# Search Algorithm for Ramsey Graphs by Union of Group Orbits

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## ABSTRACT

An algorithm for the construction of Ramsey graphs with a given automorphism group $G$ is presented. To find a graph on $n$ vertices with no clique of $k$ vertices, $K_k$, and no independent set of $l$ vertices, $\overline{K}_l$, $k, l \leq n$, with an automorphism group $G$, a Boolean formula $\alpha$ based on the $G$-orbits of $k$-subsets and $l$-subsets of vertices is constructed from incidence matrices belonging to $G$. This Boolean formula is satisfiable if and only if the desired graph exists, and each satisfying assignment to $\alpha$ specifies a set of orbits of pairs of vertices whose union gives the edges of such a graph. Finding these assignments is basically equivalent to the conversion of $\alpha$ from CNF to DNF (conjunctive to disjunctive normal form). Though the latter problem is NP-hard, we present an "efficient" method to do the conversion for the formulas that appear in this particular problem. When $G$ is taken to be the dihedral group $D_n$ for $n \leq 101$, this method matches all of the previously known cyclic Ramsey graphs, as reported by F. R. K. Chung and C. M. Grinstead ["A Survey of Bounds for Classical Ramsey Numbers," *Journal of Graph Theory*, **7** (1983), 25–38], in dramatically smaller computer time when compared to the time required by an exhaustive search. Five new lower bounds for the classical Ramsey numbers are established: $R(4,7) \geq 47$, $R(4,8) \geq 52$, $R(4,9) \geq 69$, $R(5,7) \geq 76$, and $R(5,8) \geq 94$. Also, some previously known cyclic graphs are shown to be unique up to isomorphism.

## 1. INTRODUCTION AND NOTATION

In this paper we only consider two color Ramsey numbers, and follow the definitions and notation of Chung and Grinstead [1]. The two-color Ramsey num-

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ber $R(k,l)$ is defined as the smallest integer $n$ such that, no matter how the edges of the complete graph on $n$ vertices $K_n$ are colored with 2 colors, there exists a monochromatic complete subgraph $K_k$ or $K_l$. Equivalently, $R(k,l)$ is the smallest integer $n$ such that any graph on $n$ vertices contains a clique of size $k$, $K_k$, or an independent set of size $l$, $\overline{K}_l$. We say that a graph $\Gamma$ on $n$ vertices has the $(k,l)$-property, or for short $\Gamma$ is a $(k,l,n)$-graph, if and only if the existence of $\Gamma$ proves that $R(k,l) > n$. A graph $\Gamma$ with vertex set $X = \{0, 1, 2, \ldots, n-1\}$ is *cyclic* if the mapping $g: x \to x+1$ is an automorphism of $\Gamma$, addition performed modulo $n$. Note that any cyclic graph $\Gamma$ must have as an automorphism group at least the dihedral group $D_n = \langle g, h \rangle$, where $h: x \to -x$, since $g^{(n-u-v)}: \{u, v\} \to \{-v, -u\}$. Whence the mapping $h$ is also an automorphism of $\Gamma$.

Table 1 gives all known values for $R(k,l)$, where $3 \le k \le l$, together with the best-known upper and lower bounds for the other Ramsey numbers. The data gathers information from [1] where further extensive references can be found, from [8, 12, 13], and from this paper. The *centered* numbers in the table refer to the exact known values of $R(k,l)$, whereas a pair of numbers gives the best-known lower and upper bounds. A *single* number at the *top* of an entry indicates a lower bound and is given when the above references do not cite any upper bound better than the one implied by the well-known recursive inequality $R(k,l) \le R(k,l-1) + R(k-1,l)$, [3]. We note that the majority of lower bounds of the form $n < R(k,l)$ were established by the construction of cyclic $(k,l,n)$-graphs; the exceptions are marked in Table 1 by "e" or "s." An entry preceded by "e" indicates that the graph has no evident structure; "s" precedes those entries that can be obtained by the method discovered independently by Mathon [12] and Shearer [13]. New values obtained in this paper are marked by "n." We recently learned that the bound $R(4,7) \ge 47$ was already known in

TABLE 1

| $k$ \ $l$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 6 | 9 | 14 | e18 | e23 | e28<br>29 | 36 | 39<br>44 | 46<br>54 | 49<br>63 | 58<br>73 | 63<br>84 |
| 4 | | 18 | 25<br>28 | 34<br>44 | n47 | n52 | n69 | | | | | |
| 5 | | | 42<br>55 | 57<br>94 | n76 | n94 | | | | | | |
| 6 | | | | 102<br>169 | | | | | | | | |
| 7 | | | | | s205<br>586 | | | | | | | |
| 8 | | | | | | 282 | | | | | | |
| 9 | | | | | | | s565 | | | | | |
| 10 | | | | | | | | 798 | | | | |



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in 1985 to V. Longani [all th
that this last bound improves

The regular coloring of edg
graph $K_n$ can be formulated
ulo $n$, $\mathbb{Z}_n$, and consider
$e = \{i,j\}$, $i,j \in \mathbb{Z}_n$ is define
coloring of $K_n$ is regular iff
the same color. In the case
edge and blue as a nonedge.
corresponds exactly to cyclic

Our group theoretical appr
orings, the method by Gulda
the method of sum-free-sets
Another interesting group the
Irving in [7], where they pro

Section 2 presents combina
cidence matrices for our alg
Boolean formula $\alpha$ used later
acterization of $(k,l,n)$-graph
Theorem 3. This characteriza
tion 4, where we also prese
cluding remarks useful for
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## 2. PATTERN MATRICES

We start with the definition
of $t$-designs (generalized Ste
[11] for the construction of

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**Lemma 1.** The incidence
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that a graph $\Gamma$ on $n$ vertices has
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h vertex set $X = \{0,1,2,\dots,$
is an automorphism of $\Gamma$, addi-
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| 36 | 39<br>44 | 46<br>54 | 49<br>63 | 58<br>73 | 63<br>84 |
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1978 to R. Irving, and the bounds $R(4,8) \ge 52$ and $R(3,14) \ge 64$ were known in 1985 to V. Longani [all three unpublished, private communications]. Note that this last bound improves that given in the table.

