The Number of Edges in Minimum (K_3,K_p-e,n)-good Graphs

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ABSTRACT

We derive an explicit formula for the number of edges in the minimum $(K_3,K_{k+2}-e,n)$-good graphs for $n \le 13k/4 - sign(k \bmod 4)$ and obtain the inequality $e(K_3,K_{k+2}-e,n) \ge 6n-13k$ for $k \ge 6$ and all positive integers n.

1. Introduction

We investigate the minimum number of edges in triangle-free graphs on n vertices, whose complement does not contain K_p-e. The obtained results are similar to those derived in [4] and [5], where the graphs considered were related to the classical Ramsey numbers $R(K_3,K_p)$. The change of the forbidden graph from K_p to K_p-e causes considerable changes in the set of minimum graphs and increases the complexity of the proofs, however the form of general lower bounds for minimum number of edges in such graphs remain almost identical to the classical case, at least in the ranges studied in this paper. The method used here has potential applications in evaluation and/or improvement of bounds for Ramsey numbers $R(K_3,K_p-e)$. For the state of the art of the latter problem see [3].

Throughout this paper we adopt the following notation. $\bar{G}$ denotes the complement of the graph G. A (G,H)-good graph F is defined as the graph F not containing G, nor $\bar{F}$ containing H, and a (G,H,n)-good graph is a (G,H)-good graph on n vertices. $e(G,H,n)$ is defined as the minimum number of edges in any (G,H,n)-good graph or ∞ if no such graph exists. If the number of edges in a (G,H,n)-good graph is equal to $e(G,H,n)$, then such graph is called a *minimum* (G,H,n)-graph or *minimum* (G,H)-good graph. In this paper we consider (K_3,K_p-e)-good graphs, where K_3 is a triangle and K_p-e is a complete graph on p vertices without one edge. If $G=(V,E)$ is a graph then we call $V' \subseteq V$ a *quasi-independent* set, if $|\{\{u,v\} \in E:\ u,v \in V'\}| \le 1$. As usual, $n(G)$ and $e(G)$ denote the number of vertices and edges in G, respectively. Any vertex of degree i will be called an $i-$*vertex*. For each $x \in V$, $N_G(x)=\{v:\ \{x,v\} \in E\}$ is a *neighborhood* of x in G. C_n and P_n denote a cycle and a path, respectively, of length n, and $\delta(G)$ denotes the minimal degree of vertices in graph G.

Let $G=(V,E)$ be a fixed $(K_3,K_{k+1}-e,n)$-good graph and choose some vertex $v \in V$. Following [2] define $H_2(v)$ to be the graph induced in G by the set of vertices $\{u \in V: u \ne v$ and u is not adjacent to $v\}$. The $Z-$*sum* of a vertex v in G is defined as $Z(v)=\sum\{deg_G(u): u \in N_G(v)\}$. If G is a $(K_3,K_{k+1}-e,n)$-good graph, then $H_2(v)$ is a $(K_3,K_k-e,n-deg(v)-1)$-good graph with $e(G)-Z(v)$ edges. Let $f(v)=e(H_2(v))-e(K_3,K_k-e,n-deg(v)-1)$, so we have obviously $f(v) \ge 0$, and if $f(v)=0$ then vertex v is called *full* [2].

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The construction of all (K_3,K_k-e)-good graphs for all $k\leq 6$ has been reported in [3]. To obtain general properties of minimum (K_3,K_k-e,n)-good graphs, we need to know some of them for small k, especially for $k=8$ and $k=9$. The values and lower bounds for the minimum number of edges listed in tables I and II were obtained by the algorithms used in [3].

n	16	17	18	19	20	21	22	23	24
$e(K_3,K_8-e,n)$	20	25	30	37	44	51	59	70	80

Table I.

n	18	19	20	21
$e(K_3,K_9-e,n)$	20	25	30	35

n	22	23	24	25	26	27	28	29	30
$e(K_3,K_9-e,n)\geq$	41	46	54	62	71	80	90	100	111

Table II.

In the sequel we will use the following fundamental lemma of Graver and Yackel.

Lemma 1 (variation of proposition 4 in [1]): In any $(K_3,K_{k+1}-e,n)$-good graph G, if n_i denotes the number of i-vertices in G, then

$$\Delta=\sum_{v\in V} f(v)=ne-\sum_{i\geq 0}^{k} n_i(e(K_3,K_{k+1}-e,n-i-1)+i^2)\geq 0 \qquad (1)$$

and there are at least $n-\Delta$ full vertices in G.

2. Properties of Minimum $(\mathbf{K_3,K_p-e})$-good Graphs

In lemmas 2, 3 and 4 G denotes a minimum (K_3,K_p-e)-good graph.

Lemma 2: If G has an isolated vertex, then $deg(x)\leq 2$ for all $x\in V(G)$.

Proof: Let v be an isolated vertex, and let x be a vertex of maximum degree in G. If $deg(x)\geq 3$, let $N_G(x)=\{x_1,\ldots,x_m\}$ and $N_G(x_m)=\{x,y_1,\ldots,y_l\}$, $m>l$. If $l=0$ then we define a triangle-free graph H obtained from G by deleting the edges $\{x_1,x\},\ldots,\{x_{m-1},x\}$ and joining vertices x and v by an edge. Obviously H has less edges than G. If $l>0$ then we define a triangle-free graph H obtained from G by deleting the edges $\{x_1,x\},\ldots,\{x_{m-1},x\},\{x_m,y_l\}$ and adding the edges $\{y_2,v\},\ldots,\{y_l,v\},\{x,v\}$. Since $e(H)=e(G)-(m-l)<e(G)$, H has less edges than G. In both cases we show that G and H have the same maximal quasi-independent set sizes. Let M' be any quasi-independent set in H. If $x,v\in M'$ then let $M=\{x_m\}\cup(M'-\{x\})$, and if $x,x_m\in M'$ then let $M=\{v\}\cup(M'-\{x\})$. It is easy to see that M is a quasi-independent set in G and $|M|=|M'|$. Hence H is also a (K_3,K_p-e,n)-good graph contradicting the fact G is a minimum graph. □

Lemma 3:

1) If G has a P_2 as a component, then $deg(x) \leq 2$ for all $x \in V(G)$.
2) If $\{w,x\}$, $\{x,y\}$, $\{y,z\}$ are edges of G and $deg(w)=1$, $deg(x)=deg(y)=2$, then w, x, y, z are on a path component of G and for all $v \in V(G)$ $deg(v) \leq 2$.
3) If $\{x,y\}$, $\{w,y\}$ are two edges of G and $deg(x)=deg(w)=1$, then $deg(y) \leq 2$.
4) If G has an even cycle component, then G has at most one isolated vertex.
5) If G has an isolated vertex and an even cycle component, then G has no path components.
6) If G has cycle components and an even path component, then these cycles have even length.
7) If G has an isolated vertex and a cycle component, then the length of this cycle is even.
8) If G has a path component P_l, $l \geq 2$, then l is even and G has no other path components.

