The Ramsey Numbers $R(K_4\text{-}e,K_6\text{-}e)$ and $R(K_4\text{-}e,K_7\text{-}e)$

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Abstract. The graph with vertices in $GF(16)$, whose edges connect points having difference equal to a cube, which was known to be extremal for the Ramsey numbers $R(3,3,3)$ and $R(K_3,K_6\text{-}e)$, is shown to be extremal for $R(K_4\text{-}e,K_6\text{-}e)$. The proof is obtained by using computer algorithms to analyze the properties of the family of graphs having no $K_4\text{-}e$ and having no $K_5\text{-}e$ in the complement. It is also shown that there is a unique graph, up to graph isomorphism, which is extremal for $R(K_4\text{-}e,K_7\text{-}e)$, viz., the strongly regular Schläfli graph on 27 vertices, which has an automorphism group of size 51840. This follows easily from the result that $R(K_4\text{-}e,K_6\text{-}e)$ is 17.

1. Introduction.

The two color Ramsey number $R(G,H)$ is the smallest integer n such that for any graph F on n vertices, either F contains G or the complement of F contains H. This paper considers G and H of the form $K_n\text{-}e$, the complete graph on n vertices minus an edge. The techniques used are similar to those in [2]. A graph F is called a $(K_i\text{-}e,K_j\text{-}e)$-good graph if there is no $K_i\text{-}e$ in F and no $K_j\text{-}e$ in the complement of F. Appendix A shows all the $(K_4\text{-}e,K_6\text{-}e)$-good graphs computed by the authors, using the program "fillJ4J6". The complete list has not been generated, due to the large number of graphs at some sizes. Appendix B contains descriptions of all the computer algorithms cited in this paper.

The following notation is used throughout the paper.

G = arbitrary $(K_4\text{-}e,K_6\text{-}e)$-good graph on 17 vertices
H = arbitrary $(K_4\text{-}e,K_5\text{-}e)$-good graph
x = any vertex in G
G_x = subgraph of G induced by all vertices adjacent to x
H_x = subgraph of G induced by all vertices not x and not in G_x.
support set = subset S of vertices of H satisfying:
(S1) no triangle in H has 2 vertices in S;
(S2) S induces in H a subgraph with maximum degree at most 1;
(S3) no independent 4-set in H is disjoint from S.
OKN = binary relation on the family of support sets defined as those pairs S,T with the properties:
(O1) no subgraph of H which is induced by 4 vertices and has only 1 edge is disjoint from $(S \cup T)$;
(O2) no independent 4-set in H has 3 vertices outside $(S \cup T)$ and 1 vertex in $S\text{-}T$ or $T\text{-}S$.

The process of decomposing G into the triple (x,G_x,H_x) is called preferring the vertex x in G. Note that the vertices in H_x adjacent to a vertex y in G_x form a support set, called the support set rooted at y. Note further that every G_x is a $(K_3\text{-}e,K_6\text{-}e)$-good graph and every H_x is a $(K_4\text{-}e,K_5\text{-}e)$-good graph.

It is clear that the graph on $GF(16)$ referred to above is a $(K_4\text{-}e,K_6\text{-}e)$-good graph. One of the main results of this paper is that there is no $(K_4\text{-}e,K_6\text{-}e)$-good graph on 17 vertices, establishing 17 as the Ramsey number $R(K_4\text{-}e,K_6\text{-}e)$.

2. Properties of $(K_4,e,K_6\text{-}e)$-good graphs.

Many of the results below rely on the properties of support sets. Since H is $(K_4\text{-}e,K_5\text{-}e)$-good, no support set can have more than 6 vertices. Since G_x is $(K_3\text{-}e,K_6\text{-}e)$-good, and has maximum degree at most 1, G_x has at most 8 vertices.

Moreover, if G_x has more than 5 vertices, at most 1 vertex does not belong to an edge. It is clear that support sets rooted at adjacent vertices of G_x are disjoint and support sets rooted at non-adjacent vertices of G_x are *OKN*-related.

An edge in H is called a support edge if its vertices from a support set. The first proposition characterizes support edges and shows that H has relatively few edges which can occur as subsets of support sets.

