Discrete Mathematics 92 (1991) 65–77
North-Holland

On the covering of t-sets with $(t+1)$-sets: $C(9, 5, 4)$ and $C(10, 6, 5)$

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Received 19 September 1989

Dedicated to Professor R.G. Stanton on the occasion of his 68th birthday.

Abstract

De Caen, D., D.L. Kreher, S.P. Radziszowski and W.H. Mills, On the covering of t-sets with $(t+1)$-sets: $C(9,5,4)$ and $C(10,6,5)$, Discrete Mathematics 92 (1991) 65–77.

A (v, k, t) covering system is a pair $(X, \mathscr{B})$ where X is a v-set of points and $\mathscr{B}$ is a family of k-subsets, called blocks, of X such that every t-subset of X is contained in at least one block. The minimum possible number of blocks in a (v, k, t) covering system is denoted by $C(v, k, t)$. It is proven that there are exactly three non-isomorphic systems giving $C(9, 5, 4) = 30$, and a unique system giving $C(10, 6, 5) = 50$.

1. Introduction

In this paper we determine the exact value of two set-covering numbers, $C(9, 5, 4)$ and $C(10, 6, 5)$. For more information on the classical set-covering problem, see for instance [4]. We recall that a (v, k, t) covering system is a pair $(X, \mathscr{B})$ where X is a v-set of points and $\mathscr{B}$ is a family of k-subsets, called blocks, of X such that every t-subset of X is contained in at least one block. $C(v, k, t)$ denotes the smallest possible number of blocks in a (v, k, t) covering system. A Steiner system $S(t, k, v)$ is a (v, k, t) covering system such that each t-set is

* Research supported by NSERC grant U0463.
** Research supported by NSF grant CCR-8711229.

covered exactly once. Thus, an $S(t, k, v)$ exists if and only if $C(v, k, t) = \binom{v}{t}/\binom{k}{t}$. For example, it is well known that the Steiner systems $S(2, 3, 7)$ and $S(3, 4, 8)$ exist and are unique; thus there are unique covering systems giving $C(7, 3, 2) = 7$ and $C(8, 4, 3) = 14$.

If $(X, \mathcal{B})$ is a (v, k, t) covering system and $I \subseteq X$, $|I| = i$, then the *derived system with respect to I* is $\mathcal{B}_I = \{B - I : I \subseteq B \in \mathcal{B}\}$ and we denote by $\deg(I) = \deg(\mathcal{B}; I)$ the number of blocks in $\mathcal{B}$ that contain I. We also use $\mathcal{B}[I]$ for $\{B: I \subseteq B \in \mathcal{B}\}$ *the system of blocks containing I*. Thus $\deg(I) = |\mathcal{B}_I| = |\mathcal{B}[I]|$ and $(X - I, \mathcal{B}_I)$ is a $(v - i, k - i, t - i)$ covering system. We will frequently use the following easy identity:

$$(k - i) \cdot \deg(I) = \sum_{a \in X - I} \deg(I \cup \{a\}). \tag{1}$$

2. The minimum-sized (9, 5, 4) covering systems

Lemma 1. *Let $(X, \mathcal{B})$ be a (7, 3, 2) covering system with $|\mathcal{B}| = 7$. Then there is a bijection of X onto the Galois field GF(7) that maps $\mathcal{B}$ onto the set of seven triples of the form $\{i, i + 1, i + 3\}$.*

Lemma 1 is simply the well-known result that the projective plane of order 2 is unique, together with known facts about its structure. It can be verified directly without difficulty. Also Lemma 1 follows from the proof of our next lemma.

Lemma 2. *Let $(X, \mathcal{B})$ be a (7, 3, 2) covering system with $|\mathcal{B}| = 8$. Then one of these eight triples is superfluous, i.e. the other seven blocks form a (7, 3, 2) covering system.*

Proof. We have $\sum_{a \in X} \deg(a) = 24$. Since $\deg(a) \geq C(6, 2, 1) = 3$ for all $a \in X$, we have $\deg(a) = 3$ for at least four different elements a. Let x and y be two points of degree 3. Then $\mathcal{B}$ contains triples $B_1 = \{x, y, z\}$, $B_2 = \{x, a, b\}$, $B_3 = \{x, c, d\}$, where $X = \{x, y, z, a, b, c, d\}$. Without loss we can assume that $\deg(a) = 3$. The three pairs that form blocks with a are pairwise disjoint, and thus $\{y, a, b\} \notin \mathcal{B}$; similarly $\{y, a, z\} \notin \mathcal{B}$, since $\deg(y) = 3$. Thus $\{y, a\}$ can only be covered by $\{y, a, c\}$ or $\{y, a, d\}$. Without loss we suppose that $\{y, a, c\} = B_4 \in \mathcal{B}$. This now forces $\{y, b, d\} = B_5 \in \mathcal{B}$ and $\{z, a, d\} = B_6 \in B$.

The remaining two blocks must cover the three remaining pairs, $\{z, b\}$, $\{z, c\}$ and $\{b, c\}$. Therefore, one of the last two blocks must cover at least two of these three pairs, and so must be $B_7 = \{z, b, c\}$. But now the triples B_i, $i = 1$ to 7, form an $S(2, 3, 7)$ and the remaining block B_8 is superfluous. □

We remark that if the eight triples are distinct, then the superfluous triple is the only one that contains more than one repeated pair (i.e. a pair having degree greater than one).

