The Ramsey Numbers $R(K_3, K_3 - e)$ and $R(K_3, K_9 - e)$

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Abstract. We give a general construction of a triangle free graph on 4p points whose complement does not contain $K_{p,2} - e$ for $p \geq 4$. This implies that the Ramsey number $R(K_3, K_4 - e) \geq 4k - 7$ for $k \geq 6$. We also present a cyclic triangle free graph on 30 points whose complement does not contain $K_5 - e$. The first construction gives lower bounds equal to the exact values of the corresponding Ramsey numbers for $k = 6, 7$ and 8. The upper bounds are obtained by using computer algorithms. In particular, we obtain two new values of Ramsey numbers $R(K_3, K_4 - e) = 25$ and $R(K_3, K_9 - e) = 31$, the bounds $36 \leq R(K_3, K_{10} - e) \leq 38$, and the uniqueness of extremal graphs for Ramsey numbers $R(K_3, K_6 - e)$ and $R(K_3, K_7 - e)$.

1. Introduction and Notation

The two color Ramsey number $R(G, H)$ is the smallest integer $n$ such that for any graph $F$ on $n$ vertices, either $F$ contains $G$ or the complement $\overline{F}$ contains $H$.

In this paper we consider the case $G = K_3$ and $H = K_k - e$, the complete graph $K_k$ minus an edge. Table 1 contains the values of some related Ramsey numbers.

The entries of the first two rows are given by easy equalities $R(K_3 - e, K_k - e) = 2k - 5$ and $R(K_3 - e, K_k) = 2k - 1$, which can be derived by straightforward reasoning. The value 21 of $R(K_3, K_k - e)$ for $k = 7$ was obtained by Grenda and Harborth in 1982 [5], where the authors list also all the values for $k \leq 6$. Recently, McKay and Zhang have calculated $R(K_3, K_8) = 28$ [7], other references for the classical case $R(K_3, K_k)$ can be found in [6], [7], [8], [9].

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$G$ $H$

| $K_3 - e$ | $K_k - e$ |

Table 1. Four related types Ramsey numbers $R(G, H)$

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All the graphs considered here are triangle free. Throughout this paper we adopt the following notation:

- $\overline{G}$ — complement of graph $G$
- $(G, H)$-good graph $F$ — graph $F$ does not contain $G$ and $\overline{F}$ does not contain $H$
- $(G, H, n)$-good graph — $(G, H)$-good graph on $n$ vertices
- $K_p - e$ — complete graph on $p$ vertices without one edge
- $G \equiv H$ — graphs $G$ and $H$ are isomorphic
- $e(G, H, n)$ — minimum number of edges in any $(G, H, n)$-good graph
- $E(G, H, n)$ — maximum number of edges in any $(G, H, n)$-good graph
- $G[S]$ — subgraphs of graph $G$ induced by the set of vertices $S$
- $C_p$ — cycle of length $p$

2. Constructions

Construction 1: For $p \geq 1$, let $G_p = (V_p, E_p)$ be the graph on $4p$ vertices defined by:

$$V_p = \bigcup_{i=1}^{4p} X_i, \text{ where } X_i = \{x_{in} : 1 \leq n \leq p\}, \text{ and}$$

$$E_p = \{\{x_{in}, x_{i+1,m}\} : i = 1, 3, 1 \leq n, m \leq p; \quad n \neq m\} \cup$$

$$\{\{x_{in}, x_{jn}\} : i = 1, 2, j = 3, 4, 1 \leq n \leq p\}.$$  

Observe that $G_p$ is a regular graph of degree $p + 1$ and that the induced graphs $G_p[X_1 \cup X_2]$ and $G_p[X_3 \cup X_4]$ are isomorphic to the complete bipartite graph $K_{p,p}$ with a 1-factor deleted. We say that vertex $x_{in}$ is on level $n$. The set $V_p$ is formed by $p$ levels, each of them inducing a $C_4$ in $G_p$, in particular $G_1 \equiv C_4$. We leave for the reader, as an easy but interesting and time consuming exercise, to show that the graph $G_4$ on 16 vertices is isomorphic to the well known extremal graph related to the Ramsey number $R(5, 3, 3)$, which has vertices in $G[F(16)]$ and edges connecting points whose difference is a cube $[4]$.

Theorem 1. The graph $G_p$ is a $(K_3, K_{p,2} - \epsilon, 4p)$-good graph for $p \geq 4$.

