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Simple 5–(28,6,λ) Designs from $PSL_2(27)$

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TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY

ABSTRACT

Simple 5–(28,6,λ) designs with $PSL_2(27)$ as an automorphism group are constructed for each λ, $2 \leq \lambda \leq 21$.

1. Introduction

A *t–design,* or t–(v,k,λ) *design* is a pair (X,B) with a v-set X of *points* and a family B of k–subsets of X called *blocks* such that any t points are contained in exactly λ blocks. A t–(v,k,λ) design (X,B) is *simple* if no block in B is repeated and is said to have $G \leq Sym(X)$ as an *automorphism group* if whenever K is a block $K^\alpha = \{x^\alpha : x \in K\}$ is also a block for all $\alpha \in G$.

In this paper we show that for each λ, $2 \leq \lambda \leq 21$, a 5–(28,6,λ) design exists with $G = PSL_2(27)$ as an automorphism group, leaving open only the existence of a 5–(28,6,1) design with some group other than $PSL_2(27)$. Incidently, Denniston showed that a 5–(28,7,1) design does indeed exist with $PSL_2(27)$, see [2].

2. Preliminaries

As with our construction of two disjoint nonisomorphic simple 6–(14,7,4) designs [6] as well as with the only other known examples of 6-designs with small λ [4,9], we start with the following observation of Kramer and Mesner [3]:

A t–(v,k,λ) design exists with $G \leq Sym(X)$ as an automorphism group if and only if there is a (0,1)–solution vector U to the matrix equation

$$A_{tk}U = \lambda J, \tag{1}$$

where

a. The rows of A_{tk} are indexed by the G–orbits of t–subsets of X;

b. The columns of A_{tk} are indexed by the G–orbits of k–subsets of X;

c. $A_{tk}[\Delta,\Gamma] = |\{K \in \Gamma : K \supset T_0\}|$ where $T_0 \in \Delta$ is any representative;

d. $J = [1,1,1,\ldots,1]^T$.

If we choose the group G to be $PSL_2(27)$ acting on the projective line $X = GF(3^3) \bigcup \infty$ then G has order 9828 and an isomorphic copy is generated by the permutations α, β, and γ on $\{0,1,2,\ldots 27\}$ given below:

$\alpha = (0,26,27)(1,2,10)(3,6,4)(5,15,19)(7,11,21)(8,17,14)(9,18,12)(13)(16,24,25)(20,23,22);$
$\beta = (0,13)(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)(14,25)(15,24)(16,23)(17,22)(18,21)(19,20)(26,27);$
$\gamma = (0,23,22,10,11,1,27,26,25,15,16,4,3)(2,9,13,17,24,19,8,20,14,12,6,18,7)(5)(21).$

In this action G has exactly 10 orbits of 5–subsets and exactly 54 orbits of 6–subsets, see figure 1 and 2. Hence, the $A_{5\,6}$ matrix belonging to G has 10 rows and 54 columns and is displayed in figure 3.

No.	Representative		
1	0 1 2 6 7	3	0 1 2 3 6
2	0 1 2 4 6	4	0 1 2 4 11
3	0 1 2 3 6	5	0 1 2 4 8
4	0 1 2 4 11	6	0 1 2 3 8
5	0 1 2 4 8	7	0 1 2 5 8

