# The Ramsey Multiplicity of $K_4$

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**Abstract**

With the help of computer algorithms, we improve the lower bound on the Ramsey multiplicity of $K_4$, and thus show that the exact value of it is equal to $9$.

The Ramsey multiplicity $M(G)$ of a graph $G$ is defined as the smallest number of monochromatic copies of $G$ in any two-coloring of edges of $K_{R(G)}$, where $R(G)$ is the Ramsey number of $G$, i.e. the smallest integer $n$ such that any two-coloring of edges of $K_n$ contains monochromatic copy of $G$.

The study of Ramsey multiplicity was initiated in 1974 by Harary and Prins [3] who determined $M(G)$ for all graphs $G$ of order four or less, except for $K_4$ and $K_4 - e$. The value of $M(K_4 - e)$ was later determined by Schwenk (cited in [2]). The upper bound $M(K_4) \leq 12$ was given in 1980 by Jacobson [4], and in 1988 Exoo [1] improved it by 3. The only nontrivial lower bound $M(K_4) \geq 4$ was recently presented by Olpp [7]. In this paper we improve this lower bound and thus show that $M(K_4) = 9$.

In the sequel, any two-coloring of the edges of $K_n$ containing $k$ monochromatic copies of $K_4$ is called an $(n, k)$-coloring. We say that two colorings are isomorphic if the graphs induced by the edges in the first color are isomorphic. Define $\mathcal{M}(n, k)$ to be set of all $(n, k)$-colorings. For a given $(n, k)$-coloring $C$ let $H(C)$ denote the hypergraph formed by monochromatic copies of $K_4$ in $C$. Let us define $\mathcal{M}_d(n, k)$ to be the subset of all colorings $C \in \mathcal{M}(n, k)$ such that the maximal vertex degree in $H(C)$ is equal to $d$.

Our computational approach was to generate all nonisomorphic $(18, k)$-colorings for $4 \leq k \leq 8$, by iterating an exhaustive enumeration of all possible one vertex extensions of $(n-1, k-m)$-colorings to $(n, k)$-colorings, for $m \geq 0$. Let us define $\mathcal{E}(n-1, k-m, m)$ to be the subset of all colorings from $\mathcal{M}(n, k)$ which are one vertex extensions of some coloring from $\mathcal{M}(n-1, k-m)$.

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Let $V(C)$ denote the set of vertices of coloring $C$. For each subset $W \subseteq V(C)$ let $N_3(W)$ denote the sum of the number of triangles in the first color induced by $W$ in $C$ and the number of triangles in the second color induced by $V(C) \setminus W$ in $C$. The following algorithm was used to perform the exhaustive search for all one vertex extensions $\mathcal{E}(n-1, k-m, m)$:

**Algorithm 1**

Step 1: Initialize output set $Out = \varnothing$.
Step 2: For each coloring $C$ from $\mathcal{M}(n-1, k-m)$ execute steps 3, 4, 5.
Step 3: For each subset $W \subseteq V(C)$ such that $N_3(W) = m$ execute steps 4, 5.
Step 4: Create copy $D$ of coloring $C$.
Step 5: Add a new vertex $v$ to coloring $D$. For each $w$ in $V(C)$, assign color 1 to edge $\{v, w\}$, if $w \in W$, and assign color 2 to edge $\{v, w\}$, if $w \in V(C) \setminus W$. Add this coloring to $Out$.
Step 6: Remove isomorphic copies from $Out$.

The following lemmas describe computational steps we followed in order to generate colorings of higher orders. As the initial step, we generated the set $\mathcal{M}(11, 0)$ by filtering out $(11, 0)$ colorings from all nonisomorphic graphs of order 11 (which were treated as two-colorings of $K_{11}$). The proofs of the lemmas are straightforward by considering degree sequences of all possible hypergraphs $H(C)$ in each case.