The regular coloring of edges in the sense of Kalbfleisch [8] of the complete graph $K_n$ can be formulated as follows: Label vertices of $K_n$ by integers modulo $n$, $\mathbb{Z}_n$, and consider $\lfloor n/2 \rfloor$ classes of edges, where the class of the edge $e = \{i,j\}$, $i,j \in \mathbb{Z}_n$ is defined by the number $\mathrm{dist}(e) = \min\{i-j, j-i\}$. A coloring of $K_n$ is regular iff $\mathrm{dist}(e_1) = \mathrm{dist}(e_2)$ implies that edges $e_1$ and $e_2$ have the same color. In the case of two colors, say red and blue, interpret red as an edge and blue as a nonedge. Under such an interpretation the regular 2-coloring corresponds exactly to cyclic graphs with at least a dihedral group of symmetries.

Our group theoretical approach generalizes the above method of regular colorings, the method by Guldan and Tomasta [4], and also partially generalizes the method of sum-free-sets used by Hanson [5] and Hanson and Hanson [6]. Another interesting group theoretical construction was investigated by Hill and Irving in [7], where they proved that $R(7,7) \ge 126$.

Section 2 presents combinatorial and algebraic concepts used to construct incidence matrices for our algorithm. Section 3 gives the construction of the Boolean formula $\alpha$ used later to derive $(k,l,n)$-graphs. Our main result, a characterization of $(k,l,n)$-graphs with a given automorphism group, is given in Theorem 3. This characterization leads to a natural algorithm described in section 4, where we also present some of the details of our implementation, including remarks useful for programming the method. Finally, in section 5 a report of the new lower bounds and other results is given.

## 2. PATTERN MATRICES FOR $(k,l,n)$-GRAPHS

We start with the definition of incidence matrices as applied in [9] in the theory of $t$-designs (generalized Steiner systems) and used by the authors in [10] and [11] for the construction of some simple $t$-designs with $t = 6$.

**Definition 1.** Let $X$ be a set of $n$ elements, $G$ a subgroup of the symmetric group of permutations of $X$, $G \le \mathrm{Sym}(X)$, and $t$, $k$ integers, $1 < t < k < n$. The incidence matrix $B_{tk}$ belonging to the group $G$ is defined as follows:

(a) the rows of $B_{tk}$ are indexed by the $G$-orbits of $t$-subsets of $X$;
(b) the columns of $B_{tk}$ are indexed by the $G$-orbits of $k$-subsets of $X$;
(c) $B_{tk}[I,J] = |\{F \in I : F \subseteq F_0\}|$, where $F_0$ is any fixed representative of orbit $J$.

A simple counting argument gives the first lemma:

**Lemma 1.** The incidence matrix $B_{tk}$ has column sum equal to $\binom{k}{t}$, i.e., for each $J$, a $G$-orbit of $k$-subsets of $X$, $\Sigma\{B_{tk}[I,J]: I$ is a $G$-orbit of $t$-subsets of $X\} = \binom{k}{t}$.

**Lemma 2.** On a vertex set $X$ there is a 1-1 and onto correspondence between the $(k, l, n)$-graphs with an automorphism group $G$ and the (0, 1)-vectors $U$, indexed by the $G$-orbits of 2-subsets of $X$, which simultaneously solve the inequalities

$$(U \cdot B_{2k})[J] < \binom{k}{2} \quad \text{for all } G\text{-orbits } J \text{ of } k\text{-subsets of } X\,. \tag{1}$$

$$(U \cdot B_{2l})[J] > 0 \quad \text{for all } G\text{-orbits } J \text{ of } l\text{-subsets of } X\,. \tag{2}$$

where $B_{2k}$, $B_{2l}$ are incidence matrices belonging to $G$.

***Proof.*** Suppose $\Gamma = (X, E)$ is a $(k, l, n)$-graph with automorphism group $G$. If $\{u, v\} \in E$ then for all $g \in G$ $\{u^g, v^g\} \in E$, which implies that $E$ must be a union of some $G$-orbits of 2-subsets of $X$. Define the vector $U$, indexed by $G$-orbits of 2-subsets of $X$, by $U[I] = 1$ if $I \subseteq E$, otherwise $U[I] = 0$. To show that (1) holds, consider some $G$-orbit $J$ of $k$-subsets of $X$ and take any of its representatives, say $F_0 \in J$. The subgraph induced by $F_0$ in $\Gamma$ has less than $\binom{k}{2}$ edges since $\Gamma$ has no $k$-cliques. Note that $(U \cdot B_{2k})[J]$ counts the number of edges in subgraphs of $\Gamma$ induced by any representative of $J$, in particular by $F_0$, so (1) follows. Similarly, (2) holds since $(U \cdot B_{2l})[J]$ counts number of edges in subgraphs of $\Gamma$ on $l$-vertices. Conversely, let $U$ be a vector satisfying (1) and (2). Define the graph $\Gamma$ on vertices $X$ by

$$E = \cup\{I : I \text{ is a } G\text{-orbit of 2-subsets of } X \text{ and } U[I] = 1\}\,.$$