Figure 1. Orbit representatives of 5-subsets

No.	Representative				
1	0 1 2 6 7 16	19	0 1 2 6 7 14	37	0 1 2 3 8 17
2	0 1 2 6 7 12	20	0 1 2 3 6 14	38	0 1 2 3 6 20
3	0 1 2 6 7 8	21	0 1 2 6 7 13	39	0 1 2 6 7 19
4	0 1 2 4 6 7	22	0 1 2 4 11 13	40	0 1 2 4 8 18
5	0 1 2 4 5 6	23	0 1 2 4 6 14	41	0 1 2 4 6 20
6	0 1 2 3 4 6	24	0 1 2 4 6 15	42	0 1 2 6 7 21
7	0 1 2 3 6 7	25	0 1 2 3 6 15	43	0 1 2 4 6 21
8	0 1 2 5 6 7	26	0 1 2 4 11 14	44	0 1 2 3 6 21
9	0 1 2 4 6 8	27	0 1 2 4 11 15	45	0 1 2 4 6 22
10	0 1 2 6 7 10	28	0 1 2 6 7 15	46	0 1 2 4 11 21
11	0 1 2 4 6 9	29	0 1 2 3 6 16	47	0 1 2 6 7 25
12	0 1 2 3 6 9	30	0 1 2 6 7 22	48	0 1 2 4 6 23
13	0 1 2 6 7 9	31	0 1 2 6 7 20	49	0 1 2 3 6 23
14	0 1 2 6 7 11	32	0 1 2 6 7 18	50	0 1 2 4 11 22
15	0 1 2 4 7 11	33	0 1 2 6 7 17	51	0 1 2 4 6 25
16	0 1 2 5 8 10	34	0 1 2 3 8 16	52	0 1 2 4 6 26
17	0 1 2 4 8 11	35	0 1 2 3 12 16	53	0 1 2 3 6 26
18	0 1 2 3 8 11	36	0 1 2 4 6 18	54	0 1 2 6 7 27

Figure 2. Orbit representatives of 6-subsets

1 1 1 1 0 0 1 1 0 2 0 0 1 2
0 0 2 1 2 1 1 0 1 1 1 0 0 0
1 0 0 0 2 1 1 1 0 0 1 1 2 0
0 0 0 0 0 0 0 0 0 1 0 0 0 1
0 1 1 0 0 1 0 0 1 0 1 1 1 0
1 0 2 0 0 0 0 0 0 1 0 0 0 0
2 0 0 0 0 0 0 0 0 0 1 1 0 1
0 0 0 0 0 0 0 1 0 1 0 0 0 1
0 0 0 0 0 0 0 0 1 0 2 0 2 0
1 0 0 1 2 0 0 0 0 0 0 0 0 1

Figure 3

The task of solving th
54 unknowns for a (0,1)-s
algorithm presented at the
torics Graph Theory and C
part on the L^3 algorithm
inspired by the work of J.C

3. Results

In figure 4 for each λ
whose union forms a 5–(2
design is a 5–(28,6,23-λ) de
the interest of brevity we g
to see that these solutions
can check using an algorit
5–(28,6,λ) designs.

λ	Orbit numl
2	12 15 18 28
3	7 8 9 22 26
4	2 21 22 35
5	2 8 22 28 3
6	1 10 11 13
7	1 4 7 8 11
8	1 2 8 9 10
9	3 8 9 13 14
10	1 2 3 5 6 1
11	2 3 5 9 11

rojective line $X = GF(3^3) \cup \infty$
nerated by the permutations

.12)(13)(16,24,25)(20,23,22);
.23)(17,22)(18,21)(19,20)(26,27);
20,14,12,6,18,7)(5)(21).

exactly 54 orbits of 6-subsets,
G has 10 rows and 54 columns

3 6
4 11
4 8
3 8
5 8

5-subsets

?	0 1 2 3 8 17
38	0 1 2 3 6 20
39	0 1 2 6 7 19
40	0 1 2 4 8 18
41	0 1 2 4 6 20
42	0 1 2 6 7 21
43	0 1 2 4 6 21
44	0 1 2 3 6 21
45	0 1 2 4 6 22
46	0 1 2 4 11 21
47	0 1 2 6 7 25
48	0 1 2 4 6 23
49	0 1 2 3 6 23
50	0 1 2 4 11 22
51	0 1 2 4 6 25
52	0 1 2 4 6 26
53	0 1 2 3 6 26
54	0 1 2 6 7 27