Lemma 3. $C(8, 4, 3) = 14$ *and the unique minimal* $(8, 4, 3)$ *covering system* $(X, \mathscr{B})$ *is the Steiner system* $S(3, 4, 8)$. *Furthermore, for any* $x \in X$ *there are seven blocks containing* x *and* $(X - \{x\}, \mathscr{B}_x)$ *is an* $S(2, 3, 7)$ *Steiner system. The remaining seven blocks of the* $S(3, 4, 8)$ *Steiner system* $(X, \mathscr{B})$ *are the complements of the above blocks.*

Lemma 3 is well known, and also follows from the proof of our next lemma.

Lemma 4. *Let* $(X, \mathscr{B})$ *be an* $(8, 4, 3)$ *covering system with* $|\mathscr{B}| \leq 16$. *Then some* 14 *of these quadruples form an* $(8, 4, 3)$ *covering system.*

Proof. Without loss we may suppose that no proper subset of $\mathscr{B}$ is an $(8,4,3)$ covering system. For each x in X,

$$\deg(x) \geq C(7, 3, 2) = 7 \tag{2}$$

Also, putting $q = |\mathscr{B}|$ then, by (1) we have

$$64 \geq 4q = \sum_{x \in X} \deg(x) \tag{3}$$

Now suppose that $\deg(a) = 8$ for some $a \in X$. Deleting a from the 8 quadruples containing it, we obtain a $(7,3,2)$ covering system on $X - \{a\}$ with eight triples. By Lemma 2, we see that one quadruple is of the form $\{a, b, c, d\}$ where $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are repeated triples. Since no proper subset of $\mathscr{B}$ covers all triples, it follows that $\{b, c, d\}$ is not a repeated triple. If $7 \leq \deg(b) \leq 8$ then, $(X - \{b\}, \mathscr{B}_b)$ contains an $S(2, 3, 7)$ which has the triple $\{a, c, d\}$. Since $\{a, b, c\}$ and $\{a, b, d\}$ are repeated triples there are triples $\{a, c, x\}$ and $\{a, d, y\}$ in $\mathscr{B}_b$ and they cannot possibly be part of this $S(2, 3, 7)$. This contradicts $\deg(b) = 8$. We conclude that b and similarly c, and d each have degree at least 9. By (2) and (3) we must have at least three points of degree 7.

On the other hand if $\deg(a) \neq 8$ for all a in X, then by (2) and (3) at least four points have degree 7. Thus, in either case we have at least three distinct elements, say x, y, and z, of degree 7. By the uniqueness of $S(2, 3, 7)$, we can take the quadruples of $\mathscr{B}$ that contain x to be $\{x, y, z, a\}$, $\{x, y, b, c\}$, $\{x, y, d, e\}$, $\{x, z, b, d\}$, $\{x, z, c, e\}$, $\{x, a, b, e\}$, $\{x, a, c, d\}$. It is elementary to check that this forces the remaining quadruples containing y or z to be $\{y, z, b, e\}$, $\{y, z, c, d\}$, $\{y, a, b, d\}$, $\{y, a, c, e\}$, $\{z, a, b, c\}$ and $\{z, a, d, e\}$. These 13 quadruples cover all triples of X except for $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$ and $\{c, d, e\}$. There are at most three quadruples left to cover these four triples, so there is a quadruple in $\mathscr{B}$ that covers at least two of them. It must be $\{b, c, d, e\}$. We now have a set of 14 quadruples of $\mathscr{B}$ that cover all triples of X. □

We remark that if the quadruples in $\mathscr{B}$ are distinct, then the set of 14 quadruples of Lemma 4 that form an $(8,4,3)$ covering system is uniquely determined by the fact that none of these 14 quadruples contains more than two repeated triples.

Lemma 5 *Let* $(X, \mathscr{B})$ *be a minimal* $(9, 5, 4)$ *covering system. If there are two points x and y of degree at most* 16, *then, relabeling points if necessary,* $\mathscr{B}$ *contains a set* $\dot{\mathscr{B}}$ *of* 21 *quintuples of the form*:

(A) $\{x, y, i, i+1, i+3\}$,

(B) $\{x, i+2, i+4, i+5, i+6\}$,

(C) $\{y, i+2, i+4, i+5, i+6\}$,

where $i \in \mathrm{GF}(7)$. *Furthermore,* $\deg(x, y) = 7$ *in* $\mathscr{B}$.

Proof. By Lemma 4 $\mathscr{B}_x$ contains an $S(3, 4, 8)$ system $(X - \{x\}, \mathscr{B}_x')$. Thus by the uniqueness of $S(3, 4, 8)$, we may assume that $\mathscr{B}$ contains the 14 quintuples of types (A) and (B) listed above. Similarly, $\mathscr{B}_y$ also contains an $S(3, 4, 8)$ system $(X - \{y\}, \mathscr{B}_y')$.

We claim that $\deg(x, y) = 7$ in $\mathscr{B}$. For, suppose that $Q = \{a, b, c, d\} \in \mathscr{B}_x - \mathscr{B}_x'$. Then $\{x, a, b, c\}$, $\{x, a, b, d\}$, $\{x, a, c, d\}$ and $\{x, b, c, d\}$ are repeated quadruples in $\mathscr{B}$, but since $\mathscr{B}$ is a minimal $(9, 5, 4)$ covering system, the quadruple $\{a, b, c, d\}$ is not repeated. This is possible only if a, b, c, d each have degree at least 17, using again Lemma 4 and the remark following it. In particular y is not in Q; thus $\deg(x, y) = 7$ as claimed.

