On Some Edge Folkman Numbers Small and Large*

Jenny M. Kaufmann
Princeton University
Department of Mathematics
Princeton, NJ 08544
jennak@princeton.edu

Henry J. Wickus
DeSales University
Department of Mathematics and Computer Science
Center Valley, PA 18034
hwickus@gmail.com

Stanisław P. Radziszowski
Rochester Institute of Technology
Department of Computer Science
Rochester, NY 14623
spr@cs.rit.edu

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Abstract

Edge Folkman numbers $F_e(G_1, G_2; k)$ can be viewed as a generalization of more commonly studied Ramsey numbers. $F_e(G_1, G_2; k)$ is defined as the smallest order of any K_k-free graph F such that any red-blue coloring of the edges of F contains either a red G_1 or a blue G_2. In this note, first we discuss edge Folkman numbers involving graphs $J_s = K_s - e$, including the results $F_e(J_3, K_n; n+1) = 2n-1$, $F_e(J_3, J_n; n) = 2n-1$, and $F_e(J_3, J_n; n+1) = 2n-3$. Our modification of computational methods used previously in the study of classical Folkman numbers is applied to obtain upper bounds on $F_e(J_4, J_4; k)$ for all $k > 4$.

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1 Overview

For a graph F, we say that $F \to (G_1, G_2)$ if in any red-blue coloring of the edges of F, there exists a red G_1 or a blue G_2. The classical Ramsey numbers can be defined using this arrowing notation as $R(G_1, G_2) = \min\{n \mid K_n \to (G_1, G_2)\}$. If graph F is K_k-free and $F \to (G_1, G_2)$, then we write $F \to (G_1, G_2; k)$. If graph G_i is complete, we may write $|V(G_i)|$ in place of G_i; for example, instead of $F \to (K_s, K_t; k)$ we could write $F \to (s, t; k)$. Given graphs G_1, G_2 and an integer $k > 1$, we define the set of edge Folkman graphs by

$$\mathcal{F}_e(G_1, G_2; k) = \{F \mid F \to (G_1, G_2) \text{ and } K_k \not\subseteq F\},$$

and we will denote by $\mathcal{F}_e(G_1, G_2; k; m)$ the set of such Folkman graphs with m vertices. The *edge Folkman number* $F_e(G_1, G_2; k)$ is the smallest m such that $\mathcal{F}_e(G_1, G_2; k; m)$ is nonempty. A theorem by Folkman [7] states that if $k > \max\{s, t\}$, then $F_e(s, t; k) = F_e(K_s, K_t; k)$ exists. One may easily notice that for graphs G_1 and G_2, if $k > R(G_1, G_2)$, then $F_e(G_1, G_2; k) = R(G_1, G_2)$. Henceforth, in the sequel we will focus on the cases for $k \le R(G_1, G_2)$.

In general, the Ramsey numbers $R(G, H)$ are difficult to compute, and $F_e(G, H; k)$ for $k < R(G, H)$ still more so. The graph $J_3 = P_3$, however, leads to much easier cases. The arrowing $F \to (J_3, H)$ is equivalent to the question of "Does the removal of every matching sK_2 from F leave a subgraph containing H?" In Section 2, we present constructions which witness upper bounds on $F_e(J_3; K_n; n+1)$, $F_e(J_3; J_n; n+1)$, and $F_e(J_3; J_n; n)$, and then we show that these bounds are tight.

In Section 3, we use computational methods modified from prior work on $F_e(3, 3; 4)$ to determine values of Folkman numbers $F_e(J_4, J_4; k)$ for $k > 6$, and bounds on $F_e(J_4, J_4; k)$ for $k = 5, 6$. These are obtained with the help of techniques used in satisfiability SAT and MAX-CUT, both of which are well-studied problems in computer science. The cases of $F_e(J_4, J_4; k)$ lie between the much-studied $F_e(3, 3; k)$ and little-studied $F_e(4, 4; k)$. We also present up to date history of bounds on the former, namely $F_e(3, 3; 4)$.

2 Arrowing (J_3, K_n) and (J_3, J_n)

Let the graph K_{2_n} denote the complete graph K_{2n} with removed perfect matching, i.e. $K_{2_n} = K_{2n} - nK_2$.