Proposition 1. If an edge in H has both vertices in the same support set, then it is a support edge.
Proof. Let $\{x,y\}$ be an edge in H with x and y in a support set S. It suffices to show that $\{x,y\}$ is incident with every independent 4-set I in H. Assume neither x nor y lies in I. Since the complement of H has no K_5-e, both x and y must be adjacent to at least 2 vertices in I. Since $\{x,y\}$ is not in a triangle, by (S1), there must be exactly 2 vertices in I adjacent to x and the remaining 2 vertices in I must be adjacent to y. One of the vertices of I lies in S, however, by (S3), causing 2 edges in S to be incident, which is a contradiction.

The second proposition relates to vertices in G of degree 4, 5, or 6.

Proposition 2.
(a) If H has 12 vertices, then H has no support sets.
(b) If H has 11 vertices, then
  (1) H has at most 4 support edges;
  (2) H has at most 3 support sets which are pairwise *OKN*;
(c) If H has 10 vertices, then
  (1) H has at most 9 support edges;
  (2) H has no pairwise *OKN* collection of 4 support sets S,T,U,V satisfying:
    (i) S and T have size at least 5;
    (ii) U and V have size at least 4;
    (iii) U and V contain at least 1 edge each.

Proof. Four computer programs, described in Appendix A, have been written to do the counting required to establish this result. All 4 programs examine all graphs in an input file consisting of all (K_4-e,K_5-e)-good graphs, which were found in [3]. The programs are: "countS", which counts all support sets; "countE", which counts all support edges; "hxOKN", which counts all pairwise *OKN* collections of support sets; and "OKN4E5", which counts those pairwise *OKN* collections of support sets satisfying the conditions in (c2).

3. Proofs.

Theorem 1. If G exists, then G has minimum degree at least 5.
Proof. The Ramsey number $R(K_4\text{-}e, K_5\text{-}e)$ is 13, see [1], so each H_x has size at most 12. Proposition 2(a) shows that no H_x has size 12. Thus the maximum size of H_x is at most 11 and the minimum degree in G is at least 5.

Theorem 2. If G exists, then G has minimum degree at least 6.
Proof. Assume that some vertex x has degree 5. Then H_x has 11 vertices. If x belongs to fewer than 2 triangles, then G_x has an independent set of size 4 and H_x has 4 pairwise *OKN* support sets, which is not allowed by Proposition 2(b2). Therefore each vertex of degree 5 belongs to exactly 2 triangles. The properties of G_x mentioned above then imply that all vertices of G belong to at least 2 triangles. Now let y be a vertex of degree 5. Consider the 5 support sets in H_y rooted at the vertices adjacent to y. These vertices must each belong to a triangle not containing y, so the 5 support sets they generate must each contain one or more edges. This causes H_y to have at least 5 support edges, contradicting Proposition 2(b1). Thus no vertex in G has degree 5.

Theorem 3. If G exists, then G has minimum degree equal to 6.
Proof. If the minimum degree is greater than 6 then the only degrees are 7 and 8, since no G_x has size greater than 8. If every degree is 8, then every vertex belongs to 4 triangles, and in every H_x the 8 support sets break up into 4 pairs of support sets, with each pair consisting of disjoint support sets containing 3 support edges each. This requires 12 vertices in an H_x with 8 vertices and cannot happen.

Therefore some vertex y has degree 7. Its H_y has 3 pairs of support sets with each pair consisting of disjoint support sets having at least 5 vertices each. This requires 10 vertices in an H_y with 9 vertices, again impossible. Thus the minimum degree is neither 8 nor 7.

Theorem 4. If G exists, then every vertex of G belongs to at least 3 triangles.
Proof. The only vertices which can belong to fewer than 3 triangles are the vertices of degree 6. Assume x is such a vertex and y,z are the 2 vertices in G_x which do not lie in any triangle with x. The support sets in H_x rooted at y and z have size at least 5. Choose 2 nonadjacent vertices u,v in G_x distinct from y and z. The 2 support sets rooted at u and v each have size at least 4 and at least 1 edge. The 4 support sets rooted at y,z,u,v satisfy the conditions of Proposition 2(c2) and hence cannot exist. Therefore all degree 6 vertices belong to 3 triangles, implying the theorem.