It now follows that in $\mathscr{B}_y$ the quadruples through x are $\{x, i, i+1, i+3\}$, $i \in \mathrm{GF}(7)$. Since an $S(3, 4, 8)$ contains the complement of any block in it, it follows that the quintuples of type C are also in $\mathscr{B}$. □

Lemma 6. *Let* $(X, \mathscr{B})$ *be a minimal* $(9, 5, 4)$ *covering system with two points x and y of degree at most* 16. *Then* $\deg(a, b) \geq 9$ *for all* $a, b \in X - \{x, y\}$.

Proof. We may assume that $X = \{x, y\} \cup \mathrm{GF}(7)$ and that $\mathscr{B}$ contains the 21 blocks $\dot{\mathscr{B}}$ of Lemma 5. For any pair $\{a, b\}$ of distinct elements of GF(7), there are four elements c_1, c_2, c_3, c_4 so that $\{a, b, c_i\}$ is not of the form $\{i, i+1, i+3\}$. For each i there must be a d_i such that the quintuples $\{x, a, b, c_i, d_i\}$ and $\{y, a, b, c_i, d_i\}$ are both in $\dot{\mathscr{B}}$. This implies that for each i, $\deg(a, b, c_i) \geq 4$, since $\deg(a, b, c_i) \geq C(6, 2, 1) = 3$ but the unique minimal $(6, 2, 1)$ covering system consists of disjoint pairs. Thus, given two distinct a and b in GF(7), we have $\deg(a, b, c) \geq 4$ for at least four values of c in GF(7), and $\deg(a, b, c) \geq 3$ for the remaining c in X. Thus by (1) $3\deg(a, b) = \sum_{c \neq a,b} \deg(a, b, c) \geq 4 \cdot 4 + 3 \cdot 3 = 25$ and so $\deg(a, b) \geq 9$. □

Proposition 7. $C(9, 5, 4) \geq 30$.

Proof. Let $(X, \mathscr{B})$ be a minimal $(9, 5, 4)$ covering system, with $|\mathscr{B}| = m \leq 29$. We have by (1)

$$145 \geq 5m = \sum_{a \in X} \deg(a).$$

Since $\deg(a) \geqslant 14$ for all a it follows that at least 3 points x, y and z have degree at most 16. However, by Lemma 6 we have $4\deg(z) = \sum_{a \neq z} \deg(a, z) \geqslant 6 \cdot 9 + 2 \cdot 7 = 68$ and so $\deg(z) \geqslant 17$, a contradiction. □

We will shortly exhibit some $(9, 5, 4)$ covering systems with 30 quintuples, so that indeed $C(9, 5, 4) = 30$. We now mention another consequence of Lemma 6.

Proposition 8. *Let* $(X, \mathscr{B})$ *be any* $(9, 5, 4)$ *covering system with at most* 31 *blocks. Then there are at most two points of degree at most* 16.

Proof. We may assume without loss that $\mathscr{B}$ is minimal (in the sense that there is no superfluous block), so that Lemma 6 applies. Suppose that x, y and z have degree at most 16. Then by Lemma 6, every pair except perhaps $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ has degree at least 9. Also the exceptional pairs have degrees at least 7. Therefore, $310 \geqslant |\mathscr{B}|\binom{5}{2} = \sum_{a,b} \deg(a, b) \geqslant 9 \cdot \{\binom{9}{2} - 3\} + 3 \cdot 7 = 318$ which is a contradiction. □

Proposition 9. *Let* $(X, \mathscr{B})$ *be a* $(9, 5, 4)$ *covering system with* 30 *blocks. Then it either has*:

(i) *one point of degree* 14 *and* 8 *points of degree* 17, *or*
(ii) 2 *points of degree* 14, 4 *of degree* 17, *and* 3 *of degree* 18.

Proof. We have

$$\sum_{a \in X} \deg(a) = 150. \tag{4}$$

It follows easily from (4) and Proposition 8 that there are at least four points of degree precisely 17. Also, (4) implies that if there is a unique point of degree at most 16, then it has degree 14 and the remaining points each have degree 17. It turns out in this case that there is precisely one such system, up to isomorphism. This will be presented shortly.

Hence, we may assume for the remainder of this proof that the points x and y both have degree at most 16. By Lemma 5, $\mathscr{B}$ contains the system $\dot{\mathscr{B}}$ of 21 quintuples. Let $\ddot{\mathscr{B}}$ be the remaining 9 blocks. We know that four points, say a_1, a_2, a_3 and a_4 have degree 17. Thus $\deg(\ddot{\mathscr{B}}, a_i) = 6$ for $i = 1$ to 4, since $\deg(\dot{\mathscr{B}}; a_i) = 11$. Note that $\deg(\dot{\mathscr{B}}; u, v) = 5$ for any $u \neq v$ in GF(7). Since $\deg(u, v) \geqslant 9$, it follows that $\deg(\ddot{\mathscr{B}}; u, v) \geqslant 4$. Suppose next that $\deg(b) \geqslant 19$ for some b. Then we have $\deg(\ddot{\mathscr{B}}; b) \geqslant 8$, and so by the pigeonhole principle $\deg(\ddot{\mathscr{B}}; b, a_i) \geqslant 5$ for all i. Hence,

$$4\deg(\ddot{\mathscr{B}}; a_1) = \sum_{u \neq a_1} \deg(\ddot{\mathscr{B}}; a_1, u) \geqslant 0 + 0 + 5 + 5 \cdot 4 = 25.$$