Proof: One can easily verify that $G_p$ has no triangles. Let $S$ be any set of vertices $S \subseteq V_p, |S| = p + 2$. We will show that for $p \geq 4$ the induced graph $G_p[S]$ has at least two edges. If $S$ has at least three vertices on the same level, then $G_p[S]$ has clearly at least two edges; otherwise $S$ has at least two levels $n$ and $m$ with two vertices, say $a$ and $b$ on level $n$ and $c$ and $d$ on level $m$. Since $p \geq 4$, $S$ has at least two more vertices, $u$ and $v$, on other levels. Suppose that $G_p[S]$ has at most one edge. Then without loss of generality we can assume that $u$ is not connected to any vertex in $\{a, b, c, d\}$ and $v \in X_3$. Hence $\{a, b, c, d\} \subseteq X_1 \cup X_2 \cup X_3$.  

one can easily check that $G_p[\{a, b, c, d\}]$ has at least two edges.
Corollary 1. \( R(K_3, K_k - e) \geq 4k - 7 \) for \( k \geq 6 \).

Proof: Using Theorem 1, the lower bound is established by the graph \( G_{k-2} \).

Construction 2: Define graph \( H = (Z_{20}, E) \) by

\[
E = \{(i, j) : i, j \in Z_{20}, \ i - j = \pm 1, \pm 3, \pm 9, \pm 14\}.
\]

It is not very difficult, but again tedious, to check that the graph \( H \) is triangle free, has exactly 30 independent sets of size 8, namely the neighborhoods of vertices, and finally two different neighborhoods intersect in less than 7 points. Consequently the graph \( \overline{H} \) does not contain \( K_9 - e \), since the opposite would imply the existence of two independent sets of size 8 intersecting in seven points. Thus we can formulate the next Corollary.

Corollary 2. \( R(K_3, K_9 - e) \geq 31 \).

3. Enumerating small Graphs

In [8] the construction of a data base of all triangle free graphs with maximal independent set of size not larger than 5 was reported. This data base contains all \((K_3, K_k - e)\)-good graphs for \( k \leq 6 \). These were extracted and the number of them is shown in the following tables for \( k = 3, 4, 5 \) and 6. A blank entry in a table denotes 0. Note that the values of \( e(K_3, K_k - e, n) \) and \( E(K_3, K_k - e, n) \) can be easily read by finding the location of the first and last nonzero entries in column \( n \) of the corresponding table. Observe also that \( G_4 \) is the unique \((K_3, K_6 - e, 16)\)-good graph.

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Table II. Number of \((K_3, K_7 - e)\)-good graphs

The graphs contributing to the entries of Table II were constructed independently by hand. The correctness of the data in Tables III, IV and V was double checked by running extension algorithm used in the next section, i.e. the set of graphs obtained by extraction from the data base of \((K_3, K_k)\)-good graphs was identical to the set of \((K_3, K_4 - e)\)-good graphs obtained by consecutive extensions followed by elimination of isomorphic copies of graphs. We also observe
that column 10 of Table IV corresponds to Lemma 2 in [1], likewise the graph $G_4$ was also identified as a $(K_3, K_6 - e)$-good graph by Faudree, Rousseau and Schelp in [2] and it is represented by a 1 in column 16 of Table V. Finally, we note a "curiosity" in column 10 of Table IV, namely the nonexistence of $(K_3, K_5 - e, 10)$-good graphs for $16 \leq e \leq 19$ edges. This is the first such hole known to the author (for additional data see [8], [9]).

In Tables II-VI some particular graphs of special interest have been marked as follows: $a$ — square $K_{2,2}, b$ — $K_{3,3}, c$ — $K_{4,4}, d$ — graphs from Lemma 2 in [1], $e$ — Petersen graph, $f$ — $K_{5,5}, g$ — graph on $GF(16), \{i, j\} \in E$ iff $i - j = x^2$, isomorphic to $G_4$, and $h$ — unique $(K_3, K_7 - e, 20)$-good graph found by Grenda and Harborth in [5], isomorphic to $G_5$.

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Table III. Number of $(K_3, K_4 - e)$-good graphs

4. Extensions

The system of algorithms with their implementations to construct all $(K_3, K_k, n)$-good graphs with $e$ edges was described in [8] and used extensively in [9]. This technique requires the previous knowledge of all $(K_3, K_{k-1}, \pi)$-good graphs with $\pi$ edges, for $\pi < n$ and $\pi$ ranging over the set of values, which can be determined by the method of Graver and Yackel [3]. The key to this method in our case is contained in the following Lemma.

Lemma 1 (variation of proposition 4 in Graver and Yackel 3 - 1968). For any $(K_3, K_k - e, n)$-good graph $G$ with $e$ edges

$$
\Delta = ne - \sum_{i=0}^{k-1} \pi_i(e(K_3, K_{k-1} - e, n - i - 1) + i^2) \geq 0;
$$

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Table IV. Number of $(K_3, K_5 - e)$-good graphs

where $n_i$ is the number of vertices of degree $i$ in $G$, $n = \sum_{i=0}^{k-1} n_i$ and $2 e = \sum_{i=0}^{k-1} i \cdot n_i$.

