Lower bounds for Multi-Colored Ramsey Numbers From Group Orbits

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ABSTRACT

In this paper the algorithm developed in [RK] for 2-color Ramsey numbers is generalized to multi-colored Ramsey numbers. All the cyclic graphs yielding the lower bounds $R(3,3,4) \geq 30$, $R(3,3,5) \geq 45$ and $R(3,4,4) \geq 55$ were obtained. The two last bounds are apparently new.

1. Introduction

The classical *multi-color Ramsey number* $R(r_1, r_2, ..., r_m)$ is defined to be the smallest integer n, such that no matter how the edges of K_n are colored with m colors, 1, 2, 3, ... m, there exists some i such that there is a complete subgraph K_{r_i}, all of whose edges have color i. It is said to be a multi-color Ramsey number when $m \geq 3$.

The concrete lower bounds are usually established by an explicit construction of a coloring of K_n, the complete graph on n vertices containing no monochromatic complete subgraph $K_{r_i}, 1 \leq i \leq m$, in the ith color. A coloring of K_n that establishes a lower bound on $R(r_1, r_2, ..., r_m)$ is said to an $(r_1, r_2, ..., r_m)-$*Ramsey graph*.

Only a few exact values and nontrivial bounds are known, and most of them are for $R(k,l)$, the so called *two−color* Ramsey numbers. The only known non-trivial exact value for $m-$*color* Ramsey numbers with $m \geq 3$ is $R(3,3,3) = 17$ [GG]. The only known nontrivial bounds on $m-$*color* Ramsey numbers with $m \geq 3$ are

$51 \leq R(3,3,3,3) \leq 65$	[C2,F1],
$159 \leq R(3,3,3,3,3) \leq 322$	[F2,W],
$128 \leq R(4,4,4) \leq 254$	[H1,G],
$30 \leq R(3,3,4)$	[K].

In this paper we establish that $R(3,3,5) \geq 45$ and $R(3,4,4) \geq 55$

2. Notation, Concepts and Algorithm

In this section we introduce the necessary notation, concepts, and facts about incidence matrices belonging to permutation groups used in the construction of our algorithm for finding multi-color Ramsey graphs. At the end of this section our algorithm is presented.

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If V is a set, then $Sym(V)$ denotes the full symmetric group on V. A group G is said to act on a set V if there is a function $V \times G \to V$ (usually denoted by $(v, g) \mapsto v^g$) such that for all $g, h \in G$ and $v \in V$: $v^1 = v$ and $v^{(gh)} = (v^g)^h$. We denote an action by $G \mid V$. Thus G may be thought of as being mapped homomorphically onto a permutation group of V, and v^g is the image of $v \in V$ under $g \in G$. If $v \in V$, the *stabilizer* in G of v is the subgroup $G_v = \{g \in G : v^g = v\}$ and the *orbit* of $v \in V$ under G is $v^G = \{v^g : g \in G\}$. We note that $|G| = |v^G| \cdot |G_v|$. If $|v^G| = |V|$ then the group action $G \mid V$ is said to be *transitive*. A group action $G \mid V$ induces an action on the collection $\binom{V}{t}$ of t-subsets of V. For, if $T \subseteq V$, and $g \in G$, then we define T^g by $T^g = \{v^g : v \in T\}$.

For even a relatively small number of vertices, an exhaustive computer search among all $(r_1, \ldots, r_m)$-Ramsey graphs is infeasible. However, if symmetry is imposed on the colorings, then exhaustive searches do become practical for moderate values of m and n.