Proof: Suppose that G violates some of the assertions in the lemma. In each case we define a triangle-free graph H from G with less edges than G and the same maximal quasi-independent set size, which will contradict the minimality of G.

1) Assume that x, y are the endpoints of a P_2 component. If G has an r-vertex v, $r \geq 3$, we define H from G by deleting all the edges adjacent to v and adding the edges $\{x,v\}$, $\{y,v\}$.

2) If w, x, y, z are on a path component of G, then w is an endpoint of this path. Assume that w' is another endpoint of the same path. If G has a vertex v with $deg(v) \geq 3$, we define H from G by deleting all the edges adjacent to v and adding the edges $\{w,v\}$, $\{w',v\}$. If the component which includes w, x, y, z is not a path, then we can assume that $\{u_i,u_{i+1}\}$, for $i=0,1,\ldots,m-1$, are edges of G ($m \geq 3$, $w=u_0$, $x=u_1$, $y=u_2$, $z=u_3$), $deg(u_i)=2$ for $1 \leq i < m$ and $deg(u_m) \geq 3$. We define H from G by deleting all the edges adjacent to u_m except the edge $\{u_{m-1},u_m\}$ and joining the vertices w and u_m.

3) If $deg(y) \geq 3$ then define H from G by deleting an edge $\{z,y\}$, for which $z \notin \{x,w\}$.

4) Assume that C with vertices $x_1,x_2,\ldots,x_{2m}$ is an even cycle component of G. If G has two isolated vertices u and v, then we define H from G by deleting all the edges of C and then adding the edges $\{u,v\}$ and $\{x_i,x_{i+1}\}$, for $i=1,3,\ldots,2m-1$.

5) Assume that C is an even cycle as in 4) and v is an isolated vertex of G. If G has a path component P, let $y_1,y_2,\ldots,y_l$ be all the vertices of P. We define H from G by deleting all the edges of C and P and then adding the edges $\{v,y_l\}$, $\{x_i,x_{i+1}\}$ and $\{y_j,y_{j+1}\}$ for $i=1,3,\ldots,2m-1$ and $j=1,3,\ldots,2\lfloor l/2 \rfloor-1$.

6) Assume that P is a path component of G and $y_1,y_2,\ldots,y_{2l+1}$ are all the vertices of P. If G has an odd cycle component C, assume that $x_1,x_2,\ldots,x_{2m+1}$ are all the vertices of C ($m \geq 2$). We define H from G by deleting all the edges of P and C and then adding the edges $\{x_{2m+1},y_{2l+1}\}$, $\{x_i,x_{i+1}\}$, $\{y_j,y_{j+1}\}$ for $i=1,3,\ldots,2m-1$ and $j=1,3,\ldots,2l-1$.

7) Let C be a cycle component of G and w be an isolated vertex of G. If the length of C is odd, and $v \in V(C)$, we define graph H from G by deleting all the edges adjacent to v and joining the vertices v and w by an edge.

8) Assume that G has a path component P_l, $l \geq 2$, with vertices $x_0,x_1,\ldots,x_l$. If l is odd, then there are even vertices on P_l. We define H as a triangle-free graph obtained from G by deleting all the edges of P_l and then adding the edges $\{x_0,x_1\}$, $\{x_2,x_3\}$, ..., $\{x_{l-1},x_l\}$. One can easily check that H has less edges than G but the same maximal quasi-independent set size, which contradicts the minimality of G. Thus l is even. Assume that G has another path component P_m with ver-

tices $y_0, y_1, \ldots, y_m$, then by the previous argument m is even. We define H as a triangle-free graph obtained from G by deleting all the edges in P_l and P_m and adding the edges $\{x_0,x_1\}, \ldots, \{x_{l-2},x_{l-1}\}$, $\{y_0,y_1\}, \ldots, \{y_{m-2},y_{m-1}\}$, $\{x_l,y_m\}$. This leads to a contradiction similarly as before. □

Lemma 4: If G is a minimum $(K_3, K_{k+1}-e, n)$-good graph and $n \geq 2k$, then G has no isolated vertices.

Proof: If G has isolated vertices, then by lemma 2 $deg(x) \leq 2$ for any $x \in V(G)$. If G has cycle components, then each cycle of G is even by lemma 3.7 and G has only one isolated vertex by lemma 3.4. By lemma 3.5 $k=(n-1)/2+1$, i.e. $n=2k-1$ contradicting the fact $n \geq 2k$. If G has no cycle components, then G has only isolated vertices and path components. Assume G has r isolated vertices. If there is a path component P_l, $l \geq 2$, by lemma 3.8 this is the only path of G. Hence $k=(n-r+1)/2+r$, i.e. $n=2k-r-1 \leq 2k-2$, which is a contradiction. If G has only isolated vertices and $(n-r)/2$ isolated edges, then $k=(n-r)/2+r+1$, i.e. $n=2k-r-2 \leq 2k-3$, which contradicts the fact $n \geq 2k$ again. □

3. Main Theorems

Construction 1: For $k+1 \leq n \leq 2k-2$ and $k \geq 2$, define the graph $G=<V,E>$ by (see figure 1)

$$V=\{x_1, x_2, \ldots, x_n\} \text{ and } \quad E=\{\{x_i, x_{i+1}\} : i=1,3,\ldots,2(n-k)+1\}.$$

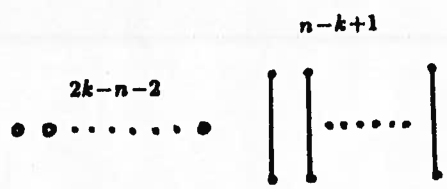

Figure 1. Construction 1.