Now, similarly as before, (1) implies that $\Gamma$ has no $k$-cliques, and (2) implies that $\Gamma$ has no $l$-independent sets. ∎

**Example 1.** Let $n = 8$, $X = \mathbb{Z}_8$, $G = D_8$, $k = 3$, and $l = 4$. We label $D_8$-orbits of 2-, 3-, and 4-subsets of $X$ by sequences describing differences between consecutive elements of a representative in given orbit, for example, the sequence "4 4" denotes the orbit of 2-subsets {04, 15, 26, 37}. The incidence matrices $B_{2,3}$ and $B_{2,4}$ belonging to group $D_8$ are as follows:

| $B_{2,3}$ | (8) 1 1 6 | (16) 1 2 5 | (16) 1 3 4 | (8) 2 2 4 | (8) 2 3 3 |
|---|---|---|---|---|---|
| (8) 1 7 | 2 | 1 | 1 | 0 | 0 |
| (8) 2 6 | 1 | 1 | 0 | 2 | 1 |
| (8) 3 5 | 0 | 1 | 1 | 0 | 2 |
| (4) 4 4 | 0 | 0 | 1 | 1 | 0 |



| $B_{2,4}$ | (8) 1 1 1 5 | (16) 1 1 2 4 | (8 1 2 |
|---|---|---|---|
| (8) 1 7 | 3 | 2 | 2 |
| (8) 2 6 | 2 | 2 | - |
| (8) 3 5 | 1 | 1 | 2 |
| (4) 4 4 | 0 | 1 | - |

The numbers in parenthesis d
orbits indexing rows and co
sequences. The only two (0
and (2) are $U_1 = (1, 0, 0, 1)$
$U_1$ and $U_2$ correspond to the t
morphic. Note also that, acc
column is equal to 3 in $B_{2,3}$ a
The incidence matrices $B_{lk}$
namely, we will show that Le
(0, 1)-matrices $\hat{B}_{lk}$. Let

$B'_{lk}[I, J]$

for all orbits $I, J$ labeling rows
fying equal columns in $B'_{lk}$ as
responding $G$-orbits of $k$-subs

**Lemma 3.** On a vertex set
the $(k, l, n)$-graphs with an a
indexed by the $G$-orbits of
inequalities

$(\overline{U} \cdot \hat{B}_{2k})[J] > 0$ fo

$(U \cdot \hat{B}_{2l})[J] > 0$ fo

where $\overline{U}$ is the binary compl

***Proof.*** We show that a
(4) and (5). Lemma 1 impli
$G$-orbit $J$ of $k$-subsets of $X$ t
$U[I] = 0$ and $B_{2k}[I, J] > 0$
by (3), iff $U$ satisfies (4). A
satisfies (5). ∎

l onto correspondence between
p $G$ and the (0, 1)-vectors $U$,
hich simultaneously solve the

$J$ of $k$-subsets of $X$. (1)

' of $l$-subsets of $X$. (2)

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)h with automorphism group $G$.
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fine the vector $U$, indexed by
$\subseteq E$, otherwise $U[I] = 0$. To
$k$-subsets of $X$ and take any of
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$\cdot B_{2k})[J]$ counts the number of
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$U$ be a vector satisfying (1) and

of $X$ and $U[I] = 1\}$.

is no $k$-cliques, and (2) implies

$= 3$, and $l = 4$. We label $D_8$-
s describing differences between
n orbit, for example, the se-
5, 26, 37}. The incidence ma-
follows:

| (8) 224 | (8) 233 |
|---|---|
| 0 | 0 |
| 2 | 1 |
| 0 | 2 |
| 1 | 0 |

| $B_{2,4}$ | (8) 1115 | (16) 1124 | (8) 1214 | (8) 1133 | (4) 1313 | (16) 1223 | (8) 1232 | (2) 2222 |
|---|---|---|---|---|---|---|---|---|
| (8) 1 7 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 0 |
| (8) 2 6 | 2 | 2 | 1 | 1 | 0 | 2 | 2 | 4 |
| (8) 3 5 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 0 |
| (4) 4 4 | 0 | 1 | 1 | 1 | 2 | 1 | 0 | 2 |

The numbers in parenthesis denote the lengths of the corresponding orbits. The orbits indexing rows and columns of matrices are denoted by their difference sequences. The only two (0, 1)-solutions to the simultaneous inequalities (1) and (2) are $U_1 = (1, 0, 0, 1)$ and $U_2 = (0, 0, 1, 1)$, and according to Lemma 2 $U_1$ and $U_2$ correspond to the two existing cyclic (3, 4, 8)-graphs, which are isomorphic. Note also that, according to Lemma 1, the sum of entries in each column is equal to 3 in $B_{2,3}$ and 6 in $B_{2,4}$.

The incidence matrices $B_{tk}$ contain redundant information for our purposes: namely, we will show that Lemma 2 can be modified to be true for simplified (0, 1)-matrices $\hat{B}_{tk}$. Let

$$B'_{tk}[I, J] = \begin{cases} 1, & \text{if } B_{tk}[I, J] > 0; \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

for all orbits $I, J$ labeling rows and columns of $B_{tk}$. Define $\hat{B}_{tk}$ from $B'_{tk}$ by identifying equal columns in $B'_{tk}$ as one column in $\hat{B}_{tk}$ labeled by the union of the corresponding $G$-orbits of $k$-subsets.

