NOTE

Enumeration of All Simple t−(t+7,t+1,2) Designs

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Abstract. We enumerate by computer algorithms all simple $t-(t+7, t+1, 2)$ designs for $1 \le t \le 5$, i.e. for all possible t, and this enumeration is new for $t \ge 3$. The number of nonisomorphic designs is equal to 3, 13, 27, 1 and 1 for $t = 1$, 2, 3, 4 and 5, respectively. We also present some properties of these designs including orders of their full automorphism groups and resolvability.

1. Introduction

A $t-(v, k, \lambda)$ *design* $D = (X, B)$ is a family B of k-subsets, called *blocks*, of a v-set X of *points*, such that every t-subset of X is contained in exactly λ blocks of B. If B has no repeated blocks then the design D is called *simple*. The family of $t-(t+7, t+1, \lambda)$ designs is perhaps one of the most investigated parameter situations, especially for $\lambda = 1$, in which case they are called Witt designs; their in depth presentation together with a number of references can be found in [1]. Only two other values of λ are of interest, namely 2 and 3, since for $\lambda \ge 4$ any such design is the complement of one with $\lambda \le 3$. A $t-(v, k, \lambda)$ design is called *resolvable* if a part of its blocks forms a $t-(v, k, \lambda_r)$ design for some $1 \le \lambda_r < \lambda$. Let $N(\lambda; t, k, v)$ denote the number of nonisomorphic simple $t-(v, k, \lambda)$ designs.

λ	t = 1	2	3	4	5
1	1	1	1	1	1
2	3	13	27	1	1
3	6	332	≥539	≥18	≥13

Table I. Values and bounds for $N(\lambda; t, t+1, t+7)$

The table I summarizes enumeration results of simple $t-(t+7, t+1, \lambda)$ designs by listing known values and bounds for $N(\lambda; t, t+1, t+7)$. The entries in column $t = 1$ count 1-(8,2,λ) designs, which are just regular graphs of degree λ on 8 points. They are usually not considered in the design theory, but are included here for completeness. They also form starting points for our extension algorithms. The uniqueness of the designs with $\lambda = 1$ is discussed in [1], Gibbons calculated $N(2; 2,3,9) = 13$ [3] and Harms, Colbourn and Ivanov obtained the value $N(3; 2,3,9) = 332$ in [4]. Both proofs in [3] and [4] relied on computer algorithms. The remaining entries of table I are obtained in this paper. For $\lambda = 2$ we were able to enumerate all of them, and for $\lambda = 3$ we have found some such designs including all for $t = 2$ (as in [4]), and all with an automorphism of order 3 not fixing any block, for all t. We postpone their description until all such designs are enumerated, i.e. when the case of $t-(t+7, t+1, \lambda)$ designs is

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closed. These special $t-(t+7, t+1, 3)$ designs were necessary in the analysis of possible 4-(12,6,6) designs [6], whose existence is still in question. The existence of $t-(t+7, t+1, \lambda)$ designs for all parameter situations was known already in 1977 (see Brouwer [2]).

2. Results

A cycle of length 8, two squares, and a triangle and a pentagon are the only 2-regular graphs on eight points, which can be treated as 1-(8,2,2) designs. Obviously the first two are resolvable into 1-(8,2,1) designs. All three of them extend to a 2-(9,3,2) design.

The 13 2-(9,3,2) designs found by Gibbons [3] have the full automorphism groups of orders 80, 18, 8(2), 6(3), 2(3) and 1(3), where the number in parenthesis shows the number of corresponding designs (if larger than 1). Exactly two of them, with groups of order 18 and 6, are resolvable into two 2-(9,3,1) designs. 11 out of them, including both resolvable ones, extend to a 3-(10,4,2) design. The two nonextendible 2-(9,3,2) designs have group orders 6 and 2.