A graph Γ with vertex set $V = \{0,1,2,\ldots,n-1\}$ is *cyclic* if the mapping $g : x \to x+1$ is an automorphism of Γ, addition performed modulo n. Note that any cyclic graph Γ must have as an automorphism group at least the dihedral group $D_n = <g,h>$, where $h : x \to -x$, since $g^{(n-u-v)}$: $\{u,v\} \to \{-v, -u\}$. It is easy to show that D_n acting on the edges of K_n has exactly $\lfloor n/2 \rfloor$ orbits. Observe that if we define the distance function $dist(e) = \min\{|i-j|, |j-i|\}$, then two edges e_1 and e_2 of K_n belong to the same orbit if and only if $dist(e_1) = dist(e_2)$. Thus an orbit of edges is completely determined by a single number k, where k is the difference of the pairs in the crbit. For example, a cyclic 3-color Ramsey graph can be completely specified by sets of distances, say *Red*, *Green* and *Blue*. That is $k \in Red$ means every edge e with $dist(e) = k$ is colored red.

Let V be an n-element set. If G is a subgroup of the symmetric group of permutations of V, $G \leq \mathrm{Sym}(V)$, and r is an integer, $2 < r < n$, the *pattern matrix* P_r *belonging to the group G is defined as follows:*

(a) the rows of P_r are indexed, by the G-orbits of 2-subsets of V;
(b) the columns of P_r are indexed by the G-orbits of r-subsets of V;
(c) $P_r[I,J] = 1$ if there are $F_i \in I$ and $F_j \in J$ such that $F_i \subseteq F_j$ and is 0 otherwise.

If V is thought of as the vertex set of the complete graph K_n, then the pattern matrix P_{r_i} describes the incidence between the orbits under G of edges and complete subgraphs of size r_i. Thus $P_{r_i}[I,J] = 1$ means that every K_{r_i} in orbit J contains at least one edge in orbit I. Hence, if we are to avoid the inclusion of a monochromatic K_{r_i} of color i, then the rows corresponding to the orbits of color i must be chosen so that no column of all 1's appears among them.

Our first theorem follows immediately from the above discussion and is a generalization of theorem 1 of [RK].

THEOREM 1. *There is a bijection between the m-color Ramsey graphs Γ with vertex set V, having $G \leq Sym(V)$ as an automorphism group and the (0,1)-vectors $U_i, 1 \leq i \leq m$, indexed by the G-orbits of 2-subsets of V, solving simultaneously the* *inequalities:*

$$(\overline{U_i}\cdot P_{r_i})[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_{r_i}, 1 \le i \le m, \quad (1)$$

$$\sum_{i=1}^{m} U_i = \bar{1}, \text{ where } \bar{1} = [1,1,\ldots,1]^T, \quad (2)$$

$$U_i \cdot U_j = 0, \text{ for } 1 \le i < j \le m, \quad (3)$$

where P_{r_i} for $1 \le i \le m$ are pattern matrices belonging to the group G.

The equations in theorem 2 can be interpreted as follows:

(1) says that for each i the coloring has no monochromatic K_{r_i} in color i,

(2) ensures that every edge is colored, and

(3) guarantees that no edge is colored twice.

In particular, to search for a (3,3,4)-Ramsey graph on n points, we need to consider the pattern matrices P_3, P_3 and P_4 for colors *red*, *green*, and *blue*, respectively. If we let U_r be the vector for color red, U_g for green, and U_b for blue, then the equations and inequalities we have to solve are

$$(\overline{U_r}\cdot P_3)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_3, \quad (1.1)$$

$$(\overline{U_g}\cdot P_3)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_3, \quad (1.2)$$

$$(\overline{U_b}\cdot P_4)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_4, \quad (1.3)$$

$$U_r + U_g + U_b = \bar{1}, \quad (2.1)$$

$$U_r \cdot U_g = 0, \quad (3.1)$$

$$U_r \cdot U_b = 0, \quad (3.2)$$

$$U_g \cdot U_b = 0. \quad (3.3)$$

The pattern matrices for large n are still too large for a computer search. The absorption law $a^*(a+b) = a$ of Boolean algebra can be used to reduce the sizes of pattern matrices, making computer search possible.