6-subsets

1 1 1 1 0 0 1 1 0 2 0 0 1 2 0 0 0 0 2 0 1 0 0 0 0 0 0 1 0 1 1 1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 0 0 0 1
0 0 2 1 2 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 1 0 1 0 1 0 1 1 0 0 1 1 0 1
1 0 0 0 2 1 1 1 0 0 1 1 2 0 0 0 0 0 1 1 1 0 0 1 1 0 0 1 1 0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 1 1 0 0 0 1 0
0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 0 2 0 1 1 0 1 1 2 1 1 1 0 2 0 0 0 0 0 0 0 0 0 0 0 0 0 2 0 0 1 0 0 1 1 1 1 0 0
0 1 1 0 0 1 0 0 1 0 1 1 1 0 0 0 1 0 0 2 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 1 1 0 0 0 0 2 1 2 0 0
1 0 2 0 0 0 0 0 0 1 0 0 0 0 0 0 1 2 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1 0 1 1 0 0 1 0 0 1 1 2 1 2 1 0 0 0 1 0 0
2 0 0 0 0 0 0 0 0 0 1 1 0 1 0 2 1 0 0 0 1 0 0 0 0 1 0 1 1 0 0 1 0 1 0 1 1 0 0 0 0 2 1 2 0 0 0 0 0 1 0 0 1 0
0 0 0 0 0 0 0 1 0 1 0 0 0 1 1 0 0 0 1 1 0 0 1 0 0 1 1 0 0 1 0 2 0 0 0 2 0 0 0 0 0 0 0 0 0 1 0 2 1 1 0 1 2 1
0 0 0 0 0 0 0 0 1 0 2 0 2 0 0 0 1 0 1 1 0 0 0 1 0 0 1 2 1 0 0 1 0 1 2 0 1 0 1 0 0 0 0 0 1 0 1 1 0 0 0 0 1 0
1 0 0 1 2 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 1 1 0 1 0 0 1 0 1 0 1 1 2 1 1 1 0 1 0 0 1 0 0 1 0

Figure 3. The $A_{5,6}$ matrix belonging to $PSL_2(27)$.

The task of solving the resulting system (1) of 10 linear Diophantine equations in 54 unknowns for a (0,1)-solution vector U was accomplished by our basis reduction algorithm presented at the 17-th Southeastern International Conference on Combinatorics Graph Theory and Computing, [5]. We remark that this algorithm is based in part on the $\mathbf{L}^3$ algorithm of A.K. Lenstra, H.W. Lenstra and L. Lovász [8], and was inspired by the work of J.C. Lagarias and A.M. Odlyzko [7].

3. Results

In figure 4 for each λ, $2 \leq \lambda \leq 11$ we give a list of orbits of 6-subsets by number whose union forms a 5-(28,6,λ) design. Noting that the complement of a 5-(28,6,λ) design is a 5-(28,6,23-λ) design, orbit number lists for $\lambda \geq 11$ were not given, and in the interest of brevity we give only one solution for each parameter situation. It is easy to see that these solutions satisfy equation (1). Furthermore, the industrious reader can check using an algorithm similar to the one found in [1] that these are indeed 5-(28,6,λ) designs.

λ	Orbit numbers
2	12 15 18 28 30 51
3	7 8 9 22 26 27 44 47
4	2 21 22 35 39 40 43 44 48 49 54
5	2 8 22 28 39 43 44 47 49 52 53
6	1 10 11 13 25 30 36 39 43 45 49 50
7	1 4 7 8 11 12 15 18 22 23 25 26 30 34 35 37 38 39 40 46 51 54
8	1 2 8 9 10 11 14 24 29 31 36 37 38 45 48 50 51
9	3 8 9 13 14 19 22 24 28 36 42 43 44 45 49 52 53
10	1 2 3 5 6 13 16 17 19 22 24 29 30 32 36 39 47 50 52 53
11	2 3 5 9 11 13 14 17 19 24 25 30 32 36 42 43 44 47 49 52 53

Figure 4. Designs from $PSL_2(27)$.

Finally, we state the following theorem.

Theorem 2.1 $PSL_2(27)$ *is an automorphism group of a simple* 5-(28,6,λ) *design for all* λ, $2 \leq \lambda \leq 21$ *and cannot be an automorphism group of a* 5-(28,6,λ) *design when* $\lambda = 1$.

Proof: The first part of the theorem is established by figure 4. To show that $G = PSL_2(27)$ cannot be the automorphism of a 5-(28,6,1) design we consider the G-orbit lengths of 6-subsets. These are: 3276, 4914 and 9828. Thus if a 5-(28,6,1) design is to be constructed as a union of G-orbits then the number of blocks, 16380, must be written as a sum of these numbers. Congruence modulo 3 and the fact that there are only 4 orbits of length 3276, namely numbers 2, 16, 18, and 35, says that exactly two of them must be used. But, this is impossible since the row sum of any two of the corresponding columns of the A_{56} matrix contains an entry greater than 1, contradicting $\lambda = 1$.

Acknowledgements

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