**Lemma 1**

$$\begin{aligned} \mathcal{M}(n,0) &= \mathcal{E}(n-1,0,0), \quad \textit{for } n \geq 2, \\ \mathcal{M}(n,k) &= \bigcup_{j=0}^{k-1} \mathcal{E}(n-1,j,k-j), \quad \textit{for } k \geq 1, \textit{ and } n \geq 2, \\ \mathcal{M}(16,4) \setminus \mathcal{M}_1(16,4) &= \bigcup_{j=0}^{2} \mathcal{E}(15,j,4-j). \end{aligned}$$

All the sets $\mathcal{M}(n, k)$, for $12 \leq n \leq 16$, and $0 \leq k \leq 3$ such that there is a nonempty entry for $n, k$ in Table 1, were obtained by running Algorithm 1 for the terms on the right hand side of the first two rules in Lemma 1. For example, $\mathcal{M}(16, 3)$ was obtained by extending colorings from $\mathcal{M}(15, 0), \mathcal{M}(15, 1)$ and $\mathcal{M}(15, 2)$.

The last identity in Lemma 1 describes the way of enumerating all $(16, 4)$-colorings except those whose monochromatic copies of $K_4$ are vertex disjoint (denoted by $\mathcal{M}_1(16, 4)$). Unfortunately, there is a frightfully large number of $(13, 1)$ and $(14, 2)$ colorings, and we were not able to complete the sequence of extensions $\mathcal{M}(12, 0) \rightarrow \mathcal{M}(13, 1) \rightarrow \mathcal{M}(14, 2) \rightarrow \mathcal{M}(15, 3) \rightarrow \mathcal{M}_1(16, 4)$. Instead, in order to generate $\mathcal{M}_1(16, 4)$, we used the following approach:

**Algorithm 2**

Step 1: Generate the set of all 2-colorings of order $8$ and extract from it $\mathcal{M}_1(8, 2)$.
Step 2: Generate $\mathcal{M}_1(12, 3)$ by exhaustively extending by $4$ vertices all colorings in $\mathcal{M}_1(8, 2)$.
Step 3: Generate $\mathcal{M}_1(16, 4)$ by exhaustively extending by $4$ vertices all colorings in $\mathcal{M}_1(12, 3)$.

In steps 2 and 3 exactly one new monochromatic $K_4$ is induced by $4$ new vertices. As a result of the above algorithm we obtained $468$ nonisomorphic $(16, 4)$ colorings.

The following lemma, together with Lemma 1, describes the remaining computational steps.

**Lemma 2**

$$\mathcal{M}(n,k) = \bigcup_{j=0}^{k-2} \mathcal{E}(n-1, j, k-j), \text{ for } k \geq 5, \text{ and } n \leq 19.$$

Using Algorithm 1 and Lemma 1 for $k \leq 4$, and Lemma 2 for $k \geq 5$, we were able to generate $\mathcal{M}(17, 0)$, ..., $\mathcal{M}(17, 6)$ and $\mathcal{M}(18, 0)$, ..., $\mathcal{M}(18, 8)$.

**Table 1. The number of nonisomorphic $(n, k)$-colorings.**

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 546356 | | | | | | | | |
| 12 | 1449166 | | | | | | | | |
| 13 | 1184231 | | | | | | | | |
| 14 | 130816 | 6144820 | | | | | | | |
| 15 | 640 | 50726 | 2491136 | | | | | | |
| 16 | 2 | 28 | 382 | 19806 | 888440 | | | | |
| 17 | 1 | 0 | 0 | 2 | 18 | 202 | 5757 | | |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1 presents the number of nonisomorphic $(n, k)$-colorings for all $n$ and $k$, which were enumerated during our computations. The emptiness of the sets $\mathcal{M}(18, 0)$, ..., $\mathcal{M}(18, 8)$ implies the main theorem:

**Theorem 1** $M(K_4) = 9$.

It is a natural goal to enumerate the set $\mathcal{M}(18, 9)$. Continuing our approach would require obtaining the whole set of colorings $\mathcal{M}(17, 7)$. The latter was unfeasible, and we were able to enumerate only the set $\mathcal{M}(17, 6)$.