Let $G \le Sym(V)$. With each color i, $1 \le i \le m$ and with each orbit I_j of pairs associate a Boolean variable x_{ij}. The assignment of true to x_{ij} will mean that jth orbit of edges is assigned color i. Also, in order to have no monochromatic K_{r_i}-subset of color i, we associate with each column h of the pattern matrix P_{r_i} of G, the clause c_{ih} given by $c_{ih} = \sum_j \{\bar{x}_{ij} : P_{r_i}[j,h]=1\}$. Whence, if $B_i = \prod_h c_{ih}$, then B_i is satisfied if and only if there is no monochromatic r_i-subset of color i. Thus the absortion law when applied to the clauses in B_i produces an equivalent Boolean expression $B_i' = \prod_{h'} c_{ih'}$ with in general far fewer clauses. We call the pattern matrix that reflects the incidence of Boolean variable and clauses in B_i' the *reduced pattern matrix.*

Although this reduction may not be significant for small matrices, for big matrices it does make a difference. For example, the pattern matrix P_5 for an

(3,3,5)-Ramsey graph on 44 points with dihedral group D_{44} has 12,446 columns. After applying the absorption property, only 1395 columns remain.

From the above discussion, theorem 1 can be rephrased as theorem 2 below.

Theorem 2. *There is a bijection between the m-color $(r_1,r_2,\ldots,r_m)$-Ramsey graphs Γ with vertex set V, having $G \le Sym(V)$ as an automorphism group and the (0,1)-vectors $U_i, 1 \le i \le m$ indexed by the G-orbits of 2-subsets of V, solving simultaneously the inequalities:*

$$(\overline{U_i} \cdot P_{r_i})[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_{r_i}, 1 \le i \le m, \tag{R1}$$

$$\sum_{i=1}^{m} U_i = \bar{1}, \text{ where } \bar{1} = [1,1,\ldots,1]^T, \tag{R2}$$

$$U_i \cdot U_j = 0, \text{for } 1 \le i < j \le m, \tag{R3}$$

where P_{r_i}, $1 \le i \le m$ are reduced pattern matrices (by Boolean absorption laws) belonging to group G.

The algorithm that naturally follows from the above discussion is

Algorithm

Step 1: Input a chosen group G, $G \le Sym(X)$, $|X| = n$, as a candidate for an automorphism group of an $(r_1,r_2,\ldots,r_m)$*-Ramsey graph.*

Step 2: Construct pattern matrices P_{r_i}, for each i, $1 \le i \le m$.

Step 3: Apply the absorption law to obtain reduced pattern matrices P'_{r_i}, for each i, $1 \le i \le m$, and their corresponding Boolean expressions B'_{r_i}, $1 \le i = m$.

Step 4: Find all assignments satisfying the Boolean expression $\alpha = \beta \cdot \prod_{i=1}^{m} B'_{r_i}$, where β expresses conditions (R2) and (R3). Each such assignment (if any) yields a $(r_1,r_2,\ldots,r_m)$*-Ramsey graph* with automorphism group G; furthermore all such graphs are obtained.

3. Results and analysis of the new Ramsey graphs

3.1. New lower bounds of some 3-color Ramsey numbers

Using the algorithm described in section 2 and the authors' experience with a similar algorithm for 2-color Ramsey numbers [RK], the three 3-color cyclic graphs given in Table I were constructed.

These three graphs give the three lower bounds on three 3-color Ramsey numbers below,

$$R(3,3,4) \ge 30; \quad R(3,3,5) \ge 45; \quad R(3,4,4) \ge 55.$$

Furthermore, these are each maximal cyclic Ramsey graphs, and consequently, these lower bounds can only be improved by non-cyclic Ramsey graphs. The lower

TABLE I

New Bound	n	Edge Orbits (Differences)		
		Red	*Green*	*Blue*
$R(3,3,4) \geq 30$	29	1 4 10 12	2 5 6 14	3 7 8 9 11 13
$R(3,3,5) \geq 45$	44	1 4 9 12 15 22	2 3 10 14 18 19	5 6 7 8 11 13 16 17 20 21
$R(3,4,4) \geq 55$	54	1 4 9 15 20 22 27	7 8 13 14 16 17 18 19 23 26	2 3 5 6 10 11 12 21 24 25

bound $R(3,3,4) \geq 30$, was found by J. G. Kalbfleisch at the University of Waterloo in his doctoral dissertation [K], but otherwise does not appear in the literature. The other two lower bounds are apparently new.

