Upper Bounds for Some Ramsey Numbers R(3,k)

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ABSTRACT

Using several computer algorithms we calculate some values and bounds for the function $e(3,k,n)$, the minimum number of edges in a triangle-free graphs on n vertices with no independent set of size k. As a consequence, the following new upper bounds for the classical two color Ramsey numbers are obtained: $R(3,10) \le 43$, $R(3,11) \le 51$, $R(3,12) \le 60$, $R(3,13) \le 69$ and $R(3,14) \le 78$.

1. Introduction

The two color Ramsey number $R(l,k)$ is defined to be the smallest integer n, such that any graph on n vertices contains either a clique of size l or an independent set of size k. In this paper we consider only the case $l=3$. A $(3,k,n,e)$-graph is a triangle-free graph on n vertices with e edges and no independent set of size k. Similarly, a $(3,k)$- or $(3,k,n)$-graph is a $(3,k,n,e)$-graph for some n and e. Let $e(3,k,n)$ be the minimum number of edges in any $(3,k,n)$-graph and define it to be ∞ if no such graph exists. Any $(3,k,n,e)$-graph is called a *minimum* graph if $e=e(3,k,n)$. The following formula was established in [4] for $k \ge 4$:

$$e(3,k+1,n) = \begin{cases} 0 & \text{if } n \le k, \\ n-k & \text{if } k < n \le 2k, \\ 3n-5k & \text{if } 2k < n \le 5k/2, \\ 5n-10k & \text{if } 5k/2 < n \le 3k. \end{cases} \tag{1}$$

Recently in [5], we also proved that

$$e(3,k+1,n) \ge 6n-13k \quad \text{for all } k,n \ge 1, \tag{2}$$

and the equality holds in (2) for all $3k \le n \le 13k/4 - sign(k \bmod 4)$.

If G is a $(3,k,n,e)$-graph and n_i denotes the number of vertices of degree i in G then by proposition 4 in [1] we have

$$ne - \sum_{i=0}^{k-1} n_i(e(3,k-1,n-i-1)+i^2) \ge 0, \tag{3}$$

where $n = \sum_{i=0}^{k-1} n_i$ and $2e = \sum_{i=0}^{k-1} i \cdot n_i$.

Equation (1) and inequalities (2) and (3) form the starting points of several computer algorithms we have implemented for the evaluation of bounds and sometimes exact values of the function $e(3,k,n)$ for $n \ge 13(k-1)/4$. Section 2 presents the progress we have done in the case $k=8$. Five new upper bounds for the classical two color Ramsey numbers $R(3,k)$, for $10 \le k \le 14$, are obtained by iterative application of inequality (3) to the results given in section 2. These new upper bounds are reported in section 3.

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2. $e(3,8,n)$

The values of $e(3,8,n)$ for $n \leq 21$ are given by equation (1). Grinstead and Roberts [2] proved that $28 \leq R(3,8) \leq 29$, $R(3,9)=36$ and established lower and upper bounds for $e(3,8,n)$, $26 \leq n \leq 28$. Using the techniques described in [1,2] we have developed several computer algorithms [4] for searching for $(3,k,n)$-graphs. The bounds and the exact values of $e(3,8,n)$ calculated by these algorithms are displayed in table I, together with the previous lower and upper bounds found in [2]. The bound $e(3,8,28) \leq 98$ is true under the assumption that $R(3,8)=29$, otherwise $e(3,8,28)=\infty$.

n	this paper	Grinstead & Roberts
22	42	
23	49	
24	56	
25	65	
26	73	71-74
27	~~83~~-85	81-87
28	~~94-98~~	~~90-98~~

Table I. Bounds and values of $e(3,8,n)$, $n \geq 22$.

To establish each of the lower bounds of the form $e(3,8,n)>e$ well tuned implementations of the algorithms described in [2,4] were used. In [4] an example of a typical procedure was given that can be followed to find all $(3,k,n)$-graphs for a given k and n. This technique requires the previous knowledge of all $(3,k-1,\bar{n},\bar{e})$-graphs for $\bar{n}$ and $\bar{e}$ ranging over some (hopefully small) set S of values, where S can be determined by the method of Graver and Yackel [1]. Using this method to obtain the lower bounds presented in table I, it is sufficient to know the following graphs:

(a) all (3,6)-graphs,

(b) all minimum (3,7)-graphs,

(c) all (3,7,22)-graphs,

(d) all $(3,7,n,e)$-graphs for $n \geq 18$ and $e = e(3,7,n)+1$,

(e) all (3,7,21,53)-graphs.