Construction 2: For $2k-1 \leq n \leq (5k-5)/2$ and $k \geq 4$, define the graph $G=<V,E>$ by (see figure 2)

$V=X_1 \cup X_2 \cup \cdots \cup X_{n-2k+2} \cup Y$, *where*

$X_t=\{x_{t,j} : j \in Z_5\}$ for $t=1,2,\ldots,n-2k+2$; $Y=\{y_1, y_2, \ldots, y_{10k-4n-10}\}$,

$E=E_1 \cup E_2 \cup \cdots \cup E_{n-2k+2} \cup E'$, *where*

$E_t=\{\{x_{t,j}, x_{t,j+1}\} : j \in Z_5\}$ for $t=1,2,\ldots,n-2k+2$,

$E'=\{\{y_i, y_{i+1}\} : i=1,3,\ldots,10k-4n-11\}$.

It is easy to check that in both constructions 1 and 2 G is a $(K_3, K_{k+1}-e, n)$-

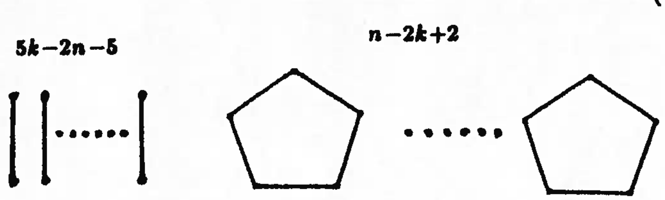

Figure 2. Construction 2.

good graph. Thus we have:

Lemma 5:

$$e(K_3,K_{k+2}-e,n)\leq \begin{cases} n-k & \text{if } k\leq n\leq 2k \text{ and } k\geq 1,\\ 3n-5k & \text{if } 2k<n\leq 5k/2 \text{ and } k\geq 3.\end{cases}$$

Theorem 1:

$$e(K_3,K_{k+2}-e,n) = \begin{cases} 0 & \text{if } n\leq k+1,\\ n-k & \text{if } k+2\leq n\leq 2k \text{ and } k\geq 1,\\ 3n-5k & \text{if } 2k<n\leq 5k/2 \text{ and } k\geq 3.\end{cases} \qquad (2)$$

Proof: The statement is obvious for $n\leq k+1$. For $k+2\leq n\leq 2k$ let H be a minimum $(K_3,K_{k+2}-e,n)$-good graph. If H has no isolated vertices, then $e(H)\geq n-k$ and by lemma 5 $e(H)=n-k$. If H has isolated vertices then, by lemma 2, $deg(x)\leq 2$ for any $x\in V(H)$. If H has no cycle components and H has a path component P_l, $l\geq 2$, then by lemma 3.8 $e(H)=l$ and $k=n-(l+1)+l/2$. By lemma 5 $l\leq n-k$, and it is easy to see that $l=2$, hence $e(H)=2=n-k$. If H has no cycle components and H has no paths components P_l with $l>1$, then H has to be as in construction 1 and the theorem holds. If H has cycle components, then by lemma 3.7 they are even cycles. By lemma 3.4 H has only one isolated vertex and by lemma 3.5 H has no isolated edges. Hence $e(H)=n-1\geq n-k$ and by lemma 5 $e(H)=n-k$.

In the case $2k<n\leq 5k/2$ and $k\geq 3$ we use induction on k. The equality (2) holds for $k=3$, since the only relevant parameter situation is $k=3$, $n=7$, and it is known that $e(K_3,K_5-e,7)=6$ [3]. Assume that H is a minimum $(K_3,K_{k+2}-e,n)$-good graph for some $k\geq 4$ and $2k<n\leq 5k/2$. Then either H has a 2-vertex or it doesn't.

If H has a 2-vertex v, then $H_2(v)$ is a $(K_3,K_{k+1}-e,n-3)$-good graph and $e(H_2(v))=e(H)-Z(v)$. Because $n-3\leq 5(k-1)/2$, so by induction the number of edges $e(H)-Z(v)$ in $H_2(v)$ is at least $3(n-3)-5(k-1)$ for $n>2k+1$ or at least $(n-3)-(k-1)$ for $n=2k+1$. In both cases, since $e(H)\leq 3n-5k$ by lemma 5, we have $2\leq Z(v)\leq 4$. If $Z(v)=4$, then $H_2(v)$ is a minimum $(K_3,K_{k+1}-e,n-3)$-good graph. For $n>2k+1$, $e(H)-Z(v)=3(n-3)-5(k-1)$, i.e. $e(H)=3n-5k$. For $n=2k+1$, $e(H)-Z(v)=(n-3)-(k-1)$, i.e. $e(H)=k+3=3n-5k$. If $Z(v)\leq 3$, then by lemmas 3.1 and 3.2 the component of H containing v is a path and $deg(x)\leq 2$ for any $x\in V(H)$. By lemma 3.8 the length of this path is even and H has no other path components. Hence by lemma 3.6 H is composed of a path and some even cycles. For $n>2k+1$ assume that the path has s vertices, then H has a quasi-independent set of size $(n-s)/2+(s+1)/2\geq k+2$, which is a contradiction. If $n=2k+1$, then for $k=3$ we have $e(H)=n-1=3n-5k$, so the theorem holds, and for $k>3$ we have $e(H)=n-1>3n-5k$, which contradicts lemma 5.