**Lemma 3.** On a vertex set $X$ there is a 1-1 and onto correspondence between the $(k, l, n)$-graphs with an automorphism group $G$ and the (0, 1)-vectors $U$, indexed by the $G$-orbits of 2-subsets of $X$, which simultaneously solve the inequalities

$$(\overline{U} \cdot \hat{B}_{2k})[J] > 0 \quad \text{for all } G\text{-orbits } J \text{ labeling a column of } \hat{B}_{2k}. \tag{4}$$

$$(U \cdot \hat{B}_{2l})[J] > 0 \quad \text{for all } G\text{-orbits } J \text{ labeling a column of } \hat{B}_{2l}. \tag{5}$$

where $\overline{U}$ is the binary complement of vector $U$.

***Proof.*** We show that a (0, 1)-vector $U$ satisfies (1) and (2) iff $U$ satisfies (4) and (5). Lemma 1 implies that a (0, 1)-vector $U$ satisfies (1) iff for each $G$-orbit $J$ of $k$-subsets of $X$ there exists a $G$-orbit $I$ of 2-subsets of $X$, such that $U[I] = 0$ and $B_{2k}[I, J] > 0$. The latter holds iff $\overline{U}[I] = 1$ and $\hat{B}_{2k}[I, J] = 1$ by (3), iff $U$ satisfies (4). A similar argument yields that $U$ satisfies (2) iff $U$ satisfies (5). ∎

**Example 1 (continued).** The simplified (0, 1)-matrices $\hat{B}_{2,3}$ and $\hat{B}_{2,4}$ are as follows:

| $\hat{B}_{2,3}$ | 116 | 125 | 134 | 224 | 233 |
|---|---|---|---|---|---|
| 17 | 1 | 1 | 1 | 0 | 0 |
| 26 | 1 | 1 | 0 | 1 | 1 |
| 35 | 0 | 1 | 1 | 0 | 1 |
| 44 | 0 | 0 | 1 | 1 | 0 |

| $\hat{B}_{2,4}$ | 1115 1232 | 1124 1214 1133 1223 | 1313 | 2222 |
|---|---|---|---|---|
| 17 | 1 | 1 | 1 | 0 |
| 26 | 1 | 1 | 0 | 1 |
| 35 | 1 | 1 | 1 | 0 |
| 44 | 0 | 1. | 1 | 1 |

According to Lemma 3, $U_1 = (1, 0, 0, 1)$ and $U_2 = (0, 0, 1, 1)$ are the only two (0, 1)-solutions to (4) and (5).

Define $S = S(\hat{B}_{tk})$ to be the set of columns of the matrix $\hat{B}_{tk}$. Consider the natural partial ordering $\leq$ on $S$, $(S, \leq)$, implied by the coordinatewise order $0 \leq 1$. We show that only the minimal elements of $S$ are important for the construction of $(k, l, n)$-graphs.

**Definition 2.** The pattern matrix $P_{tk}$, belonging to the group $G$ is the column submatrix of the (0, 1)-matrix $\hat{B}_{tk}$, belonging to the same group $G$, and has exactly those columns of $\hat{B}_{tk}$ that correspond to the minimal elements of partial order $(S(\hat{B}_{tk}), \leq)$.

We now state the final version of Lemma 2 as a theorem:

**Theorem 1.** On a vertex set $X$ there is a 1-1 and onto correspondence between the $(k, l, n)$-graphs with $G \leq \mathrm{Sym}(X)$ as an automorphism group and the (0, 1)-vectors $U$ indexed by the $G$-orbits of 2-subsets of $X$, which simultaneously solve the inequalities

$$(\overline{U} \cdot P_{2l})[J] > 0 \quad \text{for all } G\text{-orbits } J \text{ labeling a column of } P_{2l}, \tag{6}$$

$$(U \cdot P_{2k})[J] > 0 \quad \text{for all } G\text{-orbits } J \text{ labeling a column of } P_{2k}, \tag{7}$$

where $P_{2l}, P_{2k}$ are pattern matrices belonging to the group $G$.



*Proof.* By Lemma 3 we
(4) and (5) iff $U$ satisfies (6)
implies (6) and (5) implies (7
column of $P_{tk}$, by the definit
$P_{tk}$, such that $J \leq I$ in the
implies (4) and (7) implies (

**Example 1 (continued).** T

| $P_{2,3}$ | 1 |
|---|---|
| 17 | |
| 26 | |
| 35 | 0 |
| 44 | 0 |

| $P_{2,4}$ | 1 1 |
|---|---|
| 17 | |
| 26 | |
| 35 | |
| 44 | |

Although Example 1 explai
not show the amount of red
ple of typical reduction, cc
and $G = D_{56}$ there are 3,8
and the incidence matrices
respectively, while $P_{2,5}$ anc
more, using the method de
the cyclic (5, 6, 56)-graphs
Finally, assuming a fixed
define the *k-clique pattern m*
*matrix In* as $P_{2l}$. Note that
nal Ramsey numbers $Cl_k$
required for the generation

## 3. BOOLEAN CALCULU

Let $m$ be the number of G
matrices with $m$ rows. We
variables $x_1, x_2, \ldots, x_m$ with

