$ parallel to $\overrightarrow{xy}$ (we may assume that it misses all the other points of $S$). By the Lovász crossing lemma, $i$ must have at least one halving neighbor on the larger side of this line, which contradicts the fact that $H(S)$ is planar, because neither $w$ nor $x$ can be a neighbor of $i$. Therefore, $y$ must lie in region $Y$. A symmetric argument shows that any other edge defining a wedge of $S$ cannot be bad. □

**Lemma 4.7** *The unbounded face of $H(S)$ contains all the points of $S$.*

*Proof.* Label the extreme points of $S$ and the points defining the wedges of $S$ as in the proof of Proposition 4.6. Recall that all three wedges of $S$ are empty, the six points defining these wedges are distinct, and the halving segments defining the wedges cannot cross any other wedges of $S$. The three wedges account for 12 of the 14 points. The last two points must be separated by each wedge, so the triangular region bounded by the wedges (if it exists) is also empty.

First we will show that for $H(S)$ to be planar, the segments $rs$, $cd$, $yz$ must be halving. Indeed, by the Lovász crossing lemma, there must exist a halving edge emanating from $r$ to the left of $\overrightarrow{qr}$, and a halving edge emanating from $s$ to the right of $\overrightarrow{qs}$. Since the wedge of $q$ is empty, these edges would have to cross (unless they coincide), contradicting the fact that $H(S)$ is planar; hence $rs$ must be halving. Similarly, both $cd$ and $yz$ must be halving.

The last two points $i$ and $j$ must be placed in six regions bounded by the wedges and the segments $rz$, $cs$ and $dy$; see Figure 3. Label these six regions with $C$, $D$, $R$, $S$, $Y$, and $Z$, depending on which of the six wedge points are on the boundary. Points $i$ and $j$ must be separated by each wedge; hence they can be placed either in regions $C$ and $Z$, or $R$ and $D$, or $Y$ and $S$. If a free point, say $i$, were placed anywhere else, it would violate the Lovász crossing lemma, because we would always be able to find a line through $i$ (missing all the other points of $S$) such that there can be no halving edges emanating from $i$ to the smaller side of the line; then the Lovász crossing lemma would imply that $i$ has degree 1, which is impossible.

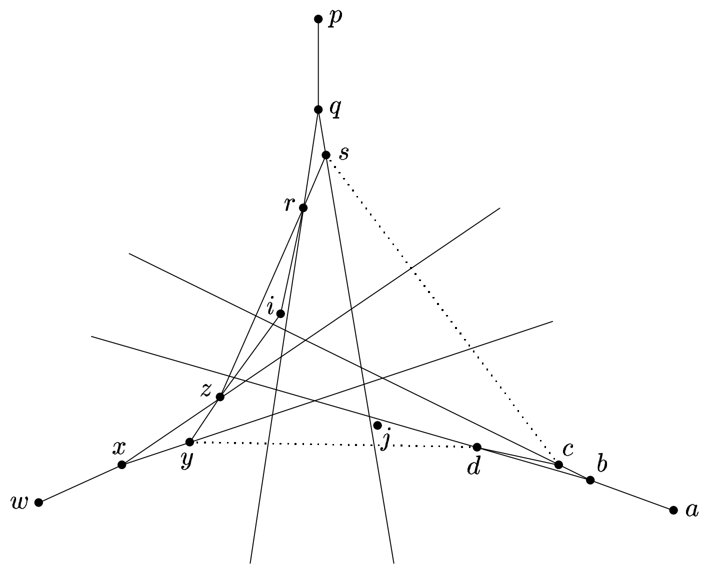


Figure 3.

Clearly, if none of $rz$, $yd$, $cs$ is a halving edge, we are done. Regardless of where we place the last two points, the unbounded face will contain all the points of $S$. Consider the case when one of the segments $rz$, $yd$, $cs$ is halving. It is easy to see that at most one can be halving. Without loss of generality we may assume that this segment is $rz$. We show below that in this case $i$ and $j$ must be placed in regions $S$ and $Y$, respectively. Suppose that the points are in regions $R$ and $D$, instead of $S$ and $Y$. Then the Lovász crossing lemma implies that the segments $ri$ and $zi$ are halving. Consider the line through $i$ parallel to $rz$; it is easy to see that $ri$ and $zi$ are the only halving segments incident to $i$ emanating into the larger side of this line (the side containing points $p$, $q$, $r$, $s$, $w$, $x$, $y$, $z$). Then the Lovász crossing lemma says that $i$ must have degree 3, which contradicts the fact that $i$ has degree 5, implied by the degree sequence of $H(S)$. This completes the proof of Lemma 4.7. □

**Theorem 4.8** $\boldsymbol{h(14) = 22}$

*Proof.* Recall that $h(14) = 23$ if and only if there exists a $(14, 23)$-set $S$ such that $H(S)$ is planar. By Euler's formula, the number $f$ of faces of $H(S)$ is

$$f = |H(S)| - n + 2 = 11.$$

We can also count the faces of $H(S)$ according to their number of sides. Let a $k$-face be a face bounded by $k$ edges, where edges that border the same face on both sides are counted twice. Let $f_k$ be the number of $k$-faces. We have $\sum_i i\, f_i = 2|H(S)| = 46$, and $\sum_i f_i = f = 11$. According to Lemma 4.7, the unbounded face is an $(n + 3)$-face. Hence

$$\sum_{i=3}^{n+2} i\, f_i \leq 2|H(S)| - (n + 3) = 29.$$

One the other hand, since each of the remaining $f - 1$ bounded faces must have at least 3 sides each, we have

$$\sum_{i=3}^{n+2} i\, f_i \geq 3(f - 1) = 30,$$

a desired contradiction. □

## 5 Enumeration and Computations

Using Knuth's notations [Knu92], let $C_n$ denote the number of nonisomorphic CC-systems on $n$ points, and let $D_n$ denote the number of topologically distinct, simple arrangements of $n$ pseudolines with a marked cell (as discussed by Goodman and Pollack [GP84]). Equivalently, $D_n$ is the number of nonisomorphic uniform acyclic oriented matroids of rank 3 on $n$ elements [Knu92].

We obtain an enumeration of isomorphism classes of marked arrangements of 10 pseudolines, which gives the value of $D_{10}$. In particular, $D_{10} = 14,320,182$. This is an additional value for the table of Knuth ([Knu92, page 35], see also [GO97, page 102]). Having completed $D_{10}$, we easily obtained $C_{10} = 2D_{10} - R_{10}$, where $R_{10}$ is the number of non-isomorphic achiral CC-systems on 10 points. By achiral we mean isomorphic to their mirror image, that is isomorphic to the system obtained by inverting the orientation of all triples. We have $R_{10} = 13,103$; hence $C_{10} = 28,627,261$.

It should also be mentioned that Felsner [Fel97] showed that the number of topologically different simple arrangements of 10 pseudolines is $B_{10} = 18,410,581,880$.

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[1] **nauty** stands for "**no** **aut**omorphisms, **y**es?"

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