3.2. Analysis of the Graphs

In order to enumerate the non-isomorphic cyclic graphs with the above parameters we introduce the group theory notion of primitivity.

A subset ω of V is called a *block of imprimitivity* (b.i.) of the transitive group action $G \mid V$, if for each $g \in G$, the set ω^g either coincides with ω or is disjoint from ω. Obviously V and the singleton subsets are b.i.'s and these are called the trivial blocks of the group action. A transitive group action $G \mid V$ is said to be *imprimitive* if it has at least one nontrivial b.i. ω, otherwise it is *primitive*. In particular, it is easy to see that a transitive group action $G \mid V$ is primitive whenever $|V|=p$ is a prime. In this case either G is isomorphic to a subgroup of $AF(p) = \{x \rightarrow \alpha x+\beta : \alpha,\beta \in \mathbb{Z}_p, \alpha \neq 0\}$, or G is 2-transitive.

In searching for a cyclic (3,3,4)-Ramsey graph on 29 vertices, the pattern matrix P_3 had 126 columns and the pattern matrix P_4 had 819 columns. After applying the absorption laws, only 56 columns in P_3 and only 63 columns in P_4 remained. The search of the remaining columns led to 14 cyclic Ramsey graphs and these appear in Table II. It will be shown that these give only two nonisomorphic solutions. The two classes of solutions are represented by No. 1 and No. 8 in Table II. Furthermore they are isomorphic if interchanging colors *red* and *green* is allowed.

THEOREM 3. *There are, up to isomorphism, only two cyclic* (3,3,4)*-Ramsey graphs on* 29 *vertices. Furthermore, the full automorphism group of each is* $G = \langle x \rightarrow x+1, x \rightarrow \omega^7 x \rangle$ *where* $\omega \in \mathbb{Z}_{29}$ *is a primitive root of unity.*

Proof: Let (V,Γ) be a cyclic (3,3,4)-Ramsey graph. Then $G \mid V$ is transitive and since $|V|=29$ is prime G acts primitively. Moreover, G cannot be 2-transitive for then there is only one orbit of edges and 3-color Ramsey graphs require at least three. Whence, G must be one of the 6 transitive subgroups of $AF(29) = \{x \rightarrow \alpha x+\beta : \alpha,\beta \in \mathbb{Z}_{29}, \alpha \neq 0\}$. These 6 subgroups are $H_d = \langle x \rightarrow x+1, x \rightarrow \omega^d x \rangle$ where $d \mid 28$ and ω is a primitive root modulo 29. Whence, H_d is an automorphism group of a cyclic (3,3,4)-Ramsey graph if and only if multiplication by ω^d preserves the coloring. A complete list of all cyclic (3,3,4)-Ramsey graphs was generated by the algorithm in section 2 and is given in Table II. It is easy to check that

TABLE II

14 cyclic (3,3,4)-Ramsey graphs on K_{29}			
No.	*Red*	*Green*	*Blue*
1	6 11 13 14	3 7 8 9	1 2 4 5 10 12
2	3 7 8 9	4 10 11 13	1 2 5 6 12 14
3	4 10 11 13	2 5 8 9	1 3 6 7 12 14
4	1 3 7 12	6 11 13 14	2 4 5 8 9 10
5	2 5 8 9	1 4 10 12	3 6 7 11 13 14
6	1 4 10 12	2 5 6 14	3 7 8 9 11 13
7	2 5 6 14	1 3 7 12	4 8 9 10 11 13
8	3 7 8 9	6 11 13 14	1 2 4 5 10 12
9	4 10 11 13	3 7 8 9	1 2 5 6 12 14
10	2 5 8 9	4 10 11 13	1 3 6 7 11 14
11	6 11 13 14	1 3 7 12	2 4 5 8 9 10
12	1 4 10 12	2 5 8 9	3 6 7 11 13 14
13	2 5 6 14	1 4 10 12	3 7 8 9 11 13
14	1 3 7 12	2 5 6 14	4 8 9 10 11 13

multiplication by -1 and by ω^7 preserves each of these 14 colorings. Also, it can be seen that the other multiplications permute the 14 colorings into two orbits $\Delta_1 = \{1,2,3,4,5,6,7\}$ and $\Delta_2 = \{8,9,10,11,12,13,14\}$. Thus up to isomorphism there are only two cyclic (3,3,4)-Ramsey graph as claimed. □