If H has no 2-vertices, then H must have an r-vertex v, $r\geq 3$, since otherwise H would have only 0- and 1-vertices and in this case $n<2k+1$. Because $n-r-1\leq 5(k-1)/2$, by the inductive assumption $H_2(v)$ is a $(K_3,K_{k+1}-e,n-r-1)$-good graph with $e(H_2(v))\geq 3(n-r-1)-5(k-1)$ for $n-r-1>2(k+1)-4$ or $e(H_2(v))\geq max((n-r-1)-(k-1),0)$ for $n-r-1\leq 2(k+1)-4$. In either case it can be easily derived that $e(H_2(v))\geq e(H)-(3r-2)$, implying that $3\leq Z(v)\leq 3r-2$. By lemma 3.3, it is sufficient to consider the case $Z(v)=3r-2$. In this case, assume that $\{v,x_i\}$, $(i=1,2,\ldots,r)$, $\{x_i,y_{ij}\}$, $(i=2,3,\ldots,r;\ j=1,2)$ are the edges of G. Hence $deg(x_1)=1$, $deg(x_i)=3$, for $i=2,3,\ldots,r$, and similarly as before, $3\leq Z(x_i)\leq 7$ for $i=2,3,\ldots,r$, and thus we have $r=deg(v)=3$. Therefore at least one of y_{21},y_{22} is a

1-vertex, and at least one of y_{31}, y_{32} is a 1-vertex, so we may assume that $deg(y_{21})=deg(y_{31})=1$. We define a triangle-free graph H' from H by deleting the edge $\{x_2, y_{22}\}$. Assume E' is a quasi-independent set in H' and $x_2, y_{22} \in E'$. If u is one of x_1, y_{21} and u does not belong to E', then we define $E=\{u\}\cup(E'-\{x_2\})$. If $x_1, y_{21} \in E'$ and $y_{31} \notin E'$, then we define $E=\{y_{31}\}\cup(E'-\{x_2\})$. If $x_1, y_{21}, y_{31} \in E'$, then both v and x_3 do not belong to E' and we define $E=\{v\}\cup(E'-\{x_2\})$. In both cases, E is a quasi-independent set in H. Hence H' has the same maximal quasi-independent set size as H but less edges than H, which contradicts the minimality of G. □

Note that $(K_3, K_4-e, 5)=4<3\cdot5-5\cdot3+5$ [3], hence the condition $k\geq3$ for $2k<n\leq5k/2$ in theorem 1 is necessary.

Lemma 6: Let G be a minimum $(K_3, K_{k+2}-e, n)$-good graph. If G has a cycle component, then the length of this cycle is equal to 4, 5, 6, 8 or 10.

Proof: Assume that G has a cycle component C_l. We first show that if l is odd, then this C_l is a minimum $(K_3, K_{(l-1)/2+1}, l)$-graph. Let H be a minimum $(K_3, K_{(l-1)/2+1}, l)$-graph and observe [4] that the size of maximal quasi-endependent set in H is equal to $(l-1)/2+1$. Define the graph G' from G by changing cycle C_l to graph H, then G' has the same size of maximal quasi-independent set as G, but less edges. This contradicts the minimality of G. Thus in the case of odd l by lemma 2 in [4] we have $l=5$. If l is even, then C_l is a minimum $(K_3, K_{l/2+2}-e, l)$-good graph. If $k\leq3$, then as found in [3] l is one of 4,6,8. If $k\geq4$, we have $l/2+5=l$ by theorem 1. Hence $l=10$. □

Theorem 2: If G is a minimum $(K_3, K_{k+2}-e, n)$-good graph and $k\geq6$ then

$$e(G)\geq5n-10k. \tag{3}$$

Proof: By theorem 1 we have $e(K_3, K_{k+2}-e, n)\geq5n-10k$ for $n\leq5k/2$ and $k\geq6$. For $5k/2<n$ we use induction on k. Inequality (3) holds for $k=6$ and $k=7$ by theorem 1 and tables I and II. Suppose G is a minimum $(K_3, K_{k+2}-e, n)$-good graph for some $k\geq8$, $5k/2<n$ and $e(G)<5n-10k$. By applying (1) and induction to G we obtain

$$\begin{aligned} 0\leq\Delta &= ne(G)-\sum_{i=0}^{k+1} n_i(i^2+e(K_3, K_{k+1}-e, n-i-1)) \\ &\leq ne(G)-\sum_{i=0}^{k+1} n_i(i^2-5i+5n-10(k+1)+15). \end{aligned}$$

Note that $\sum_{i=0}^{k+1} n_i=n$, hence

$$0\leq\Delta\leq n(e-(5n-10k-1))-\sum_{i=0}^{k+1} n_i(i-2)(i-3). \tag{4}$$

The coefficient $(i-2)(i-3)$ is nonnegative for all integers i. Hence, inequality (4) and $e(G)<5n-10k$ imply that $e(G)=5n-10k-1$, $\Delta=0$ and consequently G has only 2- and/or 3- vertices. Thus all the vertices of G are full, and we conclude that for any vertex v in $V(G)$, $Z(v)=4$ if $deg(v)=2$ and $Z(v)=9$ if $deg(v)=3$. Therefore every component of G is a cycle or a cubic graph.

We first show that $n_2=0$. If $n_2>0$ then G has a cycle component C. Assume that a vertex v is on C, so $H_2(v)$ is a $(K_3, K_{k+1}-e, n-3)$-good graph with $5n-10k-5$ edges. By the induction $e(H_2(v))\geq5(n-3)-10(k-1) =5n-10k-5$, which implies that $H_2(v)$ is a minimum graph. By lemma 6 the length of C_l is equal to 4, 5, 6, 8 or 10. It is easy to check that if $l=8$ then the graph H, obtained from G by changing C_l to two C_4's, is a minimum $(K_3,K_{k+2}-e,n)$-good graph with the same properties as G described before. If $l=10$ then the graph H, obtained from G by changing C_l to two pentagons, is also a minimum $(K_3,K_{k+2}-e,n)$-good graph. If $l=4$, i.e. C_l is a square, then $H_2(v)$ has an isolated vertex, and $n-3>5k/2-3\geq 2k$ for $k\geq 7$ contradicts lemma 4. If $l=5$, then G is a disjoint union of a pentagon and some $(K_3,K_k-e,n-5)$-graph G_1 with $5n-10(k+1)+4$ edges. By the induction $e(G_1)\geq 5(n-5)-10(k-2)=5n-10(k+1)+5>5n-10(k+1)+4$, which is impossible. If $l=6$, then let v be a vertex on C_6. Note that $H_2(v)$ is a minimum $(K_3,K_{k+1}-e,n-3)$-good graph and $H_2(v)$ has a P_2 component. By lemmas 3.7 and 3.8 G is a disjoint union of a P_2 and some even cycles. Hence $k=(n-6)/2+2$, i.e. $n=2(k+1)\leq 5k/2$, which is a contradiction, and consequently $n_2=0$.