Proposition 1. *For all $n \in \mathbb{N}$, $n \ge 2$, $K_{2_{n-1}} + K_1 \to (J_3, K_n)$.*

Proof. We will first show that, for each $n \ge 2$, in any red-blue edge coloring of $K_{2_{n-1}}$ avoiding red $J_3 = P_3$, every vertex $v \in V(K_{2_{n-1}})$ belongs to a blue K_{n-1}. We proceed by induction. The claim is obvious for $n = 2$. Next, consider any red-blue coloring of K_{2_n} avoiding red J_3. Fix any $v_1 \in V(K_{2_n})$, and let v_2 be the vertex not adjacent to v_1. If v_1 is redly adjacent to some vertex w_1, then let $\{w_1, w_2\}$ be nonadjacent; otherwise, choose independent set $\{w_1, w_2\}$ arbitrarily, but $v_1 \notin \{w_1, w_2\}$. The restriction of this coloring to

$K_{2_n} - \{v_1, v_2\} = K_{2_{n-1}}$ is a red-blue coloring avoiding red J_3, so by induction w_2 is part of some blue $K_{n-1} \subset K_{2_n} - \{v_1, v_2\}$. Since v_1 is adjacent to all vertices in $K_{2_n} - \{v_1, v_2\}$, and is bluely adjacent to all its vertices possibly except w_1, together with this blue K_{n-1} it forms a blue K_n. By induction, the statement holds for all n.

Similarly, we prove the statement of the proposition by induction. Clearly, any red-blue edge coloring of $K_{2_1} + K_1$ has either a red J_3 or a blue K_2. For $n \geq 3$, consider any red-blue coloring of the graph $K_{2_{n-1}} + K_1$ without any red J_3. Let $\{x\} = V(K_1)$. If any vertex v is redly adjacent to x, choose independent set $\{v_1, v_2\}$ so that $v_2 = v$; otherwise, choose independent set $\{v_1, v_2\}$ arbitrarily. We have shown that in the restriction of this coloring to $K_{2_{n-1}}$, v_1 is in a blue K_{n-1}. Vertex v_2 cannot be part of this K_{n-1}. Since x is adjacent to all vertices in $V(K_{2_{n-1}})$, and is bluely adjacent to all such vertices (except perhaps v_2), it is in a blue K_n. Thus, $K_{2_{n-1}} + K_1 \to (J_3, K_n)$. $\diamond$

Theorem 1. *For all $k > n \geq 2$ we have $F_e(J_3, K_n; k) = 2n - 1$.*

Proof. We notice that $R(J_3, K_n) = 2n - 1$, as listed in [20]. For $k = n + 1$, this gives the lower bound $2n - 1 \leq F_e(J_3, K_n; n+1)$, while Proposition 1 provides a witness for the upper bound. For larger k the claim follows directly from definitions since $F_e(J_3, K_n; k)$ is nonincreasing in k. $\diamond$

Theorem 2. *For all $n \geq 3$ we have*

$$F_e(J_3, J_n; k) = \begin{cases} 4 & \text{if } k = n = 3, \\ 2n - 3 & \text{if } k > n > 2, \\ 2n - 1 & \text{if } k = n \text{ and } n > 3. \end{cases}$$

Proof. For the special case of $k = n = 3$, it can be easily checked that $K_{1,3} \to (J_3, J_3)$, hence it gives the upper bound. Clearly, 3 vertices are not enough for a suitable Folkman graph, so $F_e(J_3, J_3; 3) = 4$.

For the case $k > n > 2$, as in Theorem 2, the lower bound $F_e(J_3, J_n; n+1) \geq 2n - 3$ for any $k \geq n$ follows from $R(J_3, J_n) = 2n - 3$ (cf. [20]). For the upper bound, we will prove that $K_{2_{n-3}} + K_3 \to (J_3, J_n)$. Consider any red-blue coloring of the graph $K_{2_{n-3}} + K_3$ avoiding red J_3. Let $\{x, y, z\} = V(K_3)$ and let e be the edge $\{x, y\}$. By Proposition 1, the restriction of this coloring to the subgraph $K_{2_{n-2}} + K_1 = K_{2_{n-3}} + (K_3 - e)$ must include a blue K_{n-1}. Since $K_{n-1} \not\subset K_{2_{n-3}} + K_1$, this blue K_{n-1} must include exactly one of x or y; without loss of generality it includes x and not y. But in the original coloring, y is bluely adjacent to all or all but one of the vertices in the blue K_{n-1}, so y is part of a blue J_n. Hence $F_e(J_3, J_n; k) = 2n - 3$ for all $k > n$.