This contradicts $\deg(\ddot{\mathscr{B}}; a_1) = 6$, and so $\deg(b) \leqslant 18$ for every b. If it happens, for example, that y has degree 15 or 16 then, there is a superfluous quadruple containing y and a point u of degree 17. But then, we have

$$68 = 4\deg(u) = \sum_{p} \deg(u,p) = \deg(u,x) + \deg(u,y) + \sum_{p\in \mathrm{GF}(7)} \deg(u,p)$$
$$\geq 7 + 8 + 6\cdot 9 = 69,$$

a contradiction. This clearly implies that x and y have degree 14, a_i has degree 17 for $i = 1$ to 4 and the remaining three points have degree 18. □

It is not too difficult to determine, up to isomorphism, all (9, 5, 4) covering systems $(X, \mathscr{B})$ having degree distribution of Proposition 9(ii). Indeed, let x and y be the two points of degree 14. By Lemma 5, 21 of the 30 blocks are determined and are given in the statement of that lemma. We need only determine how nine additional 5-subsets of GF(7) may be added to obtain a (9, 5, 4) covering system.

We continue with the notation above: our point-set is $\{x, y\} \cup \mathrm{GF}(7)$, the set $\dot{\mathscr{B}}$ of 21 blocks containing x or y is listed in the statement of Lemma 5. Now there are three elements of GF(7), say a, b and c, that have degree 18 in $\mathscr{B}$, and hence degree 7 in $\ddot{\mathscr{B}} = \mathscr{B} - \dot{\mathscr{B}}$. So each of a, b and c appear in 7 of the 9 blocks of $\ddot{\mathscr{B}}$. The other four points of GF(7) say d, e, f and g, appear in 6 of the 9 blocks of $\ddot{\mathscr{B}}$.

Lemma 10. *If* $i, j \in \mathrm{GF}(7)$ *and* $\deg(i,j) \geq 10$, *then* $\deg(i) = \deg(j) = 18$.

Proof. We set $F = \mathrm{GF}(7)$. Then we have

$$4\deg(i) = \sum_{x\neq i} \deg(i,z) = \deg(i,x) + \deg(i,y) + \deg(i,j) + \sum_{k\in F-\{i,j\}} \deg(i,k)$$
$$\geq 7 + 7 + 10 + 5\cdot 9 = 69.$$

Thus $\deg(i) = 18$. □

It follows from Lemma 10 that each pair of elements of GF(7), except perhaps for the pairs $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, has degree 9. (Recall that each such pair has degree at least 9 by Lemma 6.) It then follows from (1) that

$$72 = 4\deg(a) = \sum_{u\neq a} \deg(a,u) = \deg(a,b) + \deg(a,c) + 50,$$

$$72 = 4\deg(b) = \sum_{u\neq b} \deg(b,u) = \deg(b,c) + \deg(b,a) + 50,$$

$$72 = 4\deg(c) = \sum_{u\neq c} \deg(c,u) = \deg(c,a) + \deg(c,b) + 50.$$

Thus the exceptional pairs each have degree 11. Hence each of $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ appears in 6 of the 9 blocks of $\ddot{\mathscr{B}}$. In fact, elementary manipulation now forces the blocks of $\ddot{\mathscr{B}}$ to be

$$\begin{array}{lll} \{a,b,c,d,e\} & \{a,b,c,d,f\} & \{a,b,c,d,g\} \\ \{a,b,c,e,f\} & \{a,b,c,e,g\} & \{a,b,c,f,g\} \\ \{a,d,e,f,g\} & \{b,d,e,f,g\} & \{c,d,e,f,g\}. \end{array}$$

The only freedom left to us is in the choice of $\{a, b, c\}$. If we choose $\{a, b, c\}$ to be a block of the $S(2, 3, 7)$ attached to $\{x, y\}$, i.e. $\{a, b, c\} = \{i, i+1, i+3\}$ for some $i \in \mathrm{GF}(7)$, we get a certain system $\mathscr{B}'$. If we choose $\{a, b, c\}$ to be a nonblock of the $S(2, 3, 7)$ we get a system $\mathscr{B}''$. We note that the last nine blocks form the unique system giving $C(7, 5, 4) = 9$.

Proposition 11. *These are the only two possibilities that realize Proposition* 9(ii); *that is to say,* $\mathscr{B}'$ *and* $\mathscr{B}''$ *are non-isomorphic, and any* (9, 5, 4) *covering system with this degree distribution is isomorphic to either* $\mathscr{B}'$ *or* $\mathscr{B}''$.

Proof. Since x and y are the only two points of degree 14, then any isomorphism Φ between two systems $\mathscr{C}$ and $\mathscr{D}$ of this type will fix x and y or interchange them. Therefore, Φ restricts to an automorphism of the $S(2, 3, 7)$ attached to $\{x, y\}$. If α, β, γ are the points of $\mathscr{C}$ of degree 18 and α', β', γ' are the points of $\mathscr{D}$ of degree 18, then $\Phi(\{\alpha, \beta, \gamma\}) = \{\alpha', \beta', \gamma'\}$ However, the action of the automorphism group of $S(2, 3, 7)$ (which is $\mathrm{PSL}_2(7)$) on the 3-subsets of GF(7) has two orbits, namely the blocks and nonblocks of $S(2, 3, 7)$. This is sufficient information to complete the proof. □

It turns out that there is a unique $(9, 5, 4)$ covering system $(X, \mathscr{B})$ with one point of degree 14 and eight points of degree. 17. In order to show this, we will first of all prove that the distribution of degrees of pairs is uniquely determined: If x is the point of degree 14, then $\deg(x, a) = 7$ for all $a \neq x$, $\deg(a, b) = 7$ for a certain set of four disjoint pairs not including x, and the remaining pairs have degree 9 each. In what follows, the point-set will be $X = \{x, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$, where $\deg(x) = 14$ and $\deg(a_i) = 17$ for each i. Since x has degree 14, the set $\mathscr{B}_x$ of fourteen blocks through x form, after deletion of x, a Steiner $S(3, 4, 8)$ system on the remaining eight points. It is therefore clear that $\deg(x, a_i) = 7$ for all i, since each point in $S(3, 4, 8)$ lies on seven quadruples. Following the notation of this paragraph we state and prove our next lemma.