The construction of the graphs specified in (a), (b) and (c) was reported in [4]. By using the data base of all (3,6)-graphs we were able to build all the graphs in (d) and (e). For (d), the number of $(3,7,n,e)$-graphs is 15, 417, 479 and 70 for n = 18, 19, 20, 21 and e = 31, 38, 45 and 52, respectively. The number of (3,7,21,53)-graphs is 717. Somewhat surprisingly, the lower bound hardest to obtain by this method was $e(3,8,25) \geq 65$. It required about 250 hours of CPU time on a VAX780. Perhaps this is connected to the unique known so far irregularity of the form $e(3,k,n+1)-e(3,k,n) < e(3,k,n)-e(3,k,n-1)$, which occurs in this case, $k=8$ and $n=25$. For all others known exact values of $e(3,k,n)$ the latter inequality is false.

The upper bounds for $e(3,8,n)$, n = 22, 23 and 24 are achieved by the construction of (3,8,22,42)-, (3,8,23,49)- and (3,8,24,56)-graphs by applying consecutively corollary 6 of [1] three times to the unique minimum (3,8,21,35)-graph presented in [4]. Remaining upper bounds are established by examples of (3,8,25,65)-, (3,8,26,73)- and (3,8,27,85)-graphs, which are described in the appendix.

From the result of Grinstead and Roberts [2], $R(3,8)=29$ if and only if there exists a (3,8,28)-graph. Using the algorithms mentioned above and some relatively simple reasoning, we have established that any (3,8,28)-graph $G=(V,E)$ can have only vertices of degree 6 and 7. Thus there are only 5 possible degree sequences for G, one for each number of edges $94 \leq |E| \leq 98$, i.e. G has s vertices of degree 6 and $28-s$ vertices of degree 7, where $0 \leq s \leq 8$ and s is even.

We would like to point out, that further improvement of bounds in table I and calculation of $R(3,8)$ by the same method cannot be obtained unless a very powerful machine is run using probably prohibitively long time.

3. New Upper Bounds

One of the most fruitful ideas used so far to obtain upper bounds for $R(3,k)$ is the calculation of good lower bounds for $e(3,k,n)$ [1,2,3]. We also exploit this approach.

The exact values of $e(3,k,n)$, for $n \leq 13(k-1)/4 - sign((k-1)\ mod\ 4)$, are given by equations (1) and (2). The values of $e(3,k,n)$, for $k \leq 7$ and all possible n, are listed in [4], and the case $k=8$ was discussed in the previous section. For other parameter situations we proceed as follows. Inequality (3) together with simple analysis of the degree sequences produce reasonable lower bounds for $e(3,k+1,n)$ provided good lower bounds for $e(3,k,n-i)$, $0<i\leq k$, can be given. This computation is essentially a simple case of integer linear programming [4]. Table II reports the results of such calculations, which were performed to obtain lower bounds for $e(3,k,n)$ with $9\leq k\leq 13$ and $3k-1\leq n$. We note that these results improve all of the lower bounds listed in [3].

The entries in table II preceded by a "t" are obtained by applying (2), in which cases they are larger than those obtained by using (3) only. The entries preceded by an "s" are also larger than the values obtained by (3) and in these cases a straightforward checking shows that no graphs can exist for any degree sequence solving (3) with a smaller number of edges. For example, one of the three solutions to (3) for a (3,9,31,92)-graph is $n_6=29$ and $n_5=2$. This implies that there are at least $29=1\cdot 5+4\cdot 6$ edges adjacent to the neighbours of any vertex v of degree 5, and consequently also implies the existence of a $(3,8,25,x)$-graph, for some $x\leq 63=92-29$. This is impossible since $e(3,8,25)\geq 65$. Also observe that at least three values in table II are exact, namely by (2) we have: $e(3,9,26)=52$, $e(3,13,38)=72$ and $e(3,13,39)=78$.

n	k = 9	10	11	12	13
26	t 52				
27	59				
28	67				
29	75	t 57			
30	84	63			
31	s 93	70			
32	103	77	t 62		
33	114	85	68		
34	125	94	75		
35	136	103	81	t 67	
36		113	88	t 73	
37		123	96	79	
38		133	104	86	t 72
39		145	113	93	t 78
40		156	122	100	84
41		169	132	108	91
42		182	143	115	97
43			153	124	104
44			165	132	112
45			177	142	120
46			189	152	128
47			201	162	136
48			214	172	144
49			229	184	153
50			243	196	162
51				208	172
52				221	s 182
53				233	193
54				248	204
55				262	216
56				276	227
57				291	239
58				306	252
59				322	s 266
60					280
61					294
62					308
63					324
64					339
65					354
66					371
67					389
68					406

Table II. Lower bounds for $e(3,k,n)$ for $9\leq k\leq 13$ and $3k-1\leq n$.

If $n(k)$ is the row index in which the last entry of column k appears in table II, then (3) has no solution for any $n > n(k)$. Thus $R(3,k) \leq n(k)+1$. The bound $R(3,14) \leq 78$ is obtained similarly by checking that (3) has no solutions for $k=14$ and $n \geq 78$. Table III gives the new upper bounds together with the best previously known lower and upper bounds for $R(3,k)$, $10 \leq k \leq 14$.

k	lower bound	previous upper bound	new upper bound
10	39	44	43
11	46	54	51
12	49	63	60
13	58	73	69
14	64	84	78
15	71		89

Table III. Bounds for $R(3,k)$, $10 \leq k \leq 14$.