Now we can assume that G is a cubic minimum $(K_3,K_{k+2}-e,n)$-good graph with $3n/2$ edges, and thus $3n/2=5n-10k-1$, which implies that $7n=20(k+1)-18$. Note that since $e(G)=5n-10k-1<5n-10k\leq e(K_3,K_{k+1},n)$, where the last inequality uses theorem 2 in [4], G is not a minimum $(3,K_{k+1},n)$-graph. To complete the proof of theorem 2 it is now sufficient to show that no such graph G can exist.

For any vertex v in $V(G)$, $H_2(v)$ is a $(K_3,K_{k+1}-e,n-4)$-good graph with $5n-10(k+1)$ edges. On the other hand, by the induction $e(H_2(v))\geq 5(n-4)-10(k-1)=5n-10(k+1)$. Hence $H_2(v)$ is a minimum graph. We now show that $H_2(v)$ satisfies the following properties P1-P3.

(P1) The graph $H_2(v)$ has vertices of degree 2 and 3 only.

We obviously have $deg(x)\leq 3$ for any $x\in V(H_2(x))$. Since $n-4>5k/2-4\geq 2k$ for $k\geq 8$, by lemma 4 $H_2(v)$ has no isolated vertices. If $H_2(v)$ has some 1-vertex u, then $H_2(u)$ in $H_2(v)$ is a $(K_3,K_k-e,n-6)$-good graph with at most $5n-10(k+1)-1$ edges, which contradicts the inductive assumption for $H_2(u)$, namely $e(K_3,K_k-e,n-6)\geq 5(n-6)-10(k-2)=5n-10(k+1)$. Now by counting edges we conclude that

(P2) $H_2(v)$ has 6 2-vertices and $n-10$ 3-vertices.

If x is a 2-vertex in $H_2(v)$ then $Z(x)\leq 5$ with respect to the graph $H_2(v)$, since by the induction $e(K_3,K_k-e,n-7)\geq 5(n-7)-10(k-2)=e(H_2(v))-5$. Thus

(P3) Any 2-vertex in $H_2(v)$ has at least one 2-vertex as a neighbor.

Let $F(v)$ be the subgraph of $H_2(v)$ induced by it's 2-vertices. $F(v)$ cannot have isolated vertices since this would imply $Z(x)=6$ in $H_2(v)$, nor can $F(v)$ contain a pentagon, since $F(v)$ has 6 vertices and a pentagon would imply the existence of an isolated vertex. If $F(v)$ contains a P_5, then by using properties P1-P3 we can conclude that G has a component as in figure 3. But in this graph $F(x)$ has an isolated vertex y, which is impossible. If $F(v)$ is a disjoint union of P_3 and P_1, then by applying properties P1-P3 to $F(u)$ and $F(w)$ G has a component as in figure 4. But in this graph $F(x)$ has an isolated vertex y, which again is impossible. Hence one of the three cases must occur: i) $F(v)$ is a hexagon or ii) $F(v)$ is formed by three isolated edges or iii) $F(v)$ is formed by two path components P_2.

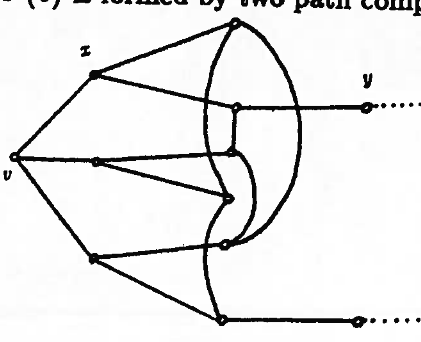

Figure 3.

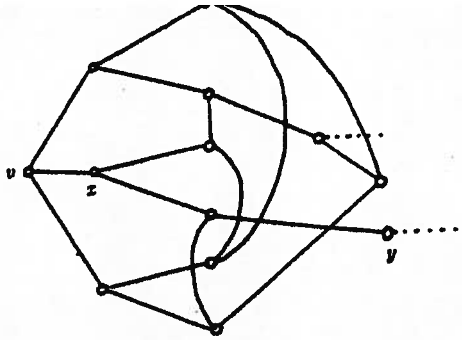

Figure 4.

Case i): If $F(v)$ is a hexagon, then $F(v)$ is a subgraph of H_1 (figure 5), which is a component of G. H_1 is a minimum $(K_3, K_5 - e, 10)$-good graph [3].

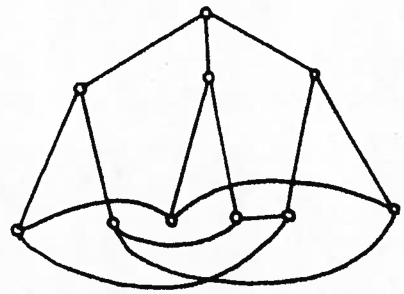

Figure 5. Graph H_1.

Case ii): If $F(v)$ has three isolated edges, then by using properties P1-P3 we can conclude that $F(v)$ is a subgraph of the dodecahedron H_2 (figure 6), which is a component of G. The dodecahedron is a minimum $(K_3, K_9 - e, 20)$-good graph [3].

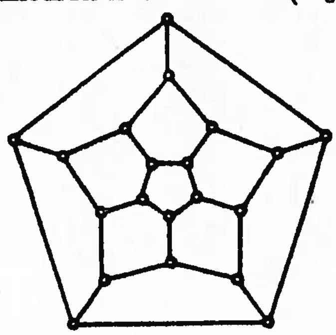

Figure 6. Graph H_2.

Case iii): If $F(v)$ consists of two P_2's, then by using properties P1-P3 we can conclude that $F(v)$ is a subgraph of the graph H_3 (figure 7), which is a component of G. H_3 is a minimum $(K_3, K_6 - e, 12)$-good graph with 18 edges.

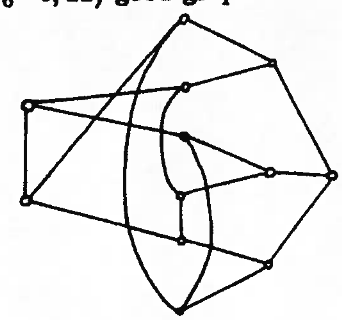

Figure 7. Graph H_3.