Lemma 12. $\deg(a_i, a_j) \leq 9$ *for all i and j*.

Proof. It is sufficient to show that $\deg(a_7, a_8) \leq 9$. We set $\delta = \deg(a_7, a_8)$ and $I = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. For any subset J of $\{x, a_7, a_8\}$ let $\mathscr{B}\{j\}$ be the set of all blocks B in $\mathscr{B}$ such that $B \cap \{x, a_7, a_8\} = J$. We let $I\{J\}$ be the set of all subsets of the form $B \cap I$, with $B \in \mathscr{B}\{J\}$. The structure of $\mathscr{B}[x]$ gives us

$$|\mathscr{B}\{x, a_7, a_8\}| = |\mathscr{B}\{x\}| = 3 \quad \text{and} \quad |\mathscr{B}\{x, a_7\}| = |\mathscr{B}\{x, a_8\}| = 4.$$

Since $\delta = \deg(a_7, a_8)$ and $\deg(a_7) = \deg(a_8) = 17$ we have

$$|\mathscr{B}\{a_7, a_8\}| = \delta - 3 \quad \text{and} \quad |\mathscr{B}\{a_7\}| = |\mathscr{B}\{a_8\}| = 13 - \delta.$$

We set $\alpha = |I\{a_7\} \cap I\{x\}|$ and $\beta = |I\{a_8\} \cap I\{x\}|$. Without loss of generality we suppose $\alpha \geq \beta$. The set I contains 15 quadruples. Three of these are the ones in $I\{x\}$. These three quadruples do not cover any of the triples in $I\{x, a_7\}$ or any of the triples in $I\{x, a_8\}$. By a counting argument we see that each of the other 12 quadruples contained in I cover exactly one of the triples in $I\{x, a_7\}$ and exactly one of the triples in $I\{x, a_8\}$.

Consider the four quadruples obtained by adjoining a_7 to each of the four triples in $I\{x, a_8\}$. None of these are covered by any block in $\mathscr{B}[x]$. At most $13 - \delta - \alpha$ are covered by the blocks in $\mathscr{B}\{a_7\}$. Therefore at least $\alpha + \delta - 9$ of these four quadruples must be covered by the blocks in $\mathscr{B}\{a_7, a_8\}$. Therefore

$$|I\{x, a_8\} \cap I\{a_7, a_8\}| \geq \alpha + \delta - 9.$$

Let Q be the set of 20 quadruples that contain a_8 and the three elements of I. Of these, four are covered by the blocks of $\mathscr{B}\{x, a_8\}$. At most $\delta - 3 - (\alpha + \delta - 9) = 6 - \alpha$ additional ones can be covered by the blocks of $\mathscr{B}\{a_7, a_8\}$. There are $13 - \delta$ quadruples in $I\{a_8\}$. Of these β are contained in $I\{x\}$. Each of the remaining $13 - \delta - \beta$ of these quadruples cover a triple in $I\{x, a_8\}$, so that the corresponding block can cover at most three additional quadruples in Q. Hence at most $4\beta + 3(13 - \delta - \beta) = 39 + \beta - 3\delta$ additional quadruples in Q can be covered by the blocks of $\mathscr{B}\{a_8\}$. Therefore

$$20 = |Q| \leq 4 + 6 - \alpha + 39 + \beta - 3\delta \leq 49 - 3\delta,$$

so that $3\delta \leq 29$ and $\delta \leq 9$. □

Since $68 = 4 \cdot 17 = \sum_{z \neq a_8} \deg(z, a_8) = 7$, $7 \leq \deg(z, a_8) \leq 9$, it is easy to see that among the seven numbers $\deg(z, a_8)$, $z \neq x, a_8$, either one is equal to 7 and the others 9 or two are equal to 8 and the others 9. We now proceed to show that this latter possibility cannot occur. For convenience we change the labeling of the points: let $\{x, y\} \cup \mathrm{GF}(7)$ be the point set. We assume that x has degree 14, so without loss the blocks through x are $\{x, y, i, i+1, i+3\}$ and $\{x, i+2, i+4, i+5, i+6\}$ for $i \in \mathrm{GF}(7)$. Assume that $\deg(y, 0) = \deg(y, 1) = 8$; we will show that this leads to a contradiction. Observe that among the above 14 blocks there is a block disjoint from $\{y, 0, 1\}$, whence by inclusion-exclusion we have

$$\begin{aligned} 29 &\geq |\{\text{blocks } B : B \cap \{y, 0, 1\} \text{ is non-empty}\}| \\ &= \deg(y) + \deg(0) + \deg(1) - \deg(y, 0) \\ &\quad - \deg(y, 1) - \deg(0, 1) + \deg(y, 0, 1) \\ &= 17 + 17 + 17 - 8 - 8 - \deg(0, 1) + \deg(y, 0, 1) \\ &= 35 - \deg(0, 1) + \deg(y, 0, 1) \end{aligned}$$