Finally we consider the case of $k = n$ for $n > 3$. Consider any K_n-free graph G with $|V(G)| = 2n - 2$. Color the edges of G as follows: take a maximum matching $R \subseteq E(G)$, color all of its edges in red, and color all edges in $G - R$ blue. This coloring contains no red J_3. We will show that either it contains no blue J_n, or that $G \subseteq K_{n-2} + nK_1$.

Suppose that G contains a blue J_n and let $S \subset V(G)$ be the vertices of the J_n. Since G does not contain K_n, there exist nonadjacent vertices $x, y \in S$. Every edge in R must be incident to a vertex in $\overline{S} = V(G) - S$, implying that $|R| \leq |\overline{S}| = n - 2$. Now consider any pair of adjacent vertices $s, t \in S$ (one of which may be x or y). Since s and t are adjacent, at least one must be incident to a red edge, since otherwise we could add the edge $\{s, t\}$ to R and obtain a matching larger than R. Since $|R| \leq |S| - 2$, there exist two vertices in S neither of which is incident to a red edge; then these vertices must be x and y. Furthermore, any other vertex in S is adjacent to x and y, so it must be incident to some red edge. Therefore, $|R| = n - 2 = |\overline{S}|$.

For any two vertices $s', t' \in \overline{S}$, there exist vertices $s, t \in S$ distinct from x and y, such that $\{s, s'\}$ and $\{t, t'\}$ are red edges. We must have that s' and t' are nonadjacent, since otherwise we could obtain a matching larger than R by taking R, removing edges $\{s, s'\}$ and $\{t, t'\}$, and replacing them with edges $\{x, s\}, \{y, t\}$, and $\{s', t'\}$. Additionally, if (without loss of generality) x is adjacent to $s' \in \overline{S}$, then we could obtain a matching larger than R by replacing edge $\{s, s'\}$ with edges $\{x, s'\}$ and $\{y, s\}$. Thus, the vertex set $\overline{S} \cup \{x, y\}$ does not induce any edges, implying that $G \subseteq K_{n-2} + nK_1$.

We can edge color $K_{n-2} + nK_1$ in a way that avoids red J_3 and blue J_n, simply by coloring only one edge in the K_{n-2} red. Thus, $K_{n-2} + nK_1 \not\rightarrow (J_3, J_n)$. Then there is no graph G on $2n - 2$ vertices such that $G \rightarrow (J_3, J_n; n)$, which gives the lower bound $F_e(J_3, J_n; n) \geq 2n - 1$. For the upper bound we consider the graph $K_{2_{n-1}} + K_1$. Let $\{x\} = V(K_1)$ and let vertices v_1, v_2 be nonadjacent. By Proposition 1, any red-blue coloring of $K_{2_{n-1}} + K_1$ with no red J_3 contains a blue K_n. This blue K_n can include at most one of v_1, v_2, and therefore at most one of $\{v_1, x\}$ and $\{v_2, x\}$. Hence, consider the subgraph $K_{2_{n-2}} + \overline{K_3} \subset K_{2_{n-1}} + K_1$ constructed by removing the edges $\{v_1, x\}$ and $\{v_2, x\}$. Next, observe that any coloring of $K_{2_{n-2}} + \overline{K_3}$ with no red J_3 therefore contains a blue J_n. So $K_{2_{n-2}} + \overline{K_3} \rightarrow (J_3, J_n)$, and thus, $F_e(J_3, J_n; n) = 2n - 1$. $\diamond$

3 Folkman Numbers $F_e(J_4, J_4; k)$

3.1 Cases for $k \geq 6$

In order to find upper bounds on $F_e(J_4, J_4; k)$ for $k \geq 6$ we reduced the corresponding arrowings to instances of the Boolean satisfiability SAT problem, which has been extensively studied. In particular, this approach had been previously used by Shetler, Wurtz, and the third author to test arrowing of (K_3, J_4). We applied it instead to the question of whether $G \not\rightarrow (J_4, J_4)$, as follows: We map the edges $E(G)$ to the variables of a Boolean formula ϕ_G, so that the color of an edge e is represented by the value of its corresponding Boolean variable. Then for each J_4 consisting of edges e_1, e_2, e_3, e_4, e_5, we add to ϕ_G two clauses

$$(e_1 + e_2 + e_3 + e_4 + e_5) \wedge (\overline{e_1} + \overline{e_2} + \overline{e_3} + \overline{e_4} + \overline{e_5}).$$

Then $G \not\rightarrow (J_4, J_4)$ if and only if ϕ_G is satisfiable. We solved many such instances of satisfiability problem for formulas ϕ_G with the SAT-solver `MiniSAT` [6]. The results of these computations lead to the next theorem.