Finally, each component of G must be isomorphic to one of the graphs H_i for $1 \leq i \leq 3$, which have maximal independent sets of sizes 4, 8 and 5, respectively. Assume that G contains s_i copies of the graph H_i. Hence we have $k+1=4s_1+8s_2+5s_3$ and $n=10s_1+20s_2+12s_3$. Note also that $7n=20(k+1)-18$, so $10s_1+20s_2+16s_3=18$. The latter equalities have no solutions in nonnegative integers s_i, which completes the proof of theorem 2. □

Observe that $e(K_3,K_7-e,14)=19<5 \cdot 14-10 \cdot 5$ [3], thus the condition $k \geq 6$ in theorem 2 is necessary.

Theorem 3:

$$e(K_3,K_{k+2}-e,n)=5n-10k \quad \text{for} \quad 5k/2<n \leq 3k,\ k \geq 6. \tag{5}$$

Proof: Since $e(K_3,K_{k+2}-e,n) \leq e(K_3,K_{k+1},n)$, by theorem 2 of this paper and by theorems 2 and 4 in [4] we have $5n-10k \leq e(K_3,K_{k+2}-e,n) \leq e(K_3,K_{k+1},n) = 5n-10k$ for $5k/2<n \leq 3k$ and $k \geq 6$. Hence the equality (5) follows. □

By using computer algorithms described in [3] and theorems 1 and 3 above, we have obtained the following lemmas 7 and 8.

Lemma 7: If G is a minimum $(K_3,K_8-e,18)$-good graph, then all the vertices in G have degree at least 3 and G is either the graph G_6 defined in [4] or the graph $H=(V,E)$, where

$V=X \cup Y \cup Z$ and $X=\{x_i: i \in Z_6\}$, $Y=\{y_i: i \in Z_6\}$, $Z=\{z_i: i \in Z_6\}$,

$E=\{\{x_i,x_{i+1}\},\{y_i,y_{i+1}\},\{z_i,z_{i+1}\} : i \in Z_6\} \cup \{\{y_i,x_{2i}\},\{y_{i+3},x_{2i}\}: i \in Z_6\}$

$\cup \{\{z_i,x_{2i+1}\},\{z_{i+3},x_{2i+1}\} : i \in Z_6\}$ (see figure 8).

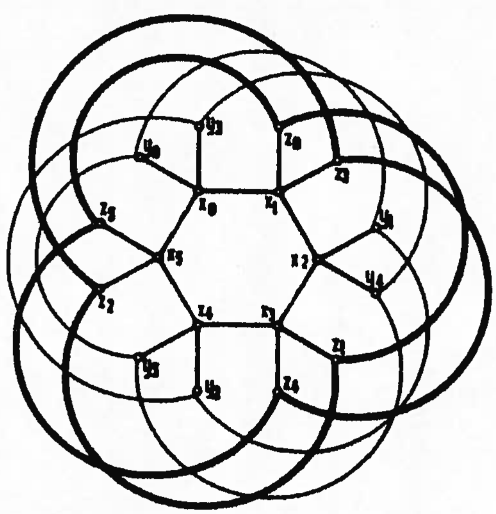

Figure 8. Graph H from lemma 7.

Lemma 8: $e(K_3,K_9-e,23) \geq 46$ and if G is a $(K_3,K_9-e,23)$-good graph with 46 edges, then G is a 4-regular graph.

Lemma 9: $e(K_3,K_9-e,23) \geq 47$.

Proof: If G is a $(K_3,K_9-e,23)$-good graph with 46 edges, then by lemma 8 G is a 4-regular graph. For any $v \in V(G)$, $H_2(v)$ is a $(K_3,K_8-e,18)$-good graph with 30 edges, and by table I $H_2(v)$ is a minimum graph. Since by lemma 7 $\delta(H_2(v))\geq 3$, G doesn't contain C_4 as a subgraph. Thus $H_2(v)$ is isomorphic to H from lemma 7, because G_6 contains C_4. Let $J(v)=\{u\epsilon V(G)$: for some $w\in N_G(v)$ $\{u,w\}\in E(G)\}$. It is easy to see that $J(v)=\{y_i,z_i: i\epsilon Z_6\}$ (see figure 8). By the definition of H, the subgraph induced by $J(v)$ is composed of two hexagons. For some two vertices in distance 3 on one of the hexagons, there is a vertex $x\epsilon N_G(v)$ adjacent to both of them. But by the definition of H, for any such two vertices in Y (or in Z) there is a vertex in X adjacent to them. Hence G contains a C_4, which is a contradiction. □

Lemma 10: $e(K_3,K_{10}-e,26)\geq 52$.

Proof: Using the algorithms from [3], table II and lemma 9, we obtain $(K_3,K_{10}-e,26)\geq 51$, furthermore the only two possible solutions for the degree sequence of a $(K_3,K_{10}-e,26)$-good graph with 51 edges are $n_4=24$, $n_3=2$ or $n_4=25$, $n_2=1$.

In the first case, let v be a 3-vertex. Then $H_2(v)$ is a $(K_3,K_9-e,22)$-good graph with at most 40 edges, which contradicts $e(K_3,K_9-e,22)\geq 41$ (table II). In the second case, let u be a 2-vertex. Then $H_2(u)$ is a $(K_3,K_9-e,23)$-good graph with 43 edges, which contradicts $e(K_3,K_9-e,23)\geq 46$ (table II), thus $e(K_3,K_{10}-e,26)\geq 52$. □

Using the algorithms from [3], table II and lemmas 3 and 4, we have obtained the following lower bounds listed in table III.

n	24	25	26	27	28	29	30
$e(K_3,K_{10}-e,n)\geq$	40	46	52	58	65	72	81

n	31	32	33	34	35	36	37	38
$e(K_3,K_{10}-e,n)\geq$	90	99	110	121	133	144	158	171

Table III.