*Proof.* By Lemma 3 we only need to show that a (0, 1)-vector $U$ satisfies (4) and (5) iff $U$ satisfies (6) and (7). Since $P_{2k}$ is a column submatrix of $\hat{B}_{2k}$, (4) implies (6) and (5) implies (7). Conversely, for any column $I$ of $\hat{B}_{2k}$ that is not a column of $P_{2k}$, by the definition of $P_{2k}$ there exists some minimal column $J$ in $P_{2k}$ such that $J \leq I$ in the sense of partial order $(S, \leq)$. Consequently, (6) implies (4) and (7) implies (5). ∎

**Example 1 (continued).** The pattern matrices $P_{2,3}$ and $P_{2,4}$ are as follows:

| $P_{2,3}$ | 116 | 134 | 224 | 233 |
|---|---|---|---|---|
| 17 | 1 | 1 | 0 | 0 |
| 26 | 1 | 0 | 1 | 1 |
| 35 | 0 | 1 | 0 | 1 |
| 44 | 0 | 1 | 1 | 0 |

| $P_{2,4}$ | 1115 1232 | 1313 | 2222 |
|---|---|---|---|
| 17 | 1 | 1 | 0 |
| 26 | 1 | 0 | 1 |
| 35 | 1 | 1 | 0 |
| 44 | 0 | 1 | 1 |

Although Example 1 explains the sequence of constructions performed, it does not show the amount of reduction that occurs for larger matrices. As an example of typical reduction, consider the following: For $n = 56$, $k = 5$, $l = 6$, and $G = D_{56}$ there are 3,819,816 5-subsets and 32,468,436 6-subsets of $\mathbb{Z}_{56}$, and the incidence matrices $B_{2,5}$ and $B_{2,6}$ have 34,111 and 289,955 columns, respectively, while $P_{2,5}$ and $P_{2,6}$ have only 6164 and 9221 columns. Furthermore, using the method described in section 4, this situation easily produces the cyclic (5, 6, 56)-graphs found in [6].

Finally, assuming a fixed automorphism group $G$ acting on $n$ vertices, let us define the *k-clique pattern matrix* $Cl_k$ to be $P_{2k}$ and the *l-independent-set pattern matrix* $In_l$ as $P_{2l}$. Note that in the case of searching for lower bounds for diagonal Ramsey numbers $Cl_k = In_l$, which *halves* the amount of calculations required for the generation of matrices.

## 3. BOOLEAN CALCULUS

Let $m$ be the number of $G$-orbits of 2-subsets of $X$, so $Cl_k$ and $In_l$ are (0, 1)-matrices with $m$ rows. We will work with propositional Boolean calculus on $m$ variables $x_1, x_2, \ldots, x_m$ with the following operators: "$-$" for negation, "$+$" (or

$\Sigma$) for disjunction, and "$\cdot$" (or $\Pi$) for conjunction. If no confusion arises conjuction is also represented by juxtaposition. The logical meaning of each variable $x_i$, $1 \leq i \leq m$, is if the $i$th $G$-orbit of 2-subsets of $X$ should or should not be included in the edge set of the $(k, l, n)$-graph under construction. Given a $(0, 1)$-vector $V = (v_1, v_2, \ldots, v_m)$ denote by $\text{neg}(V) = \sum_{v_i=1} \bar{x}_i$ and $\text{pos}(V) = \sum_{v_i=1} x_i$.

**Definition 3.** The $(k, l, n)$-graph formula $\alpha_{kln}$ belonging to group $G$ is $\alpha_{kln} = \beta_{kn} \cdot \gamma_{ln}$, where $\beta_{kn} = \Pi\{\text{neg}(J)\colon J \text{ is a column of } Cl_k\}$ and $\gamma_{ln} = \Pi\{\text{pos}\,(J)\colon J \text{ is a column of } In_l\}$.

**Theorem 2.** On a vertex set $X$ there is a 1-1 and onto correspondence between the $(k, l, n)$-graphs with an automorphism group $G$ and the $(0, 1)$-assignments to the variables $x_1, x_2, \ldots, x_m$ satisfying the $(k, l, n)$-graph formula $\alpha_{kln}$ belonging to group $G$.

***Proof.*** By Theorem 1 we need only to show a 1-1 and onto correspondence between vectors $U$ satisfying (6) and (7) and true assignments to formula $\alpha_{kln}$. The correspondence is $x_i = 1$ iff $U[I] = 1$, where $I$ is the $i$th $G$-orbit of 2-subsets of $X$, for $i = 1, \ldots, m$. Each of the inequalities in (6) with $J$, a $G$-orbit of $k$-subsets, is equivalent to the fact that at least one of the literals in the clause neg($J$) is true, so $U$ satisfies (6) iff $\beta_{kn}$ is true under the assignment to $x_i$s defined by $U$. Similarly $U$ satisfies (7) iff $\gamma_{ln}$ is true under the assignment, defined by $U$. The result follows, since $\alpha_{kln} = \beta_{kn} \cdot \gamma_{ln}$. ∎

We summarize Theorems 1 and 2 with the following characterization theorem:

**Theorem 3.** The following are all equivalent for a graph $\Gamma = (X, E)$ with $G \leq \text{Sym}(X)$ as an automorphism group:

(i) $\Gamma$ is a $(k, l, n)$-graph;
(ii) Each entry of $U \cdot B_{2k}$ is less than $\binom{k}{2}$ and each entry of $U \cdot B_{2l}$ is nonzero, where $U[I] = 1$ if $I \subseteq E$, 0 otherwise;
(iii) The Boolean formula $\alpha_{kln}$ is satisfied by the assignment $x_i = 1$ iff $E$ contains the $i$th orbit of edges.