COROLLARY 4. *There is, up to isomorphism and interchange of colors, a unique cyclic (3,3,4)-Ramsey graph on 29 vertices. Furthermore, its full automorphism group is* $G = \langle x \to x+1, x \to \omega^7 x \rangle$ *where* $\omega \in \mathbb{Z}_{29}$ *is a primitive root of unity.*

Proof: We note that the mapping induced on the colorings by interchanging the colors *red* and *green* swaps rows in Table II according to the permutation (1,8)(2,9)(3,10)(4,11)(5,12)(6,13)(7,14). Thus without fixing colors, there is a unique cyclic (3,3,4)-Ramsey graph as claimed. □

In searching for cyclic (3,3,5)-Ramsey graph on 44 vertices, the pattern matrix P_3 has 161 columns and the pattern matrix P_4 has 12446 columns. After applying the absorption laws, there are only 141 columns in P_3 and 1395 columns in P_4. In this situation 260 cyclic graphs were found with the algorithm presented in section 2. These graphs may be obtained from the 13 graphs listed in Table III by multiplying modulo 44 by numbers a relatively prime to 44 and/or interchanging *red* and *green* edges.

THEOREM 5. *There are, up to isomorphism and interchange of colors, exactly 13 cyclic (3,3,5)-Ramsey graph on 44 vertices.*

TABLE III

13 non-isomorphic cyclic (3,3,5)-Ramsey graphs on K_{44}			
No.	*Red*	*Green*	*Blue*
1	1 4 9 12 15 22	2 3 10 14 18 19	5 6 7 8 11 13 16 17 20 21
2	1 4 12 15 22	2 3 10 14 18 19	5 6 7 8 9 11 13 16 17 20 21
3	1 4 10 16 21	2 5 6 15 18 22	3 7 8 9 11 12 13 14 17 19 20
4	1 3 11 16 18 20	2 6 9 14 17 22	4 5 7 8 10 12 13 15 19 21
5	1 3 11 16 18 20	2 6 9 14 17	4 5 7 8 10 12 13 15 19 21 22
6	1 3 10 14 18	2 5 8 11 12 15	4 6 7 9 13 16 17 19 20 21 22
7	1 3 7 11 15 20	2 8 9 12 13	4 5 6 10 14 16 17 18 19 21 22
8	1 3 10 14 18 22	2 8 9 12 13	4 5 6 7 11 15 16 17 19 20 21
9	1 3 10 14 22	2 8 9 12 13	4 5 6 7 11 15 16 17 18 19 20 21
10	1 3 8 12 17 22	2 9 10 13 14	4 5 6 7 11 15 16 18 19 20 21
11	1 3 8 13 15 19	4 7 12 21 22	2 5 6 9 10 11 14 16 17 18 20
12	1 3 8 15 19	4 7 12 13 21 22	2 5 6 9 10 11 14 16 17 18 20
13	1 3 8 15 19	4 7 12 21 22	2 5 6 9 10 11 13 14 16 17 18 20

Proof: To see that these 13 graphs are non-isomorphic, fix vertex 0 in each of the graphs, and call the subgraphs which are induced by the vertices adjacent to vertex 0 by red edges, the *red subgraphs*. Similarly define the green subgraphs. By counting the number of vertices and blue edges in each of these subgraphs it is easy to distinguish between the 13 graphs except for possibly numbers 8 and 10. In the red subgraph of graph 8 there are, however, vertices with green degree 2, while graph 10 has no such vertices. Consequently, they are also non-isomorphic. Hence there are 13 non-isomorphic cyclic (3,3,5)-Ramsey graphs.

