Some Folkman Problems

chromatic vertex Folkman numbers
existence, computational challenges

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, NY

joint work with Xiaodong Xu and Meilian Liang

DisCoMathS, 18 September 2020
Henrietta NY

Plan of Talk

- On the most wanted Folkman number
  (16 min)

  Aleksander Soifer
  The Mathematical Coloring Book, first edition 2009.
  second edition in making, 2021, with expanded Folkman theme.

- Chromatic vertex Folkman problems
  (32 min)

  XLR
  The Electronic Journal of Combinatorics, 27(3) (2020) P3.53.
  first draft 2015, hick-ups, revisions, out 9/4/2020.

Erdős and Hajnal

Research Problem 2-5, JCT 2, p. 105, 1967

Construct a graph G which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color.

done by R.L. Graham, 1968

The proposers expect that for every cardinal m there is a graph G which contains no complete quadrilateral such that for every coloring of the edges by m colors there is a triangle all of whose edges have the same color.

proved for $m = 2$ by Folkman, 1970
generalized by Nešetřil and Rödl, 1976

53 Years of the Most Wanted Folkman Number

What is the smallest order n of a K_4-free graph which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	– 10^{10}?	Erdős offers \$100 for proof
1986	– 8×10^{11}	Frankl-Rödl (almost won)
1988	– 3×10^{9}	Spencer (won \$100)
1998	– 10^{6}?	Chung-Graham offer \$100 for the answer
1999	16 –	Piwakowski-R-Urbański (implicit)
2007	19 –	R-Xu
2008	– 9697	Lu
2008	– 941	Dudek-Rödl
2012	– 786	Lange-R-Xu
2016	20 –	Bikov-Nenov
2020	21 –	Bikov-Nenov
2023	– 127	somebody?

Most Wanted Folkman Number: $F_e(3,3;4)$

and how to earn $100 from RL Graham

The best known bounds:

$$21 \leq F_e(3,3;4) \leq 786.$$

- Upper bound 786 from a modified residue graph via SDP.
- Ronald Graham Challenge for $\$100$ (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \rightarrow (3,3)^e$, moreover
  - removing 33 vertices from G_{127} gives graph G_{94},
    which still looks good for arrowing, if so, worth $100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972),
  16 (PUR 1999), 19 (RX 2007), 20, 21 (Bikov-Nenov 2016, 2020).

R · I · T
Department of Computer Science

Graph G_{127}

Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \rightarrow (3, 3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

R · I · T
Department of Computer Science

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \rightarrow (s,t)^e$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \rightarrow (G,H)^e$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2
- variants: coloring vertices, arrowing general graphs, more colors

Edge Folkman graphs

$\mathcal{F}_e(s,t;k) = \{F \mid F \rightarrow (s,t)^e,\ K_k \not\subseteq F\}$

Edge Folkman numbers

$F_e(s,t;k) =$ the smallest order of graphs in $\mathcal{F}_e(s,t;k)$

on the previous slide we discussed $F_e(3,3;4)$

Theorem (Folkman 1970)

If $k > \max(s,t)$, then $F_e(s,t;k)$ and $F_v(s,t;k)$ exist.