The formula $\alpha_{kln}$ is in conjunctive normal form (CNF); as it is well known, the derivation of all the satisfying assignments is computationally equivalent to the conversion of the formula to its disjunctive normal form (DNF). The corresponding decision problem is NP-complete [2], so we do not pretend to give an efficient general algorithm for such a transformation. However, in the next section we will describe a reasonable approach, which was used successfully to solve all the pattern matrices we were able to generate so far. To avoid an explosion in the number of terms during the transformation to DNF, we use the heuristic strategy of alternately multiplying factors from $\beta_{kn}$ and $\gamma_{ln}$, preferring



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**Example 1 (continued).**

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**Example 2.** A search
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(b) $G_2 = D_{17}$. The pa
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(c) $G_3 = SAF_{17}$, the s
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those having a smaller number of terms and/or those leading to immediate cancellation of terms by the subsumption and/or contradiction rules of Boolean algebra.

**Example 1 (continued).** The formulas from Definition 3 for this example are

$$\beta_{3,8} = (\bar{x}_1 + \bar{x}_2)(\bar{x}_1 + \bar{x}_3 + \bar{x}_4)(\bar{x}_2 + \bar{x}_4)(\bar{x}_2 + \bar{x}_3),$$

$$\gamma_{4,8} = (x_1 + x_2 + x_3)(x_1 + x_3 + x_4)(x_2 + x_4).$$

Using the strategy mentioned above, with easy algebraic manipulations we obtain

$$\alpha_{3,4,8} = \beta_{3,8} \cdot \gamma_{4,8} \equiv x_1\bar{x}_2\bar{x}_3x_4 + \bar{x}_1\bar{x}_2x_3x_4,$$

which gives us exactly two satisfying assignments to $\alpha_{3,4,8}$ and by Theorem 2 they define the only two cyclic (3, 4, 8)-graphs.

The formula equivalent to $\alpha_{kln}$, but with many more clauses, could be defined directly from matrices $B'_{2k}$ and $B'_{2l}$, and still would satisfy Theorem 2. However, the sequence of constructions presented here reflects the structure of the most efficient algorithm we were able to design.

**Example 2.** A search for (4, 4, 17)-graphs is as follows: Since $k = l = 4$, the pattern matrices $Cl_4$ and $In_4$ are both equal to $P_{2,4}$ for any group chosen. Set $X = \mathbb{Z}_{17}$ and consider the three different groups:

(a) The trivial group $G_1$. The pattern matrix $P_{2,4}$ belonging to $G_1$ has $\binom{17}{2}$ = 136 rows and $\binom{17}{2}$ = 2380 columns, and solving it is equivalent to an exhaustive search among all graphs on 17 vertices.
(b) $G_2 = D_{17}$. The pattern matrix $P_{2,4}$ belonging to $G_2$ has 8 rows and 12 columns. The formula $\alpha_{4,4,17}$ can be calculated by hand, leading to two self-complementary (4, 4, 17)-graphs $\Gamma_1$ and $\Gamma_2$, where $\Gamma_i = (X, E_i)$, for $i = 1, 2$, $E_1 = \{e : \mathrm{dist}(e) \in \{1, 2, 4, 8\}\}$ and $E_2 = \{e : \mathrm{dist}(e) \in \{3, 5, 6, 7\}\}$.
(c) $G_3 = SAF_{17}$, the special affine group defined by $\{x \to ax + b : a$ is a nonzero square in $\mathbb{Z}_{17}, b \in \mathbb{Z}_{17}\}$. The pattern matrix $P_{2,4}$ belonging to $G_3$ has two rows corresponding to squares and nonsquares in $\mathbb{Z}_{17}$ and one column. The same solutions as in (b) are obtained immediately without effort.

Thus it is clear that the choice of automorphism group $G$ is crucial in obtaining a tractable Boolean formula $\alpha_{kln}$. The choice of the correct group is perhaps one of the most difficult steps. It is easy to see that the chosen group cannot act more than transitively, for otherwise there would be only one orbit of edges. Although the dihedral groups appear profitable (all new lower bounds obtained in this paper use a dihedral group), we believe further improvements of lower bounds will be obtained with groups that do not lead to cyclic graphs.

## 4. ALGORITHM

The algorithm that follows naturally from Theorem 3 is

### Algorithm

(1) Input a chosen group $G.G \leq \mathrm{Sym}(X)$, $|X| = n$, as a candidate for an automorphism group of a $(k, l, n)$-graph.
(2) Construct the incidence matrices $\hat{B}_{2k}$ and $\hat{B}_{2l}$ belonging to $G$.
(3) Construct the pattern matrices $P_{2k}$ and $P_{2l}$ belonging to $G$ by finding minimal columns of matrices $\hat{B}_{2k}$ and $\hat{B}_{2l}$.
(4) Build the $(k, l, n)$-graph formula $\alpha_{kln}$ belonging to $G$ and find all the satisfying assignments for $\alpha_{kln}$. Each such assignment (if any) yields a $(k, l, n)$-graph with automorphism group $G$; furthermore, all such graphs are obtained.