and thus $\deg(0, 1) \geq 6 + \deg(y, 0, 1)$. It then follows by Lemma 12 that $\deg(0, 1) = 9$ and $\deg(y, 0, 1) = 3$, since trivially $\deg(y, 0, 1) \geq 3$. By Lemma 2, the set of eight triples that form blocks with $\{y, 0\}$ contain seven triples that form an $S(2,3,7)$. Already the triples $\{x,1,3\}$, $\{x,2,6\}$ and $\{x,4,5\}$ form blocks with $\{y,0\}$. The only ways to complete these three triples to an $S(2,3,7)$ is either by adding the triples

$$\Phi_1 = \{\{1,2,5\},\{1,4,6\},\{2,3,4\},\{3,5,6\}\}$$

or by adding the triples

$$\Phi_2 = \{\{1,2,4\},\{1,5,6\},\{2,3,5\},\{3,4,6\}\}.$$

Similarly, $\{y,1\}$ already forms blocks with $\{x,0,3\}$, $\{x,2,4\}$, $\{x,5,6\}$. The only ways to complete these triples to an $S(2,3,7)$ are by adding the triples

$$\Psi_1 = \{\{0,2,5\},\{0,4,6\},\{2,3,6\},\{3,4,5\}\}$$

or by adding

$$\Psi_2 = \{\{0,2,6\},\{0,4,5\},\{2,3,5\},\{3,4,6\}\}.$$

Only by using $\Phi_1 \cup \Psi_1$ do we satisfy the condition that $\deg(y,0,1)=3$. Thus, in addition to the 14 blocks through x already given, we have forced six new blocks: $\{y,0,1,2,5\}$, $\{y,0,1,4,6\}$, $\{y,0,2,3,4\}$, $\{y,0,3,5,6\}$, $\{y,1,2,3,6\}$, $\{y,1,3,4,5\}$. This makes a total of twenty forced blocks.

The assumption that $\deg(y,0)=\deg(y,1)=8$ also implies that there is some $u \in \{2,3,4,5,6\}$ with $\deg(0,u)=8$. Thus some set of seven triples in $\mathcal{B}_{0,u}$ must form an $S(2,3,7)$. Examination of the twenty forced blocks shows that this is impossible for any u. We have now proved our claim about the distribution of degrees of pairs and summarize this as a lemma.

Lemma 13. *Let* $(X,\mathcal{B})$ *be a* $(9,5,4)$ *covering system on the point-set* $\{x,1,2,3,4,A,B,C,D\}$ *such that* $\deg(x)=17$, $\deg(z)=14$ *for* $z \neq x$. *Then*:

(i) $\deg(x,z)=7$ *for* $z \neq x$;

(ii) *there are four other disjoint pairs of degree* 7, *say without loss of generality the pairs* $\{1,A\}$, $\{2,B\}$, $\{3,C\}$, $\{4,D\}$, (*let us call these the special pairs*);

(iii) *all other pairs have degree* 9.

It is not to difficult to deduce from this lemma that such a covering system is unique; however, this partly rests on knowledge of the automorphisms of $S(3,4,8)$.

If every $B \in \mathcal{B}[x]$ contains an even number of special pairs we may assume without loss that the 14 quintuples through x are

$$\mathcal{B}[x] = \left\{\begin{array}{l}\{x,1,2,3,4\},\{x,A,B,C,D\},\{x,1,2,A,B\},\{x,1,2,C,D\},\\ \{x,3,4,C,D\},\{x,1,3,A,C\},\{x,1,3,B,D\},\{x,2,4,A,C\},\\ \{x,1,4,A,D\},\{x,1,4,B,C\},\{x,2,3,A,D\},\{x,2,3,B,C\},\\ \{x,3,4,A,B\},\{x,2,4,B,D\},\end{array}\right\}.$$

Now since $\{1, A\}$ has degree 7, the triples that form blocks with $\{1, A\}$ are an $S(2, 3, 7)$. The triples $\{x, 2, B\}$, $\{x, 3, C\}$ and $\{x, 4, D\}$ are present; the two ways to complete this to an $S(2, 3, 7)$ are by adjoining the four triples

$$Q_{1A}\colon \{2, 3, 4\}, \{2, C, D\}, \{B, 3, D\}, \{B, C, 4\}$$

or the four triples

$$R_{1A}\colon \{2, 3, D\}, \{2, C, 4\}, \{B, 3, 4\}, \{B, C, D\}.$$

Similarly for $\{2, B\}$:

$$Q_{2B}\colon \{1, 4, 3\}, \{1, D, C\}, \{A, 4, C\}, \{A, D, 3\}$$

or

$$R_{2B}\colon \{1, 4, C\}, \{1, D, 3\}, \{A, 4, 3\}, \{A, D, C\};$$

and for $\{3, C\}$:

$$Q_{3C}\colon \{1, 2, 4\}, \{A, B, 4\}, \{1, B, D\}, \{A, 2, D\}$$

or

$$R3C\colon \{1, B, 4\}, \{2, A, 4\}, \{1, 2, D\}, \{A, B, D\};$$

and for $\{4, D\}$:

$$Q_{4D}\colon \{1, 2, 3\}, \{3, A, B\}, \{2, A, C\}, \{1, B, C\}$$

or

$$R_{4D}\colon \{2, 3, A\}, \{1, 3, B\}, \{1, 2, C\}, \{A, B, C\}.$$

It is straightforward to check that the only ways to get a covering system is by adding, to $\mathcal{B}[x]$, an even number of Q's and thus an even number of R's. (Example: Q_{1A}, R_{2B}, R_{3C}, R_{4D} does not work since the quadruple $\{1, B, C, D\}$ is left uncovered.) These eight ways (1 way for 4 Q's 1 way for 4 R's, 6 ways for 2 Q's and 2 R's) are all isomorphic. Example: The permutation $\pi = (2B)(3C)$ gives an isomorphism between $\mathcal{B}[x] \cup \{\text{four } Q\text{'s}\}$ and $\mathcal{B}[x] \cup Q_{1A} \cup R_{2B} \cup R_{3C} \cup Q_{4D}$. Again, the inherent symmetry makes the isomorphisms obvious (after a bit of practice anyway!). Let us call the blocks of this system $\mathcal{B}'''$, for use in Section 3.

If on the other hand there is a B in $\mathcal{B}[x]$ containing exactly one special pair, then we may assume without loss it is $\{x, 1, 2, 3, C\}$. There are six labelled ways to complete $\{1, 2, 3, C\}$ to an $S(3, 4, 8)$. But we note that there are precisely four permutations, namely those generated by $(4, D)$ and $(3, C)$, that fix the special pairs and $\{1, 2, 3, C\}$. Using these it is easy to see that there are up to isomorphism two ways to complete $\{1, 2, 3, C\}$ to an $S(3, 4, 8)$. There are thus two cases to consider for $\mathcal{B}[x]$.

$$\textit{Case } 1\colon \mathcal{B}[x] = \left\{\begin{array}{l} \{x, 1, 2, A, B\}, \{x, 1, 2, 4, D\}, \{x, 1, 3, A, 4\}, \{x, 1, 3, B, D\}, \\ \{x, 1, C, B, 4\}, \{x, 3, 4, C, D\}, \{x, 3, A, B, C\}, \{x, 2, B, C, D\}, \\ \{x, 2, 3, 4, B\}, \{x, 2, 3, A, D\}, \{x, 1, 2, 3, C\}, \{x, 4, A, B, D\}, \\ \{x, 1, C, A, D\}, \{x, 2, 4, A, C\} \end{array}\right\}$$

and

$$\text{Case 2: } \mathcal{B}[x] = \left\{ \begin{array}{l} \{x,1,2,A,4\}, \{x,1,2,B,D\}, \{x,1,3,A,B\}, \{x,1,3,4,D\}, \\ \{x,1,C,B,4\}, \{x,3,B,C,D\}, \{x,3,A,4,C\}, \{x,2,4,C,D\}, \\ \{x,2,3,4,B\}, \{x,2,3,A,D\}, \{x,1,2,3,C\}, \{x,4,A,B,D\}, \\ \{x,1,C,A,D\}, \{x,2,B,A,C\} \end{array} \right\}.$$

In Case 1 note that $\{1, A\}$ forms blocks with $\{x, 2, B\}$, $\{x, 3, 4\}$ and $\{x, C, D\}$. The only way to complete this to an $S(2, 3, 7)$ is to take triplets of the form $\{i, j, k\}$ where $i \in \{2, B\}$, $j \in \{3, 4\}$ and $k \in \{C, D\}$. In particular each new block will have either a 2–3 or 3–2 distribution with respect to the sets $\{1, 2, 3, 4\}$ and $\{A, B, C, D\}$. The same is true for $\{2, B\}$, $\{3, C\}$ and $\{4, D\}$. The 16 new blocks obtained this way are all distinct and this accounts for all 30 blocks. However, the above observation on the distribution of points in the new blocks shows that the quadruples $\{1, 2, 3, 4\}$ and $\{A, B, C, D\}$ are left uncovered. Case 2 can also be disposed of in a similar fashion.

Thus $\mathcal{B}'''$, in which every block contains an even number of special pairs, is the unique $(9, 5, 4)$ covering system satisfying Lemma 13.

3. $C(10, 6, 5) = 50$

We are now in a position to determine $C(10, 6, 5)$ and show that there is a unique covering system giving equality. Let $(X, \mathcal{B})$ be any $(10, 6, 5)$ covering system. We have

$$6|\mathcal{B}| = \sum_{a \in X} \deg(a) \geq 10 \cdot C(9, 5, 4) = 300 \tag{5}$$

and so $|\mathcal{B}| \geq 50$. We now exhibit a system with 50 blocks. Let $S = \{a, b, c, d, e\}$ and $S' = \{a', b', c', d', e'\}$ be disjoint five-element sets. Let $X = S \cup S'$ and let

$$\mathcal{B}_* = \{S \cup \{x'\} : x' \in S'\} \cup \{S' \cup \{x\} : x \in S\}$$
$$\cup \{\{\alpha, \beta, \gamma, u', v', w'\} : |\{\alpha, \beta, \gamma\} \cap \{u, v, w\}| \text{ is odd}\}.$$

It is easy to check that $(X, \mathcal{B}_*)$ has 50 blocks and is a $(10, 6, 5)$ covering system. Thus we have the following.

Proposition 14. $C(10, 6, 5) = 50$.