In searching for a cyclic (3,4,4)-Ramsey graph on 54 vertices, the pattern matrix P_3 has 243 columns and the pattern matrix P_4 has 1807 columns. After applying the absorption laws, there are only 196 columns in P_3 and 950 columns in P_4. Here the algorithm presented in section 2 found 18 solutions. These 18 solutions are listed in Table IV.

Similarly to the proof of theorem 3 we have the following theorem.

THEOREM 6. *There are, up to isomorphism, exactly two cyclic* (3,4,4)*-Ramsey graph on 54 vertices. They are listed in Table V.*

Also, it is again easy to see that interchanging green and blue swaps these two graphs.

COROLLARY 7. *There is, up to isomorphism and interchange of colors, a unique cyclic* (3,4,4)*-Ramsey graph on 54 vertices.*

TABLE IV

18 cyclic (3,4,4)-Ramsey graphs on K_{54}			
No.	*Red*	*Green*	*Blue*
1	1 4 9 15 20 22 27	7 8 13 14 16 17 18 19 23 26	2 3 5 6 10 11 12 21 24 25
2	2 5 8 9 20 21 27	7 11 13 14 16 18 19 22 23 26	1 3 4 6 10 12 15 17 24 25
3	2 9 10 13 16 21 27	4 5 7 8 14 17 18 20 23 25	1 3 6 11 12 15 19 22 24 26
4	2 9 14 15 19 22 27	1 4 5 8 10 17 18 20 23 25	3 6 7 11 12 13 16 21 24 26
5	3 4 9 10 11 26 27	7 8 14 16 17 18 19 20 23 25	1 2 5 6 12 13 15 21 22 24
6	3 7 8 9 22 26 27	1 2 4 5 10 11 17 18 20 25	6 12 13 14 15 16 19 21 23 24
7	3 8 9 10 14 25 27	1 2 7 11 13 16 18 19 22 26	4 5 6 12 15 17 20 21 23 24
8	4 9 14 15 16 17 27	1 2 5 10 11 13 18 19 22 26	3 6 7 8 12 20 21 23 24 25
9	9 16 20 21 23 26 27	1 2 4 5 10 11 13 18 22 25	3 6 7 8 12 14 15 17 19 24
10	1 4 9 15 20 22 27	2 3 5 6 10 11 12 21 24 25	7 8 13 14 16 17 18 19 23 26
11	2 5 8 9 20 21 27	1 3 4 6 10 12 15 17 24 25	7 11 13 14 16 18 19 22 23 26
12	2 9 10 13 16 21 27	1 3 6 11 12 15 19 22 24 26	4 5 7 8 14 17 18 20 23 25
13	2 9 14 15 19 22 27	3 6 7 11 12 13 16 21 24 26	1 3 4 8 10 17 18 20 23 25
14	3 4 9 10 11 25 27	1 2 5 6 12 13 15 21 22 24	7 8 14 16 17 18 19 20 23 25
15	3 7 8 9 22 26 27	6 12 13 14 15 16 19 21 23 24	1 2 4 5 10 11 17 18 20 25
16	3 8 9 10 14 25 27	4 5 6 12 15 17 20 21 23 24	1 2 7 11 13 16 18 19 22 26
17	4 9 14 15 16 17 27	3 6 7 8 12 20 21 23 24 25	1 2 5 10 11 13 18 19 22 26
18	9 16 20 21 23 26 27	3 6 7 8 12 14 15 17 19 24	1 2 4 5 10 11 13 18 24 25

TABLE V

two non-isomorphic cyclic (3,4,4)-Ramsey graphs on K_{54}			
No.	*Red*	*Green*	*Blue*
1	1 4 9 15 20 22 27	7 8 13 14 16 17 18 19 23 26	2 3 5 6 10 11 12 21 24 25
10	1 4 9 15 20 22 27	2 3 5 6 10 11 12 21 24 25	7 8 13 14 16 17 18 19 23 26

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