We have found that there are exactly 27 nonisomorphic 3-(10,4,2) designs; each of them has 60 blocks and each point appears in 24 blocks. Their full automorphisms groups have orders 400, 40, 20, 16, 8(5), 4(6), 2(11) and 1. The 3-(10,4,2) design with group of order 20 is the only one resolvable into two 3-(10,4,1) designs, and also it is the only one extendible to a 4-(11,5,2) design. Furthermore, up to isomorphisms, it extends uniquely to the well known 4-(11,5,2) design. Consequently, by Alltop's extension theorem, there exists a unique 5-(12,6,2) design. We note that one can easily see that any $5-(12,6,\lambda)$ design must be closed under block complementation.

Theorem 1. *There exist unique, up to isomorphism,* $4-(11,5,2)$ *and* $5-(12,6,2)$ *designs, and both of them are resolvable.*

Kramer and Mesner [5] gave the precise analysis of mutually disjoint $S(t, t+1, t+7)$ Steiner systems, in particular in the case of resolvable 4-(11,5,2) designs. Since the unique 4-(11,5,2) and 5-(12,6,2) designs (both resolvable) that occur are the ones studied in [5], we refer the reader to this paper for more information, as well as to [1]. We remark that their automorphism groups have orders 110 and 1320, respectively, and recall that the orders of groups for the Witt design $t-(t+7, t+1, 1)$ are 384, 432, 1440, 7920 and 95040, for $t = 1$, 2, 3, 4 and 5, respectively. We also observe that our enumeration implies the uniqueness of 4-(11,5,5) and 5-(12,6,5) designs (the complements of unique designs), which permits an answer to a question posed by Kramer and Mesner [5] formulated in the next theorem.

Theorem 2. *The* $4-(11,5,\lambda)$ *and* $5-(12,6,\lambda)$ *designs are not resolvable for* $\lambda = 3$ *and* $\lambda = 5$.

Proof: Assume that a 4-(11,5,5) design D can be partitioned into a $4-(11,5,\lambda_1)$ design D_{λ_1} and a $4-(11,5,\lambda_2)$ design D_{λ_2}, for some $1 \le \lambda_1 < \lambda_2$, $\lambda_1 + \lambda_2 = 5$. If $\lambda_1 = 1$ then resolved $\bar{D}$ and D_{λ_1} form three mutually disjoint $S(4,5,11)$'s. If $\lambda_1 = 2$ then both $\bar{D}$ and D_{λ_1} can be resolved to a total of four mutually disjoint $S(4,5,11)$'s. Similarly, the uniqueness and resolvability of the 4-(11,5,2) design imply that any resolvable 4-(11,5,3) design is formed by three mutually disjoint $S(4,5,11)$'s. This contradicts a theorem of Kramer and Mesner [5] stating that there can be at most two mutually disjoint Steiner systems $S(4,5,11)$. The same reasoning is valid for the cases of 5-(12,6,5) and 5-(12,6,3) designs. □

After a moment of thought one can easily see that theorems 1 and 2 leave open only one nontrivial resolvability question concerning these designs. We formulate this question in three equivalent forms as follows.

Does there exist a resolvable $4-(11,5,4)$ *design ?*
Do there exist a Steiner system $S(4,5,11)$ *and a* $4-(11,5,3)$ *design which are disjoint ?*

Do there exist two disjoint 4–(11,5,3) *designs ?*

In order to obtain the above enumeration we used natural algorithms starting from graphs ($t = 1$) and then performing consecutive extensions to ($t+1$)-designs. Each specific design extension and resolvability step was completed by solving appropriate systems of 0-1 integer linear equations. For automorphism groups and design isomorphism we did the calculations twice: with the software developed by the author, and independently with the program *nauty* written by B.D. McKay [7]. The completion of enumeration of all $t-(t+7, t+1, 3)$ designs with our current approach would require an extreme man/machine effort, unless a more efficient method is deviced or some strong properties of such designs are discovered.

References

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