Linking edge- and vertex- Ramsey arrowing

- $G \in \mathcal{F}_v(R(s-1,t), R(s,t-1); k-1) \Rightarrow$

  $G + x \in \mathcal{F}_e(s,t;k)$

  is equivalent to

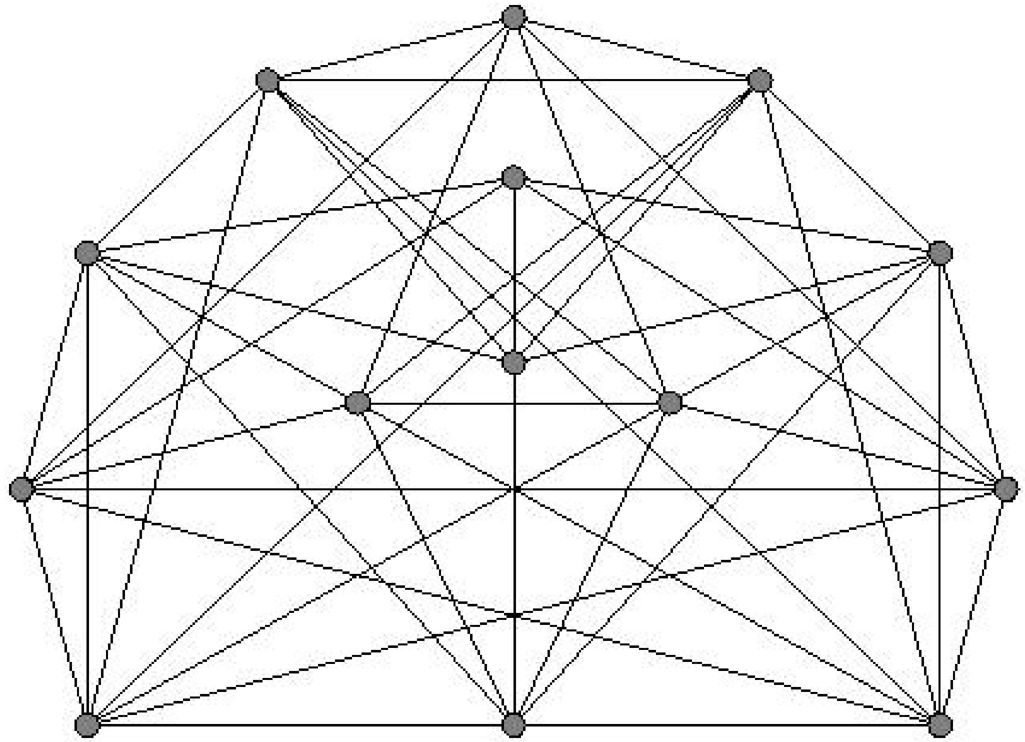
- $G + x \not\rightarrow (s,t)^e \Rightarrow$

  $G \not\rightarrow (R(s-1,t), R(s,t-1))^v$

Clearly, $cl(G+x) = cl(G) + 1$

R · I · T
Department of Computer Science

$F_e(3,3;5) = 15$, and $F_v(3,3;4) = 14$



unique 14-vertex bicritical $F_v(3,3;4)$-graph G [PRU 1999]
$cl(G) = 3$, $\chi(G) = 5$, $|Aut(G)| = 2$ and $G \to (3,3)^v$, $G + x \to (3,3)^e$

A pearl of vertex Folkman numbers

Theorem (ancient folklore)
$F_v(\underbrace{2, \cdots, 2}_{r}; r) = r + 5$ for $r \geq 5$

Sketch of the proof

for the upper bound $G = K_{r-5} + C_5 + C_5$
$n(G) = r + 5$, $cl(G) = r - 1$, $\chi(G) = r + 1$

for the lower bound take any
K_r-free graph H on $r + 4$ vertices, then
assemble matchings in $\overline{H}$ to show $\chi(H) \leq r$

Bounds from Chromatic Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$.

Theorem (Nenov 2001, Lin 1972, others)

If $G \to (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \to (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