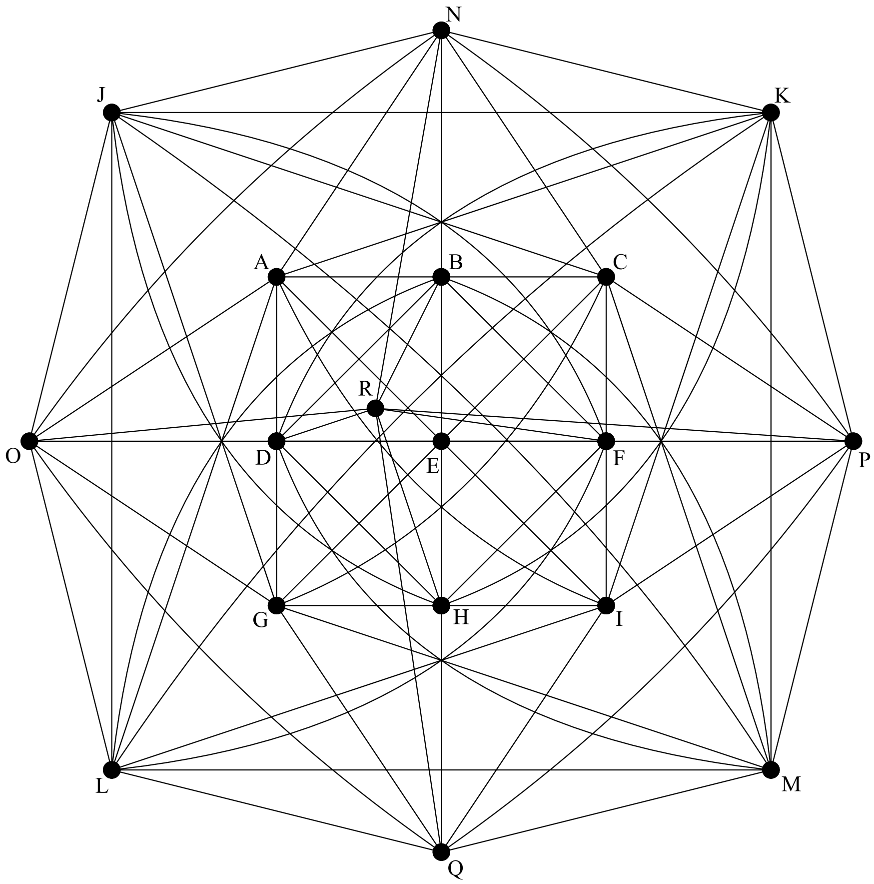


**Figure 1. The new $(18, 9)$-coloring**

Since, similar to the previous lemmas, we easily have

$$\mathcal{M}(18, 9) \setminus \mathcal{M}_2(18, 9) = \bigcup_{j=0}^{6} \mathcal{E}(17, j, 9 - j),$$

we enumerated all $(18, 9)$-colorings such that not every vertex belongs to exactly two monochromatic copies of $K_4$. There are 4 such colorings, where two of them come from the other two by exchanging the colors. Of the two essentially different colorings, one was presented in [1] and the other is presented in Figure 1, where only the edges in one color are shown. There are seven $K_4$ in the first color induced by vertex sets: $\{A, B, D, E\}$, $\{B, C, E, F\}$, $\{D, E, G, H\}$, $\{E, F, H, I\}$, $\{J, C, G, M\}$, $\{K, A, I, L\}$, $\{J, K, L, M\}$ and two $K_4$ in the second color induced by $\{B, O, P, H\}$

and $\{N, D, F, Q\}$. Notice that the labels $Q$ and $R$ in the Figure 2 in [1] are mistakenly switched. It results in serious complications with decoding the $(18, 9)$-coloring by the reader.

The question about contents of the set $\mathcal{M}_2(18, 9)$ remains open; however we conjecture that it is empty.

Three powerful programs, *nauty*, *makeg*, and *autoson*, implemented by Brendan McKay [5] were used in our work. All the algorithms specific for this project were written independently by both authors, and then a very large number of intermediate and final graphs were tested for isomorphism between the two implementations. Moreover, the cardinalities of all sets $\mathcal{M}(n, 0)$, for $n = 11, ..., 18$ agreed with the previous enumeration described in [6].

# References


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