The programs were written in the programming language C for the supermicrocomputer MASSCOMP MC 500 running UNIX.* In our implementation, rather than building one big program, we have followed the spirit of UNIX by writing a package of programs for separate tasks and then using them as tools together with system facilities. We will comment briefly on each of the steps (2) through (4) above, stressing the more important algorithms and techniques used.

(2) We found more problems with huge memory required to store matrices rather than with the time of computation. The columns of binary matrices were packed into integers, which saves memory and permits an extensive use of fast word bitwise operations. For the dihedral group, it is fairly easy to write a program with an output stream formed by the columns of matrix $B'_{2k}$. Further memory savings were obtained at this stage of computation by buffering this output and performing local subsumption of columns, i.e., if inside the buffer, $I \leq J$ for some columns $I$ and $J$, column $J$ was eliminated. This reduced stream of columns was distributed over different files according to the number of ones in each column. Finally, the matrix $\hat{B}_{2k}$ was produced by sorting files and deleting identical elements. This was done on each file separately.
(3) The form of data obtained after step (2) permits efficient construction of matrices $P_{2k}$, since to find minimal columns we only need to execute a

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*UNIX is a trademark of Bell Laboratories.



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## 5. RESULTS

### 5.1 New Lower Boun

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$R(4, 7) \geq 47$, $n = 46$

DIST = {1, 2, 4, 12, 13,
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subsumption algorithm for each file of columns against only those files with a larger number of ones in each column.

(4) To find satisfying assignments to formula $\alpha_{k,l,n}$ a recursive backtracking algorithm is used. During the recursion we keep track of statistics of number and location of columns (clauses of the formula $\alpha_{k,l,n}$) with a fixed number of ones. Once the value of some variable $x_i$ is assigned to 0 or 1, the $i$th rows in $Cl_k$ and $In_l$ are eliminated. If $x_i = 0$ then all the columns of $Cl_k$ satisfying (6) are eliminated; if $x_i = 1$ then all the columns of $In_l$ satisfying (7) are eliminated. In both cases the statistics are updated and then used to choose next variable $x_j$. When both matrices $Cl_k$ and $In_l$ vanish then a satisfying assignment(s) to the formula $\alpha_{k,l,n}$ has been found. The creation of a column with no ones returns recursion to the higher level. This technique, with the heuristic strategy mentioned in section 3, permits the implementation of an efficient branch and bound recursive algorithm with a surprisingly narrow recursion tree. For example, in the calculation of $R(4,7) \geq 47$ the formula $\alpha_{4,7,46}$ has $2^{23} = 8{,}388{,}608$ possible assignments, but in our backtrack algorithm only 1127 assignments were considered to find all of the 11 cyclic (4, 7, 46)-graphs.

It seems that we are able to solve practical problems in which there is a large number of columns in the incidence matrices. The bottleneck of the method is the number of rows, i.e., the number of orbits of edges. With our current implementation we can handle matrices with up to 64 rows.

## 5. RESULTS

### 5.1 New Lower Bounds

For each of the new lower bounds obtained, $(k, l, n)$-graphs were constructed on vertex set $\mathbb{Z}_n$ with the dihedral group $D_n$ as an automorphism group. For each parameter situation the graphs are specified by a set of values DIST $\subseteq \mathbb{Z}_n$, where the graph has an edge $e$ iff dist($e$) $\in$ DIST. For notational convenience $pr(n)$ denotes the set of positive integers relatively prime to $n$ and smaller than $n/2$, and the set obtained by multiplying DIST by $s \in pr(n)$ is {dist({0, $sx$}): $x \in$ DIST}.

#### $R(4, 7) \geq 47$, $n = 46$

DIST = {1, 2, 4, 12, 13, 17, 19, 20}. This is the unique cyclic (4, 7, 46)-graph, up to isomorphism. All of the 11 cyclic (4, 7, 46)-graphs can be obtained from this one by multiplying DIST by the 11 numbers in $pr(46)$. There does not exist a cyclic $(4, 7, n)$-graph for $n = 47, 48$, and 49.

*R(4, 8) ≥ 52, n = 51*

There are exactly 4 nonisomorphic cyclic (4, 8, 51)-graphs given by

(a) $DIST_1 = \{1, 2, 5, 6, 8, 12, 15, 17, 25\}$.
(b) $DIST_2 = \{1, 2, 5, 6, 9, 12, 17, 19, 25\}$.
(c) $DIST_3 = \{1, 2, 5, 9, 11, 12, 17, 19, 25\}$,
(d) $DIST_4 = \{1, 2, 5, 9, 12, 17, 19, 25\}$.

All of the 64 cyclic (4, 8, 51)-graphs are obtained by multiplying $DIST_i$, $i = 1, 2, 3, 4$, by the 16 numbers from *pr*(51). The graph defined by $DIST_4$ is a subgraph of those defined by $DIST_2$ and $DIST_3$. There does not exist a cyclic (4, 8, $n$)-graph for $n = 52, 53$.

*R(4, 9) ≥ 69, n = 68*

There are exactly 7 nonisomorphic cyclic (4, 9, 68)-graphs given by

(a) $DIST_1 = \{1, 2, 6, 7, 9, 10, 15, 18, 22, 32, 33\}$.
(b) $DIST_2 = \{1, 2, 9, 10, 15, 16, 18, 23, 24, 28, 32\}$.
(c) $DIST_3 = \{1, 2, 6, 7, 9, 18, 19, 24, 28, 32, 33\}$,
(d) $DIST_4 = \{1, 4, 5, 10, 12, 21, 22, 24, 27, 28, 33\}$,
(e) $DIST_5 = \{1, 2, 7, 11, 12, 17, 18, 20, 27, 28, 32\}$.
(f) $DIST_6 = \{1, 4, 5, 10, 12, 22, 24, 27, 28, 33\}$.
(g) $DIST_7 = \{1, 4, 5, 10, 12, 22, 24, 25, 27, 28, 33\}$.