Lemma 11: If G is a (K_3,K_p-e)-good graph and $\delta(G)\geq 2$, then

1) $Z(v)\geq 4$ for any $v\epsilon V(G)$,

2) If u is a 2-vertex in G, $Z(u)\leq 5$ and $n_2\leq 4$, then G has a C_4 component or there is a 2-vertex v such that $Z(v)=5$.

Proof: 1) The assertion holds since $\delta(G)\geq 2$.

2) Assume that G has no C_4 components. If $Z(u)=5$, then the assertion holds. If $Z(u)=4$, let x,y be the two neighbors of u. Since $Z(x),Z(y)\leq 5$ and G has no C_4 components, there are $x_1,y_1\epsilon V(G)$, $x_1\neq u$, $y_1\neq u$ such that $deg(x_1)\leq 3$ and $deg(y_1)\leq 3$. By the assumption $n_2\leq 4$, one of $deg(x_1)$ or $deg(y_1)$ is equal to 3. Hence one of $Z(x)$ or $Z(y)$ is equal to 5. □

Lemma 12: If G is a $(K_3,K_{k+2}-e,n)$-good graph with $6n-13k$ edges for k=6,7 or 8, then $\delta(G)\geq 3$.

Proof: By checking tables I, II and III, we find that the relevant parameter situations are $k=6$, $n=18$; $k=7$, $21\leq n\leq 23$; and $k=8$, $24\leq n\leq 27$. The lemma holds for $k=6$ by lemma 7. By using lemma 4 any minimum $(K_3,K_{k+2}-e,n)$-good graph has no isolated vertices in the remaining parameter situations. Since

$e(K_3,K_{k+2}-e,n)>e(K_3,K_{k+1},n-2)$ for $k=7$, $21\le n\le 23$ and $k=8$, $24\le n\le 27$, G has no 1-vertices. Using lemmas 1, 4, 7, 11 and data in table I, II and III, in each case one can derive that G cannot have vertices of degree 2. The tedious details are omitted. □

Theorem 4: For all $k\ge 6$

$$e(K_3,K_{k+2}-e,n)\ge 6n-13k. \tag{6}$$

The proof of theorem 4 is broken into few lemmas. We start with a technical definition used in these lemmas.

Definition: Let k_0 be the largest integer (or ∞) such that (6) is true for $6\le k<k_0$ and $n\ge 0$. Define Ψ to be the class of minimum $e(K_3,K_{k+2}-e,n)$-good graph with $6n-13k$ edges such that $0\le k\le k_0$. A smallest counterexample to (6) is a minimum $(K_3,K_{k_0+2}-e,n)$-good graph with less than $6n-13k_0$ edges. Define Λ to be the set of smallest counterexamples to (6).

Note that $\Lambda=\phi$ if $k_0=\infty$, in which case theorem 4 is true. By checking tables I, II and III we observe that $k_0\ge 9$.

Lemma 13: If $G\epsilon\Psi\cup\Lambda$ is a minimum $(K_3,K_{k+2}-e,n)$-good graph, then

(a) $n\ge 3k$,

(b) For any $v\epsilon V(G)$ $e(H_2(v))\ge 6n(H_2(v))-13(k-1)$, furthermore the equality holds if and only if $H_2(v)\epsilon\Psi$,

(c) if $G\epsilon\Psi$ then for all $v\epsilon V(G)$ $Z(v)\le 6deg(v)-7$,

(d) if $G\epsilon\Lambda$ then G is connected.

Proof: The proofs of (a), (b) and (c) are the same as the proof of proposition 5.1.4. in [5], and the proof of (d) is the same as the proof of proposition 5.1.3.(b) in [5]. □

Lemma 14: If G $\epsilon\Psi\cup\Lambda$ then $\delta(G)\ge 3$.

Proof: Assume that G is a minimum $(K_3,K_{k+2}-e,n)$-good graph. If $G\epsilon\Psi$ then G has $6n-13k$ edges, and we use induction on k. The lemma holds for $k=6,7,8$ by previous lemmas, so we may assume that $k\ge 9$. If G has some 1-vertex v, then $H_2(v)$ is a $(K_3,K_{k+1}-e,n-2)$-good graph with at most $6n-13k-1$ edges. By the induction $e(H_2(v))\ge 6(n-2)-13(k-1)=6n-13k+1$, which is a contradiction, thus we have $\delta(G)\ge 2$. If G has some 2-vertex u, then $e(H_2(u))\ge 6(n-3)-3(k-1)=6n-13k-5=e(G)-5$, so $Z(u)\le 5$. If $Z(u)=5$ then there is a 2-vertex x adjacent to u. Let $y\epsilon V(G)$, y adjacent to x and $y\ne u$, then $deg(y)\le 3$. Note that $H_2(u)$ is a minimum $(K_3,K_{k+1}-e,n-3)$-good graph with $6n-13k-5=6(n-3)-13(k-1)$ edges, i.e. $H_2(u)\epsilon\Psi$ and $deg_{H_2(u)}(y)\le 2$, which contradicts the inductive assumption. Thus we can suppose that each 2-vertex u has $Z(u)=4$, and consequently if u is a 2-vertex, then u is on some cycle component C_l in G. By lemma 6 and the proof of theorem 2 we can assume that $4\le l\le 6$. If $l=4$ then G has a component H_1, which is a minimum $(K_3,K_k-e,n-4)$-good graph with $6n-13(k+1)+9$ edges. Now by the definition of Ψ, $e(H_1)\ge 6(n-4)-13(k-2)=6n-13(k+1)+15>6n-13(k+1)+9$. If $l=5$ then G has a component H_2, which is a minimum $(K_3,K_{k-1},n-5)$-graph with $6n-13(k+1)+8$ edges. But by theorem 5.1.1 in [5], $e(H_2)\ge 6(n-5)-13(k-2)=6n-13(k+1)+9>6n-13(k+1)+8$. If $l=6$ then G has a component H_3, which is a minimum $(K_3,K_{k-1}-e,n-6)$-good graph with $6n-13(k+1)+9$ edges. Now by the definition of Ψ, $e(H_3)\ge 6(n-6)-13(k-3)=6n-13(k+1)+16>6n-13(k+1)+9$. Hence $\delta(G)\ge 3$.