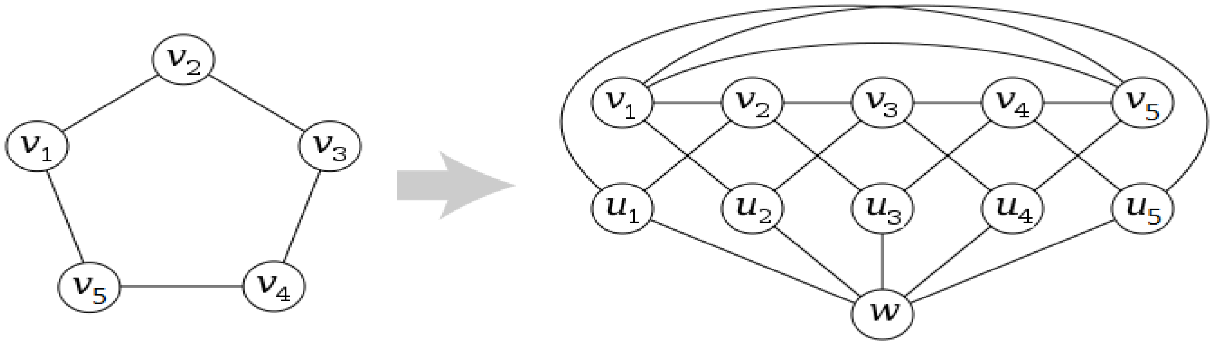
Special Case of Folkman Numbers

is just about graph chromatic number $\chi(G)$

Note: $G \rightarrow (2 \cdots_r 2)^v \iff \chi(G) \geq r + 1$

For all $r \geq 1$, $F_v(2^r; 3)$ exists and it is equal to
the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1}; 3) \leq 2F_v(2^r; 3) + 1$, Mycielski construction, 1955



Small Cases of Chromatic Folkman Numbers

$F_v(2^2;3) = 5, \quad C_5$, Mycielskian, 1955

$F_v(2^3;3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4;3) = 22 < 2*11+1$, Jensen and Royle, 1995

$F_v(2^5;3) \leq 44$, Droogendijk, 2015

$32 \leq F_v(2^5;3) \leq 40$, Goedgebeur, 2020

Bounds for the smallest k-chromatic graphs of given girth,
Exoo, Goedgebeur, 2018, 2020

R · I · T
Department of Computer Science

Chromatic Folkman Graphs and Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$

Theorem (Nenov 2001, Lin 1972, others)

If $G \rightarrow (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \rightarrow (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

Definition. Chromatic vertex/edge Folkman graphs/numbers:

graphs

$$\mathcal{F}_v^{\chi}(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_v(a_1, \cdots, a_r; s), \text{ and } \chi(G) = m\},$$
$$\mathcal{F}_e^{\chi}(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_e(a_1, \cdots, a_r; s), \text{ and } \chi(G) = M\},$$

numbers

$$F_v^{\chi}(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v^{\chi}(a_1, \cdots, a_r; s)\},$$
$$F_e^{\chi}(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e^{\chi}(a_1, \cdots, a_r; s)\}.$$

Existence

Edge chromatic Folkman numbers exist. It follows from a construction by Nešetřil and Rödl, 1976. Their computation looks mostly hopeless.

Vertex chromatic Folkman numbers exist. It follows from some past and our present work. Their computation can be wrestled a little.

All nice so far, and more ...

Theorem
For all r, we have $F_v^\chi(2^r;3) = F_v(2^r;3)$

But things can break ...

Let G be the unique witness to $F_v(3,4;5) \leq 13$,
we have $\chi(G) = 7$, $m = 6$, thus

$$F_v^\chi(3,4;5) > 13 = F_v(3,4;5), \quad 17 \leq F_v(4,4;5) \leq 23$$

Problem and Conjecture

Problem

For which $s > a \geq 2$, it is true that $F_v^\chi(a, a; s) = F_v(a, a; s)$?

Conjecture

For $s \geq 2$, we have $F_v^\chi(s, s; s+1) = F_v(s, s; s+1)$.

For $s = 2$: $m = 3$, yes, C_5 is a witness
For $s = 3$: $m = 5$, yes, implied by PRU'1999
For $s = 4$: $m = 7$, open

RIT Department of Computer Science

Main Theorem

Notation for r-color diagonals $F^{\chi}(r,s,t) = F_v^{\chi}(s^r;t)$.