Theorem 3. *It holds that*

$$F_e(J_4, J_4; k) = \begin{cases} 10 & \text{for } k \geq 8, \\ 11 & \text{for } k = 7, \end{cases}$$

and $11 \leq F_e(J_4, J_4; 6) \leq 14$.

Proof. It is known that $R(J_4, J_4) = 10$ (cf. [2]), hence $F_e(J_4, J_4; k) \geq 10$ for all $k \geq 4$, and $F_e(J_4, J_4; k) = 10$ for $k \geq 11$. A computation using `MiniSAT` determined that the graph $G = K_4 + K_{2,2,2}$ satisfies $G \to (J_4, J_4)$. Since $|V(G)| = 10$ and G is K_8-free, then using previous comments we obtain that $F_e(J_4, J_4; 8) = 10$. Because $F_e(J_4, J_4; k)$ is nonincreasing in k, we also obtain that $F_e(J_4, J_4; k) = 10$ for $k = 9$ and $k = 10$.

To find the lower bound for $F_e(J_4, J_4; 7)$, we tested all nonisomorphic graphs on 10 vertices found with `nauty` [16]. We ignored graphs containing K_7 and those which are K_5-free (since it would contradict $F_e(3, 3; 5) = 15$ [19]). Testing exhaustively all 1806547 such graphs via ϕ_G with `MiniSAT` revealed that $\mathcal{F}_e(J_4, J_4; 7; 10) = \varnothing$, and thus $F_e(J_4, J_4; 7) \geq 11$. A computation using `MiniSAT` determined that the graph $F = K_2 + K_{3,2,2,2}$ satisfies $F \to (J_4, J_4)$. Since $|V(F)| = 11$ and F is K_7-free, then similarly as before we obtain $F_e(J_4, J_4; 7) \leq 11$. Lastly, we determined using `MiniSAT` that the graph $H = C_5 + K_{3,3,3}$ satisfies $H \to (J_4, J_4)$. Since $|V(H)| = 14$ and H is K_6-free, we have that $F_e(J_4, J_4; 6) \leq 14$. $\diamond$

The exact value of $F_e(J_4, J_4; 6)$ possibly could be determined as above with a larger effort using similar computational techniques.

3.2 $F_e(J_4, J_4; 5)$ and MAX-CUT

Our attempts to use `MiniSAT` to find a graph G witnessing an upper bound on $F_e(J_4, J_4; 5)$ were unsuccessful, as the SAT-solver slowed down significantly when we tested larger graphs. However, we managed to obtain the bound $F_e(J_4, J_4; 5) \leq 1297$ using a modification of an idea and computational approach of Dudek and Rödl [3] for studying $F_e(3, 3; 4)$, which itself is based on an idea of Goodman [9].

For a red-blue coloring of a graph G, we define $T_{\text{diff}}(v)$ and $T_{\text{same}}(v)$, respectively, to be the number of triangles containing v in which the edges incident to v are different colors or the same color, respectively. Let t be the number of triangles in G, and let m be the number of monochromatic triangles in G. In each non-monochromatic triangle, there are two vertices v_1, v_2 for which the edges incident to it are different colors. Then $\sum_{v \in G} T_{\text{diff}}(v) = 2(t - m)$ counts each non-monochromatic triangle in G twice. Furthermore, $\sum_{v \in G} T_{\text{same}}(v) = t + 2m$ gives the number of non-monochromatic triangles plus three times the number of monochromatic triangles. Therefore,

$$6m = 2 \sum_{v \in G} T_{\text{same}}(v) - \sum_{v \in G} T_{\text{diff}}(v). \tag{1}$$

Observe that if $3m > |E(G)|$, then the ratio of edges in monochromatic triangles to edges is greater than 1, implying that there is some edge e which is part of two distinct monochromatic triangles. Therefore, if for every red-blue coloring of G we have

$$2|E(G)| < 2\sum_{v\in G} T_{\text{same}}(v) - \sum_{v\in G} T_{\text{diff}}(v), \tag{2}$$

then $G \to (J_4, J_4)$.