Theorem 5. The Ramsey number $R(K_4\text{-}e,K_6\text{-}e)$ is equal to 17.
Proof. It was noted above that there is a $R(K_4\text{-}e,K_6\text{-}e)$-good graph on 16 vertices, establishing 17 as a lower bound for $R(K_4\text{-}e,K_6\text{-}e)$. Therefore it remains to show that no graph G exists. Assume that G exists and that x is a vertex in G of degree 6. Theorem 4 implies that the 6 support sets in H_x have at least 2 edges each, requiring 12 support edges. Proposition 2(c1) shows this is impossible, since H_x has size 10.

Theorem 6. The Ramsey number $R(K_4\text{-}e,K_7\text{-}e)$ is equal to 28. Furthermore, there is only one $R(K_4\text{-}e,K_7\text{-}e)$-good graph on 27 vertices.
Proof. If y is a vertex in a $(K_4\text{-}e,K_7\text{-}e)$-good graph F and F is decomposed into (y,G_y,H_y) by preferring y, then G_y is a $(K_3\text{-}e,K_7\text{-}e)$-good graph and H_y is a $(K_4\text{-}e,K_6\text{-}e)$-good graph. Therefore G_y has at most 10 vertices and H_y has at most 16 vertices, implying F has at most 27 vertices. Thus 28 is an upper bound for the Ramsey number $R(K_4\text{-}e,K_7\text{-}e)$.

The Schläfli graph, see [4], has 27 vertices, each of degree 10 and each in 5 edge-disjoint triangles, implying there are no $K_4\text{-}e$ subgraphs. In the complement of the Schläfli graph each vertex has degree 16 and belongs to 16 K_6's. The largest intersection between 2 K_6's is a K_3, so there are no $K_7\text{-}e$ subgraphs in the complement. Thus the Schläfli graph is $(K_4\text{-}e,K_7\text{-}e)$-good, establishing 28 as a lower bound for the Ramsey number $R(K_4\text{-}e,K_7\text{-}e)$.

The computer program "fillJ4J6", modified to construct $(K_4\text{-}e,K_7\text{-}e)$-good graphs, was used to extend all 4 of the $(K_4\text{-}e,K_6\text{-}e)$-good graphs on 16 vertices to all possible $(K_4\text{-}e,K_7\text{-}e)$-good graphs on 27 vertices. Only the Schläfli graph was produced, proving its uniqueness as an extremal graph for the Ramsey number $R(K_4\text{-}e,K_7\text{-}e)$.

4. Acknowledgement.

The authors would like to thank Geoffrey Exoo for pointing out the existence of $(K_4\text{-}e,K_6\text{-}e)$-good graphs on 16 vertices with more than 47 edges. There are 3 graphs of this type with 48, 49, and 50 edges. Their automorphism groups have orders 48, 24, and 48, respectively.

REFERENCES.

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2. S.P. Radziszowski, The Ramsey Numbers $R(K_3,K_8-e)$ and $R(K_3,K_9-e)$, JCMCC 8(1990), pp. 137-145
3. S.P. Radziszowski, On the Ramsey Number $R(K_5-e,K_5-e)$, to appear
4. J.J. Seidel, Strongly Regular Graphs (Chapter 7 in Bollobas, Surveys in Combinatorics)

APPENDICES. Appendix A contains a listing of the number of non-isomorphic (K_4-e,K_6-e)-good graphs, broken down by the number of vertices, n, and the number of edges, e. These graphs were generated by the program "fillJ4J6", which uses the graph isomorphism program described in [2] and [3]. The letter "x" denotes an uncomputed number.

Appendix B contains outlines of the computer programs used to prove Proposition 2 and the program "fillJ4J6".