Theorem

For integers $r \geq 2$ and $s \geq 3$, let $b_i = i(s-1)+1$ for $i \in [r-1]$, and $B = \prod_{i=1}^{r-1} b_i$. Then $F^{\chi}(r,s,s+1)$ exists and

$$F^{\chi}(r,s,s+1) \leq 1 + s + \sum_{i=2}^{r-1} F^{\chi}(i,s,s+1) + B \cdot F^{\chi}(r,s-1,s).$$

In particular, for all $s \geq 3$, the chromatic vertex Folkman number $F^{\chi}(2,s,s+1)$ exists and $F^{\chi}(2,s,s+1) \leq 1 + s + sF^{\chi}(2,s-1,s)$.

Proof: double constructive induction.

Main Theorem

sketch of the proof, construction

- Construct the target graph $G(r,s) \in \mathcal{F}^{\chi}(r,s,s+1)$ given
  - $G_0 = K_1$, $G_1 = K_s$, $V_0 = V(G_0)$, $V_1 = V(G_1)$,
  - any graphs $G_i \in \mathcal{F}^{\chi}(i,s,s+1)$ for $2 \leq i \leq r-1$, G_i with vertices $|V_i| = F^{\chi}(i,s,s+1)$,
  - any graph H in $\mathcal{F}^{\chi}(r,s-1,s)$.
- $G(r,s)$ has vertices

$$V = V_0 \; \cup \; V_1 \; \cup \bigcup_{i=2}^{r-1} V_i \; \cup \bigcup_{(j_0,\cdots,j_{r-1})} V(H(j_0,\cdots,j_{r-1})),$$

  where $1 \leq j_k \leq \chi(G_k)$ for $0 \leq k \leq r-1\}$, $\chi(G_i) = b_i = i(s-1)+1$, $B = \prod_{i=1}^{r-1} i(s-1)+1$ copies $H(j_0,\cdots,j_{r-1})$ of H are used.
- A proper corona of edges linking all parts is added.

Main Theorem

sketch of the proof, correctness

- Basis of induction is formed by the sets $\mathcal{F}^{\chi}(i, 2, 3)$.
- Prove the required properties of $G(r, s)$:
  - $cl(G(r, s)) < s + 1$,
  - $G(r, s) \to (K_s \cdots_r K_s)^v$, and
  - $\chi(G(r, s)) = m = r(s - 1) + 1$.
- The second part of the theorem is just an instantiation of the first part for two colors, $r = 2$.

Extension Theorem

Theorem

For any integers a, b and s such that $2 \leq a, b \leq s$, $F_v^\chi(a, b; s+1)$ exists and we have

$$F_v^\chi(a, b; s+1) \leq \frac{a+b-1}{2s-1} F_v^\chi(s, s; s+1).$$

Proof: constructive.

Another Sample Theorem

Theorem

For any integer $s \geq 2$, we have

$$F_v^{\chi}(2s, 2s; 2s+1) \leq (4s-1)F_v^{\chi}(s, s; s+1).$$

Proof: constructive.

$F_v(r, s, s+1) \leq C_r s^2 \log^2 s$, Hàn-Rödl-Szabó 2018

Conjecture

Turán graph $T_{n,s}$ is a complete multipartite graph on n vertices whose s partite sets have sizes as equal as possible.

For any integer $s \geq 3$, let $n = F_v(s, s; s+1)$.

Conjecture (somewhat stronger than one on slide 16)
There exists an n-vertex K_{s+1}-free subgraph G of the Turán graph $T_{n,2s-1}$, such that $G \rightarrow (s, s)^v$.

If true, then it implies that

$$F_v^{\chi}(s, s; s+1) = F_v(s, s; s+1).$$

Some references

- Xiaodong Xu, Meilian Liang, SPR
  Chromatic Vertex Folkman Numbers
  The Electronic Journal of Combinatorics, 27(3) (2020) P3.53

- Xiaodong Xu, Meilian Liang, SPR
  On the Nonexistence of Some Generalized Folkman Numbers
  Graphs and Combinatorics, 34 (2018) 1101-1110

- Many papers by Bikov, Dudek, Erdős, Folkman, Graham, Li, Lin, Lu, Nenov, Nešetřil, Rödl, Ruciński, Soifer, Xu, and others ...

Thanks for listening!