We now recall a method for linking arrowing triangles to MAX-CUT problem, first proposed by Dudek and Rödl [3]. Let H_G be the graph created as follows: We map every edge e of G to vertex v_e of H, so that $V(H_G) = E(G)$. Then for any two vertices v_e, v_f in $V(H_G)$, we add the edge $\{v_e, v_f\}$ if and only if their corresponding edges e and f are a part of some triangle in G. Note that any red-blue coloring of $E(G)$ corresponds to a bipartition $V(H_G) = B \cup R$ of vertices of H_G, inducing an edge cut C, for which any non-monochromatic triangle in G has exactly two edges in C. For any graph F, let $MC(F) = \text{MAX-CUT}(F)$ denote the maximum number of edges in F between the partite sets of any bipartition of $V(F)$. Letting $M_C(H_G)$ be the size of the cut C, we have

$$M_C(H_G) = \sum_{v\in G} T_{\text{diff}}(v) \le MC(H_G). \tag{3}$$

Clearly, any edge in H_G has both endpoints in the same partite set B or R if and only if it is not in C. The above considerations lead to the following theorem.

Theorem 4. *If $MC(H_G) < 2t(G) - 2|E(G)|/3$, then $G \to (J_4, J_4)$.*

Proof. For any graph G, whose edges are arbitrarily colored red and blue, consider the cut C of H_G as described above. Using (1) and (3), one can easily that

$$\sum_{v\in G} T_{\text{same}}(v) = |E(H_G)| - M_C(H_G) = 3t - M_C(H_G).$$

Now from the assumption we have $2|E(G)| < 2(3t - M_C(H_G)) - (M_C(H_G))$. Finally, using (2) and its implication we conclude that $G \to (J_4, J_4)$. $\diamond$

For large graphs H, tight upper-bounding $MC(H)$ is computationally expensive. For this reason, we used the following weakening of Theorem 4 for vertex-transitive graphs G. Its advantage is that it allows to detect conditions for which Theorem 4 can be applied much faster.

Theorem 5. *Let G be a vertex-transitive d-regular graph, where G_v denotes the graph induced in G by the neighbors of vertex v. If we have*

$$MC(G_v) < \frac{2}{3}|E(G_v)| - \frac{d}{3},$$

then $G \to (J_4, J_4)$.

Proof. This is following the same argument as in an alternative approach to bounding Folkman numbers used by Lu [14] and Spencer [24]. Here, however, with an additional term $d/3$, we need to use the observation made above between equalities (1) and (2). $\diamond$

MAX-CUT is among Karp's original 21 NP-hard problems [11]. In order to find good bounds on $MC(H_G)$ and $MC(G_v)$ for graphs G of our interest, we used the eigenvalue and semi-definite programming approximations of MAX-CUT. This approach was used by several authors, including Lu [14], Dudek and Rödl [3], and Lange et al. [12] to obtain upper bounds on $F_e(3,3;4)$ (see Section 4 for historical summary).

We applied Theorems 4 and 5 to many graphs of different types. We found an interesting positive instance using the following construction described by Lu [14]. For positive integers n and s, $s < n$, define $S = \{s^i \pmod{n} \mid i = 0, 1, ...n-1\}$. Then, if $n-1 \in S$, let $L(n,s)$ be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{x,y\} \mid x-y \in S\}$. Clearly, the graphs $L(n,s)$ are vertex-transitive.

Theorem 6. $F_e(J_4, J_4; 5) \leq 1297$.

Proof. For the graph $L(1297, 8)$, which is 216-regular, we found that it satisfies the assumptions of both Theorems 4 and 5, using two MAX-CUT bounding methods: the eigenvalue method and SDP approach. We used our Java library and associated programs, including the `eigs` function in `Matlab` [15] and the SDP solver `SDP-LR` [10], An easy (computer) test shows that the graph $L(1297, 8)$ is K_5-free, and hence it is a witness of the upper bound. $\diamond$

We wish to note that recently (and after this work was completed) a much better bound of 51 on $F_e(J_4, J_4; 5)$ was obtained by Xu et al. [25]. The latter bound did not require any computations. We also would like to recall the bound on $F_e(J_4, J_4; 4)$ obtained by Lu [14], as follows.