If $G\epsilon\Lambda$ then G has $6n-13(k+1)+12$ edges, and by theorems 1 and 3 $n\geq 3k$. Since $k\geq 9$ then by lemma 5 it is easy to see that G has no isolated vertices. If G has a 1-vertex v, then $H_2(v)$ is a $(K_3,K_{k+1}-e,n-2)$-good graph with at most $6n-13(k+1)+11$ edges. By the induction, $e(H_2(v)) \geq 6(n-2)-13(k-1)=6n-13(k+1)+14$, which is a contradiction. If G has a 2-vertex u, then $H_2(u)$ is a $(K_3,K_{k+1}-e,n-3)$-good graph with $6n-13(k+1)+12-Z(u)$ edges. By the induction $e(H_2(u)) \geq 6(n-3)-13(k-1)=e(G)-4$, hence $Z(u)\leq 4$. Since G has no 1-vertices, u is on a cycle component of G. As in the proof of theorem 2, we can see that G has no 2-vertices. Hence $deg(x)\geq 3$ for every $x\epsilon V(G)$. □

Lemma 15: If $G\epsilon\Lambda$ then G is a connected 4-regular graph.

Proof: Suppose that $G\epsilon\Lambda$, i.e. G is a minimum $(K_3,K_{k_0+2}-e,n)$-good graph for some $k_0\geq 9$ and $e(G)<6n-13k_0$, furthermore by theorems 1 and 3, $n>3k_0$. Using the definition of Λ, and applying (1) and induction to G we obtain

$$0\leq\Delta=ne(G)-\sum_{i=0}^{k_0+1} n_i(i^2+e(K_3,K_{k_0+1}-e,n-i-1))$$
$$\leq ne(G)-\sum_{i=0}^{k_0+1} n_i(i^2-6i+6n-13(k_0+1)+20).$$

Note that $\sum_{i=0}^{k_0+1} n_i=n$, hence

$$0\leq\Delta\leq n(e(G)-(6n-13(k_0+1)+12))-\sum_{i=0}^{k_0+1} n_i(i-2)(i-4). \tag{7}$$

We consider three possible cases of solutions to (7) using $e=e(G)<6n-13k_0$: *case* i) $e=6n-13(k_0+1)+11$ and $n=n_3$, *case* ii) $e=6n-13(k_0+1)+12$ and G has some 3-vertices, and *case* iii) $e=6n-13(k_0+1)+12$, $n_3=0$ and G has only 2- and/or 4-vertices. By lemmas 14 and 13(d), to complete the proof of lemma 15 it is sufficient to show that the cases i) and ii) are impossible.

case i) Since $e(G)=3n/2$ and $e=6n-13(k_0+1)+11$, we have $9n=26(k_0+1)-22$. Note that $k_0\geq 9$ and $n>3k_0$, thus $9n>27k_0\geq 26(k_0+1)-22$, which is a contradiction.

case ii) In this case $G\epsilon\Psi$. For any 3-vertex $v\epsilon V(G)$, $H_2(v)$ is a $(K_3,K_{k_0+1}-e,n-3)$-good graph and $e(H_2(v))=e(G)-Z(v)$. Since $e(K_3,K_{k_0+1}-e,n-3)\geq 6(n-3)-13(k_0-1) = e(G)-10$, we have $Z(v)\leq 10$. If $Z(v)=10$, then by lemma 14, there is a vertex $u\epsilon V(G)$ such that u is adjacent to v and $deg(u)=3$. Since $H_2(v)$ is a minimum $(K_3,K_{k_0+1},n-3)$-good graph with $6(n-3)-13(k_0-1)$ edges, hence $H_2(v)\epsilon\Psi$, and by lemma 14 $\delta(H_2(v))\geq 3$. Therefore $Z(u)>10$, which contradicts that $Z(v)\leq 10$ for any 3-vertex v. Hence $Z(v)\neq 10$, which by lemma 14 implies $Z(v)=9$ and by lemma 13(d) G is connected, hence G is a cubic graph. Now $e(G)=3n/2=6n-13(k_0+1)+12$, so we have $9n=26(k_0+1)-24$. Note finally that $k\geq 9$ and $n>3k_0$, hence we have $9n>27k_0\geq 26(k_0+1)-24$, which completes the proof of the lemma. □

Corollary 1: If G is a minimum $(K_3,K_{k+2}-e,n)$-good graph and $G\epsilon\Lambda$, then $n\geq 3k+2$.

Proof: Using the statement and the proof of lemma 15, we see that $e(G)=6n-13(k+2)+12=2n$, which implies that for $k\geq 9$, $n=13(k+1)/4-3\geq 3k+2$. □

Proof of theorem 4: To prove theorem 4, it is now sufficient to show that no 4-regular graph in Λ can exist. By lemmas 13,14,15 and corollary 1, one can prove this theorem by the same sequence of steps as in section 5 of [5], if one replaces k with $k+1$ and k_0 with k_0+1. □

Note that $(K_3, K_7-e, 15)=24<6\cdot 15-13\cdot 5$ [3], hence the condition $k\geq 6$ in theorem 4 is necessary.

Theorem 5:

$$e(K_3, K_{k+2}-e, n)=6n-13k \quad \text{for} \quad k\geq 6 \ \text{and} \ 3k\leq n\leq 13k/4-sign(k \ mod \ 4).$$

Proof: Since $e(K_3, K_{k+2}-e, n)\leq e(K_3, K_{k+1}, n)$, by theorem 4 here and corollary 5.3.4 in [5], we have $6n-13k\leq e(K_3, K_{k+2}-e, n)\leq e(3, K_{k+1}, n)=6n-13k$ for $3k\leq n\leq 13k/4-\text{sign}(k \ mod \ 4)$ and $k\geq 6$. □

Finally we note that $e(K_3, K_{k+2}-e, n)=e(K_3, K_{k+1}, n)$ for $n\leq 13k/4-sign(k \ mod \ 4)$, except 7 nontrivial pairs of values n and k, which are listed in Table IV.

k, n	2,5	3,8	4,11	4,12	4,13	5,14	5,15
$e(K_3, K_{k+2}-e, n)$	4	8	14	18	24	19	24
$e(K_3, K_{k+1}, n)$	5	10	15	20	26	20	25

Table IV.

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