All of the 112 cyclic (4, 9, 68)-graphs are obtained by multiplying $DIST_i$, $i = 1, \ldots, 7$ by the 16 numbers from *pr*(68). The graph defined by $DIST_6$ is a subgraph of those defined by $DIST_4$ and $DIST_7$. There does not exist a cyclic (4, 9, 69)-graph.

*R(5, 7) ≥ 76, n = 75*

There are exactly 4 nonisomorphic cyclic (5, 7, 75)-graphs given by

(a) $DIST_1 = \{1, 2, 3, 5, 9, 10, 12, 16, 19, 22, 24, 26, 27, 31, 32, 33\}$.
(b) $DIST_2 = \{1, 2, 3, 5, 8, 9, 10, 17, 19, 20, 28, 30, 33, 34, 36\}$.
(c) $DIST_3 = \{1, 2, 3, 5, 8, 9, 10, 17, 19, 20, 21, 28, 30, 33, 34, 36\}$.
(d) $DIST_4 = \{1, 2, 3, 6, 7, 8, 15, 16, 17, 19, 22, 27, 28, 31, 33, 34\}$.

All of the 80 cyclic (5, 7, 75)-graphs are obtained by multiplying $DIST_i$, $i = 1, 2, 3, 4$, by the 20 numbers from *pr*(75). The graph defined by $DIST_2$ is a subgraph of that defined by $DIST_3$. There does not exist a cyclic (5, 7, 76)-graph.



*R(5, 8) ≥ 94, n = 93*

Two of the several cyclic

DIST = {1, 2, 3, 11, 12
$DIST_2$ = DIST ∪ {37}

The full search was not co

## 5.2. Uniqueness of Gr

*R(5, 6) ≥ 57, n = 56*

The cyclic (5, 6, 56)-graph

DIST = {2, 3, 6, 9, 14,

was given in [6]. We have
to isomorphism. All of the
one by multiplying DIST

*R(6, 6)*

A cyclic (6, 6, 101)-graph
$a < 51\}$, which proves
algorithm we have found t

(a) The above graph is t
(b) There does not exis

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existence of a cyclic ($k$, $l$
there exists a cyclic (4, 4,
graph]. It would be intere

## 5.3. Note

Searching for the answer
case: The cyclic graph or
(3, 5, 12)-graph. Modular
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$k \neq l$, by assuming $1 \in$ D

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(51). The graph defined by
DIST$_3$. There does not exist

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32},
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ph defined by DIST$_6$ is a sub-
There does not exist a cyclic

)-graphs given by

26, 27, 31, 32, 33}.
0, 33, 34, 36}.
, 28, 30, 33, 34, 36},
27, 28, 31, 33, 34}.

ed by multiplying DIST$_i$, $i$ =
graph defined by DIST$_2$ is a
s not exist a cyclic (5, 7, 76)-

*R(5, 8)* $\geq$ *94,* n = *93*

Two of the several cyclic (5, 8, 93)-graphs we found are

$$\text{DIST}_1 = \{1, 2, 3, 11, 12, 14, 16, 17, 18, 20, 22, 24, 27, 29, 31, 32, 39, 40, 46\},$$
$$\text{DIST}_2 = \text{DIST}_1 \cup \{37\}.$$

The full search was not completed.

### 5.2. Uniqueness of Graphs that Appear in the Literature

*R(5, 6)* $\geq$ *57,* n = *56*

The cyclic (5, 6, 56)-graph with

$$\text{DIST} = \{2, 3, 6, 9, 14, 16, 18, 19, 23, 24, 25, 27, 28\}$$

was given in [6]. We have found that it is the unique cyclic (5, 6, 56)-graph, up to isomorphism. All of the 12 cyclic (5, 6, 56)-graphs can be obtained from this one by multiplying DIST by the 12 numbers in $pr(56)$.

*R(6, 6)*

A cyclic (6, 6, 101)-graph with DIST = $\{a: a$ is nonzero square in $\mathbb{Z}_{101}$ and $a < 51\}$, which proves that $R(6, 6) \geq 102$, was given in [8]. Using our algorithm we have found that

(a) The above graph is the unique up to isomorphism cyclic (6, 6, 101)-graph.
(b) There does not exist a cyclic $(6, 6, n)$-graph for $n$ = 100, 102, and 103.

Thus the existence of a cyclic $(k, l, n + 1)$-graph does not always imply the existence of a cyclic $(k, l, n)$-graph [a simpler counterexample is as follows: there exists a cyclic (4, 4, 17)-graph, but there does not exist a cyclic (4, 4, 16)-graph]. It would be interesting to find other such counterexamples.

### 5.3. Note

Searching for the answer to a question in [6], we have noted an interesting case: The cyclic graph on $\mathbb{Z}_{12}$ defined by DIST = {2, 3} is the unique cyclic (3, 5, 12)-graph. Modular multiplication of DIST by $pr(12) = \{1, 5\}$ fixes the graph defined by DIST. Hence, in the search for cyclic $(k, l, n)$-graphs with $k \neq l$, by assuming $1 \in$ DIST the generality can be lost.

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