The lower bound $R(3,14) \geq 64$ was established by Longani in 1985 (private communication), the upper bound $R(3,10) \leq 44$ was given in 1968 by Walker [6]. All of the other lower and previous upper bounds in table III were derived by Kalbfleisch in 1966 [3].

Appendix

We have found 396 nonisomorphic minimum (3,8,25,65)-graphs, 62 minimum (3,8,26,73)-graphs and 4 (3,8,27,85)-graphs, however possibly there are more of each of them.

(a) The following minimum (3,8,25,65)-graph H_{25} has the largest group of automorphisms among groups of symmetries for all of these graphs. It's full automorphism group Γ of order 10 is isomorphic to the dihedral group on 5 symbols and is generated by permutations

$$\alpha_1 = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)(10\ 11\ 12\ 13\ 14)(15\ 16\ 17\ 18\ 19)(20\ 21\ 22\ 23\ 24),$$

$$\alpha_2 = (0)(1\ 4)(2\ 3)(5\ 14)(6\ 13)(7\ 12)(8\ 11)(9\ 10)(15\ 24)(16\ 23)(17\ 22)(18\ 21)(19\ 20).$$

The set of edges of H_{25} is the union of orbits of pairs under Γ, whose representatives are: {0,1}, {0,10}, {0,20}, {5,10}, {5,19}, {15,17} and {15,20}. The first orbit has length 5, all the others have length 10 which totals 65 edges. The orbit of singleton {5} forms 10 vertices of degree 4; the remaining 15 vertices have degree 6.

Since no graph with automorphism group larger than 3 was found for the parameter situations (3,8,26,73) and (3,8,27,85), we present examples of the corresponding graphs by their incidence matrices.

(b) A (3,8,26,73)-graph with C_3 as a full automorphism group generated by permutation

$$\alpha = (0\ 4\ 2)(1\ 5\ 3)(6\ 9\ 10)(7\ 8\ 11)(12\ 15\ 16)(13\ 17\ 14)(18\ 22\ 20)(19\ 21\ 23)(24)(25),$$

where the vertices are labeled from 0 to 25 according to the order of rows, is defined by the matrix below:

0 1 0 1 0 0 1 0 0 0 0 0 0 1 0 1 0 0 1 0 0 0 0 0 0 0
1 0 0 0 1 0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0
0 0 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0 0 0 0 1 0 0 0 0 0
1 0 1 0 0 0 0 1 1 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 1 1 0 0 0 0 1 0 0 0
0 0 1 0 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
1 0 0 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 1 0 0 0 0
0 1 0 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 1 1 0 0 0 0 0 0
0 0 0 1 0 1 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1 1 0 0 0
0 0 0 1 1 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0
0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 1 0 0
0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 0 0 1 0 0
0 0 1 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 1 0
1 0 0 0 0 0 0 0 1 1 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 1
0 0 1 0 0 0 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 1
1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1 0
0 0 0 0 1 0 0 0 0 0 1 1 0 0 1 0 0 0 1 0 0 0 0 0 1 0
0 0 0 0 1 0 0 0 0 0 1 1 0 0 0 1 0 0 1 0 0 0 0 0 0 1
1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 0 0 0 1 0 0 0 0
0 0 0 1 0 0 1 1 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 1 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0
0 1 0 0 0 0 1 0 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
0 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 1 0 0
0 0 0 0 0 1 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 0 0 0 0 0 1 0

(c) A (3,8,27,85)-graph with trivial automorphism group is defined by the following matrix:

0 0 0 1 1 0 1 0 1 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 1 0
0 0 0 1 1 0 1 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 1
0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1 0
1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0 0 0 0 0
1 1 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 1 0 0
0 0 1 0 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1
1 1 0 0 0 0 0 0 1 1 0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 0
0 0 1 0 0 0 0 0 0 0 1 1 1 0 1 0 0 1 0 0 0 1 0 0 0 0
1 1 1 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 0 1 0 0 0
0 0 1 0 0 0 1 0 0 0 0 1 0 1 0 0 0 1 0 0 1 1 0 0 0 0
0 0 0 0 0 1 1 0 0 0 0 1 0 1 0 1 0 0 1 1 0 0 0 0 0 0
0 0 0 1 0 1 0 1 0 0 0 0 0 1 0 0 0 0 1 1 0 0 0 1 0 0
0 0 0 1 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
0 0 0 1 1 0 1 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 0 1 0 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 1 0 0
0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 0
1 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 1 0
0 0 0 0 0 0 0 1 0 1 1 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1
0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0 0 0 0 0
0 0 0 0 1 0 0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 0
0 0 0 0 0 1 1 0 1 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0
0 0 0 0 1 0 1 0 0 0 0 1 1 0 0 0 1 0 0 0 0 0 0 0 0 0
1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1
0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0

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