We now show that $\mathcal{B}_*$ is, up to isomorphism, the only $(10, 6, 5)$ covering system with 50 blocks. According to (5), any such system $(X, \mathcal{B})$ has the property that for every point x, there are 30 blocks containing x and after deleting x from these blocks we have a $(9, 5, 4)$ covering system. Denoting by $\mathcal{B}_x$ this derived system, we thus have that each $\mathcal{B}_x$ must be isomorphic to one of the three $(9, 5, 4)$ covering systems $\mathcal{B}'$, $\mathcal{B}''$ and $\mathcal{B}'''$, given in Section 2. We now show that the first two systems (the ones with two points of degree 14) do not occur. We remark that for the above system $\mathcal{B}_*$ each derived system is indeed isomorphic to the third (9, 5, 4) covering system $\mathcal{B}'''$.

Lemma 15. *The derived system with respect to any point of a minimal* (10, 6, 5) *is isomorphic to* $\mathcal{B}'''$.

Proof. Let $(Y, \mathcal{A})$ be a (10, 6, 5) covering system with 50 blocks. Fix any point $\infty \in Y$ and suppose that $\mathcal{A}_\infty$ is isomorphic to $\mathcal{B}'$ or $\mathcal{B}''$. Then there are two points x and y in Y of degree 14 in $\mathcal{A}_\infty$. It is easy to see that there are 30 blocks through ∞, 30 through x, 14 through $\{x, \infty\}$ and thus, precisely $50-30-30+14=4$ blocks missing $\{x, \infty\}$ entirely. On the other hand, consider the point y. It appears in 14 blocks with ∞, and at most 11 additional blocks with x, since $\deg(\infty, x, y)=7$ and $\deg(x, y) \leq 18$. Hence y appears in at most 25 of the 46 blocks meeting $\{\infty, x\}$. Thus, y appears in at least 5 of the 4 blocks missing $\{x, \infty\}$, a contradiction. □

Proposition 16. *The minimum-sized* (10, 6, 5) *covering system is unique, up to isomorphism.*

Proof. Let $(X, \mathcal{B})$ be such a system. Then by Lemma 15, for each x the derived system $\mathcal{B}_x$ is a (9, 5, 4)-covering system isomorphic to $\mathcal{B}'''$, having one point $\hat{x}$ of degree 14 and the others of degree 17. Symmetrically, x has degree 14 in $B_{\hat{x}}$. Thus, we may put $X=\{1, \hat{1}, 2, \hat{2}, 3, \hat{3}, 4, \hat{4}, 5, \hat{5}\}$ with $\deg(i, \hat{\imath})=14$ for $i=1$ to 5.

Now put $\mathcal{D}=\mathcal{B}-(\mathcal{B}[5] \cup \mathcal{B}[\hat{5}])$. We note that $\mathcal{D}$ has $50-(30+30-14)=4$ blocks. Further, each pair $\{i, \hat{\imath}\}$, $i=1$ to 4, is contained in $14-(7+7-3)=3$ of the blocks of $\mathcal{D}$. This means that the four blocks of $\mathcal{D}$ are uniquely determined: $\{1, \hat{1}, 2, \hat{2}, 3, \hat{3}\}$, $\{1, \hat{1}, 2, \hat{2}, 4, \hat{4}\}$, $\{1, \hat{1}, 3, \hat{3}, 4, \hat{4}\}$, $\{2, \hat{2}, 3, \hat{3}, 4, \hat{4}\}$. It is easy to see that $\mathcal{B}[5]$ determines $\mathcal{B}[\hat{5}]$. This is sufficient to complete the proof, since $\mathcal{B}=\mathcal{B}[5] \cup \mathcal{B}[\hat{5}] \cup \mathcal{D}$. □

4. Additional remark

The system $\mathcal{B}_*$ of Section 3, giving $C(10, 6, 5) \leq 50$, was found independently by Guy Giraud in the context of Turán numbers (letter to de Caen dated August 1987). Recall that the Turán number $T(n, l, k)$ is the minimum possible number of k-subsets of an n-set such that every l-set contains at least one of these k-sets. See for example [1] for more information. It is not hard to see that $T(n, l, k)=C(n, n-k, n-l)$, simply by complementing blocks. Thus $C(9, 5, 4)=T(9, 5, 4)$ and $C(10, 6, 5)=T(10, 5, 4)$. Kalai [3] has an intriguing question concerning the Turán numbers $T(n, 5, 4)$: Let (X, Q), $|X|=n$, $|Q|=T(n, 5, 4)$, be a minimum system of quadruples such that every quintuple contains a member of Q. Does it follow that every quintuple contains an odd number of members of Q? We observe that Kalai's question has an affirmative answer for $n \leq 10$: this is easy for $n \leq 8$ (the minimal systems are unique and easily tested); for $n = 9$ there are three minimal systems (Section 2) and for $n = 10$ a unique system (Section 3); in all cases one can verify that the parity condition is satisfied. Thus any counterexample must have at least 11 points. We have not determined $T(11, 5, 4) = C(11, 7, 6) \leq 84$. Also, for $T(12, 5, 4) = C(12, 8, 7)$ we have $120 \leq C(12, 8, 7) \leq 126$. The upper bounds 84 and 126 can be derived from the results in [2]. Details are omitted.

References

[1] A.E. Brouwer and M. Voorhoeve, Turán theory and the lotto problem, Math. Centre Tracts 106 (1979) 99–105.
[2] D. de Caen, D.L. Kreher and J. Wiseman, On constructive upper bounds for the Turán numbers $T(n, 2r + 1, 2r)$, Congr. Numer. 65 (1988) 277–280.
[3] G. Kalai, A new approach to Turán's conjecture, Graphs Combin. 1 (1985) 107–109.
[4] W.H. Mills, Covering Designs I: Coverings by a small number of subsets, Ars Combin. 8 (1979) 199–315.