Proposition 2. $F_e(J_4, J_4; 4) \leq 30193$.

The bound in Proposition 2 is mentioned by Lu [14] in his paper on $F_e(3,3;4)$ as a side result, without any comments on the approach. However, we communicated with the author who confirmed that the main idea of his approach was similar to one in this work.

3.3 History of the Folkman Number $F_e(3,3;4)$

Table 1 below summarizes the history of bounds on the edge Folkman number $F_e(3,3;4) = F_e(K_3, K_3; 4)$, which is the smallest unknown classical Folkman number, sometimes also called *the most wanted*. This table builds up on an earlier Table 5 by Xu and the third author [22], where further extensive comments about the progress related to $F_e(3,3;4)$ can be found. The new entries in Table 1 here are lower bounds 13, 14 and 20. The bound $F_e(3,3;4) \geq 14$ can be obtained as follows: removal of any independent set of 3 vertices from any graph in $\mathcal{F}_e(3,3;4)$ must yield a 5-chromatic K_4-free graph, but Nenov [18] proved (without using computer algorithms) that any such graph has at least 11 vertices. $F_e(3,3;4) \geq 13$ is implied in the same way by an earlier result of Nenov [17]. In contrast, the currently best known lower bound of 20 was obtained by Bikov and Nenov [1] using cpu-intensive computations.

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal [5]
1970	exist	Folkman [7]
1972	11 –	Lin, implicit in [13], implied by $F_e(3,3;5) \geq 10$
1975	– 10^{10}?	Erdős offers \$100 for proof [4]
1983	13 –	implied by a result of Nenov [17]
1984	14 –	implied by a result of Nenov [18]
1986	– 8×10^{11}	Frankl-Rödl [8]
1988	– 3×10^{9}	Spencer [24]
1999	16 –	Piwakowski-Radziszowski-Urbański, implicit in [19]
2007	19 –	Radziszowski-Xu [21]
2008	– 9697	Lu [14]
2008	– 941	Dudek-Rödl [3]
2012	– 100?	Graham offers \$100 for proof
2014	– 786	Lange-Radziszowski-Xu [12]
2017	20 –	Bikov-Nenov [1]

Table 1. History of bounds on the Folkman number $F_e(3,3;4)$.

For any graph G with t triangles and graph H_G as defined in Section 3.2, one can easily observe that $G \to (K_3, K_3)$ if and only if $MC(H_G) < 2t$ (see also [3]). Thus, computational techniques to upper-bound MAX-CUT may lead to good upper bounds on $F_e(3,3;4)$, including the first such result by Dudek and V. Rödl [3]. Lange, Xu, and the third author used the SDP MAX-CUT approximation to obtain an upper bound on $MC(H_G)$ for a particular K_4-free graph G on 786 vertices, and used it to show that $G \to (K_3, K_3)$.

We made numerous attempts to lower this bound by trying to find a smaller K_4-free graph G for we could obtain the bound $MC(H_G) < 2t$. Among the graphs tested were the graphs $G(n,r)$ considered in [3], the graphs $L(n,s)$ from [14], and their variations. In particular, we tested a generalization of $L(n,s)$ to Galois fields $GF(p^k)$, in addition to graphs constructed by adjoining various pairs of circulant graphs in a variety of ways. Our efforts have convinced us that these methods are unlikely to yield any major improvement on this bound.

The well known K_4-free graph $G_{127} = L(127, 5)$ was studied by several authors (cf. [21, 22]). In particular, it was conjectured by Exoo that $G_{127} \to (K_3, K_3)$. Needless to say, we were not successful in proving Exoo's conjecture, because otherwise it would imply that $F_e(3, 3; 4) \leq 127$.

Computations

Some of the results in this paper were found through the use of various computational methods. This involved a large library of functions, including graph manipulation, construction of various types of graphs, and tests for graph arrowing. Graphs were represented in a variety of ways, including two-dimensional Boolean arrays, lists of edges for sparse graphs, and McKay's `g6`-format [16]. Our code was written in Java and executed on Unix and Windows systems. For our final results, `Matlab` [15] and `SDP-LR` [10, 23] were used to calculate eigenvalue and SDP MAX-CUT approximations, respectively. `MiniSAT` [6] was used to solve satisfiability problems. We also made use of lists of nonisomorphic graphs with special properties found with `nauty` [16].

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