APPENDIX A

n=	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
e																
0	1	1	1	1	1											
1		1	1	1	1											
2			1	2	2	2										
3			1	3	4	5	1									
4				2	6	9	6	1								
5					5	14	16	2								
6					3	17	34	15	1							
7						12	49	49	4							
8						6	55	122	25	1						
9						2	45	210	101	5						
10							22	260	355	23	1					
11							6	223	853	104	3					
12							1	136	1399	529	12	1				
13								49	1537	2066	49	1				
14								12	1163	5567	x	4				
15								2	582	9713	x	14				
16								1	187	11072	x	x				
17									38	8261	x	x				
18									9	4020	x	x				
19									1	1238	x	x				
20									1	252	x	x	2			
21										41	x	x	26			
22										7	x	x	447			
23										2	x	x	x			
24										1	x	x	x			
25										1	47	x	x			
26											4	x	x	40		
27											1	x	x	605		
28												x	x	x		
29												78	x	x		
30												7	x	x	7	
31													x	x	24	
32													x	x	151	
33													250	x	589	
34													16	x	1645	
35														x	3063	
36														x	4105	
37														831	4030	
38														71	3100	
39														7	1812	
40															747	1
41															207	0
42															36	0
43															5	0
44																0
45																0
46																0
47																0
48																1
49																1
50																1

APPENDIX B

program: countS(H)
arguments: $H = R(K_4\text{-}e, K_5\text{-}e)$-good graph
purpose: compute the number of support sets in H
code:
1. call support(U,H);
2. return number of nonzero entries in U.

program: countE(H)
arguments: $H = R(K_4\text{-}e, K_5\text{-}e)$-good graph
purpose: compute the number of support edges in H
code:
1. call support(U,H);
2. for each edge E in H:
    2a. if E belongs to some set in U increment NUM
3. return NUM.

program: OKN4E5(H)
arguments: $H = R(K_4\text{-}e, K_5\text{-}e)$-good graph
purpose: compute the maximum size of a family of support sets in H satisfying
(a) 2 sets have size 5
(b) all sets have size 4 or 5 and at least 1 edge
(c) all sets are pairwise *OKN*
code:
1. call support(U,H);
2. remove from U support sets of size less than 4 or greater than 5;
3. remove from U support sets without an edge;
4. for each pair (S_1,S_2) from U with S_1 *OKN* S_2 and size (S_1) = size(S_2) = 5:
    4a. form the array C of all support sets from U of size 4 which are *OKN* with S_1 and S_2
    4b. define length(S_1,S_2) = hxOKN(C)
5. return the maximum value of length(S_1,S_2)

program: support(U,H)
arguments: U = array to hold all support sets in H
$H = R(K_4\text{-}e, K_5\text{-}e)$-good graph
purpose: compute the family of support sets in H
code:
1. build array A of all adjoining edges in H;
2. build array T of all triangles in H;
3. build array I of all independent 4-sets in H;
4. for each set S of vertices of H:
    4a. if S contains no A[i] and S meets each T[i] in fewer than 2 vertices and S meets each I[i] in at least 1 vertex then adjoin S to the array U

program: hxOKN(C)
arguments: C = array of support sets
purpose: compute $MAXOKN$ = the maximum number of support sets in C which are pairwise *OKN*
code:
1. define MAX = current value of the maximum number of support sets in C which are pairwise *OKN*
2. build array flag of 0's of same length as C
3. call cluster(&*MAX*,*flag*,*C*,1);
4. return *MAX*

program: cluster *(ptr,flag,C,index)*

arguments:
ptr = pointer to integer variable *MAX*
flag = array of 0's and 1's showing families of support sets which are pairwise *OKN*
C = array of support sets
index = index in array *C*

purpose: recursively construct all families of support sets which are pairwise *OKN* and record the maximum size of such families

code:
```
1. if index > length(C) { update MAX;return; }
2. if C[index] is OKN with all preceding flagged
   support sets { C[index] = 1;
   call cluster(ptr,flag,C,index + 1); }
3. C[index] = 0;
4. call cluster(ptr,flag,C,index + 1);
```

program: fillJ4J6*(min,H)*

arguments:
min = integer
H = $(K_4\text{-}e,K_5\text{-}e)$-good graph

purpose: construct all $(K_4\text{-}e,K_6\text{-}e)$-good graphs with preferred triple (y,G_y,H_y) using G_y with size *min*, *H* as H_y, and minimum degree *min*

code:
```
1. call support(U,H)
2. for each number of edges in Gy and each assignment
   of support sets from U to the vertices of Gy:
      2a. test if the support sets for adjacent vertices
          are disjoint and the support sets for
          independent vertices are OKN
      2b. test if the resulting graph is (K4-e,K6-e)-
          good with minimum degree min
```