Lemma 1 gives reasonable lower bounds for $e(K_3, K_k - e, n)$ provided good lower bounds for $e(K_3, K_{k-1} - e, n - i - 1)$ are given. Furthermore, it permits the design of extension algorithms based on the ones used by Grinstead and Roberts in 1982 [6] to evaluate $R(3, 9)$. Similarly as in [8], [9] we have implemented these algorithms for the case of $(K_3, K_k - e)$-good graphs and they have produced the results gathered in Tables VI and VII.

Let $e_k(n) = e(K_3, K_k - e, n)$ and let $N_k(n, e)$ be the number of nonisomorphic $(K_3, K_k - e, n)$-good graphs with $e$ edges. Table VI presents all nonzero values of $e_3(n)$, and $N_3(n, e)$ for some values of $n$ and $e$. Table VII contains similar data for $(K_3, K_k - e, n)$-good graphs. In the case of $(K_3, K_7 - e, n)$-good graphs we have found all of them for $n \geq 18$: there are 225 such graphs for
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Table V. Number of $(K_5, K_\epsilon - \epsilon)$-good graphs
\( n = 18 \) with the number of edges ranging from 43 to 51, and unique graphs for \( n = 19 \) and 20. The graph \( G_5 \) is the unique \((K_3, K_7 - e, 20)\)-good graph and obviously it is isomorphic to the graph defined by Grenda and Harborth in [S]. Also, there exist a unique \((K_3, K_7 - e, 19)\)-good graph, which can be obtained from \( G_5 \) by the deletion of one vertex. The nonexistence of a \((K_3, K_8 - e, 25)\)-good graphs implies, by Corollary 1, that \( R(K_3, K_8 - e) = 25 \). We note that \( G_6 \) has 84 edges, thus it is not a minimum graph. For further calculation of \( R(K_3, K_9 - e) \) we need only the graphs in column \( n = 22 \) in Table VII and the values of \( e_8(n) \) for \( 22 \leq n \leq 24 \).

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Table VI. Number of \((K_3, K_7 - e, n)\)-good graphs

**Theorem 2.** \( R(K_3, K_8 - e) = 25 \) and \( R(K_3, K_9 - e) = 31 \).

**Proof:** Corollaries 1 and 2 establish that 25 and 31 are lower bounds for \( R(K_3, K_8 - e) \) and \( R(K_3, K_9 - e) \), respectively. The fact that these values are also upper bounds follows from the calculations described above. For example, to prove \( R(K_3, K_9 - e) \leq 31 \) assume that \( G \) is a \((K_3, K_9 - e, 31)\)-good graph with \( e \) edges. Then \( G \) can have vertices of degree 6, 7 and 8, and by Lemma 1 we have:

\[
\Delta = 31e - (n_6(36+80) + n_7(49+70) + n_8(64+59)) = 31(e - 116) - 3n_7 - 7n_8 \geq 0
\]
There are three solutions in nonnegative integers for the latter, which are listed in Table VIII. One can easily conclude that $G$ must be an extension of a $(K_3, K_8 - e, 22)$-good graph with 59 or 60 edges. There are 15 such graphs (see column 22 in Table VII). Running extension algorithm on these graphs did not produce $G$. Thus $R(K_3, K_9 - e) \leq 31$.

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<th>number of vertices $n$</th>
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<td>$\geq 169$ 7 2 1 1</td>
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<td>60 71 81</td>
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Table VII. Number of $(K_3, K_8 - e, n)$-good graphs

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Table VIII. Theorem 2

Using only Lemma 1 and Table VII we obtain:

$$e(K_3, K_9 - e, 30) \geq 111,$$
$$e(K_3, K_9 - e, 29) \geq 100,$$ and
$$e(K_3, K_9 - e, 28) \geq 90.$$

The latter inequalities and Lemma 1 imply the nonexistence of a $(K_3, K_{10} - e, 39)$-good graph, hence $R(K_3, K_{10} - e) \leq 39$. If we could prove $e(K_3, K_9 - e, 28) > 90$ then $R(K_3, K_{10} - e) \leq 38$. We have $36 = R(K_3, K_9) \leq R(K_3, K_{10} - e)$, so the lower bound also seems to be weak. There exists a good chance to calculate the exact value of $R(K_3, K_{10} - e)$. We conclude by stating the following Theorem.

Theorem 3. $36 \leq R(K_3, K_{10} - e) \leq 39$.

References


