Chromatic Folkman Problems

chromatic vertex Folkman numbers
existence, computational challenges

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joint work with Xiaodong Xu and Meilian Liang

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Erdős and Hajnal

Research Problem 2-5, JCT 2, p. 105, 1967

Construct a graph G which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color.

done by R.L. Graham, 1968

The proposers expect that for every cardinal m there is a graph G which contains no complete quadrilateral such that for every coloring of the edges by m colors there is a triangle all of whose edges have the same color.

proved for $m = 2$ by Folkman, 1970
generalized by Nešetřil and Rödl, 1976

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51 Years of the Most Wanted Folkman Number

What is the smallest order n of a K_4-free graph
which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	– 10^{10}?	Erdős offers \$100 for proof
1986	– 8×10^{11}	Frankl-Rödl (almost won)
1988	– 3×10^{9}	Spencer (won \$100)
1998	– 10^{6}?	Chung-Graham offer \$100 for the answer
1999	16 –	Piwakowski-R-Urbański (implicit)
2007	19 –	R-Xu
2008	– 9697	Lu
2008	– 941	Dudek-Rödl
2012	– 786	Lange-R-Xu
2012	– 100?	Graham offers \$100 for proof
2016	20 –	Bikov-Nenov

Most Wanted Folkman Number: $F_e(3,3;4)$

and how to earn \$100 from RL Graham

The best known bounds:

$$20 \leq F_e(3,3;4) \leq 786.$$

- Upper bound 786 from a modified residue graph via SDP.
- Ronald Graham Challenge for $\$100$ (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \rightarrow (3,3)^e$, moreover
  - removing 33 vertices from G_{127} gives graph G_{94},
    which still looks good for arrowing, if so, worth \$100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972),
  16 (PUR 1999), 19 (RX 2007), 20 (Bikov-Nenov 2016).

Graph G_{127}

Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x,y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \rightarrow (3,3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \rightarrow (s,t)^e$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \rightarrow (G,H)^e$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2
- variants: coloring vertices, arrowing general graphs, more colors

Edge Folkman graphs

$\mathcal{F}_e(s,t;k) = \{F \mid F \rightarrow (s,t)^e,\ K_k \not\subseteq F\}$

Edge Folkman numbers

$F_e(s,t;k) =$ the smallest order of graphs in $\mathcal{F}_e(s,t;k)$

on the previous slide we discussed $F_e(3,3;4)$

Theorem (Folkman 1970)

If $k > \max(s,t)$, then $F_e(s,t;k)$ and $F_v(s,t;k)$ exist.

Bounds from Chromatic Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$.

Theorem (Nenov 2001, Lin 1972, others)

If $G \rightarrow (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \rightarrow (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

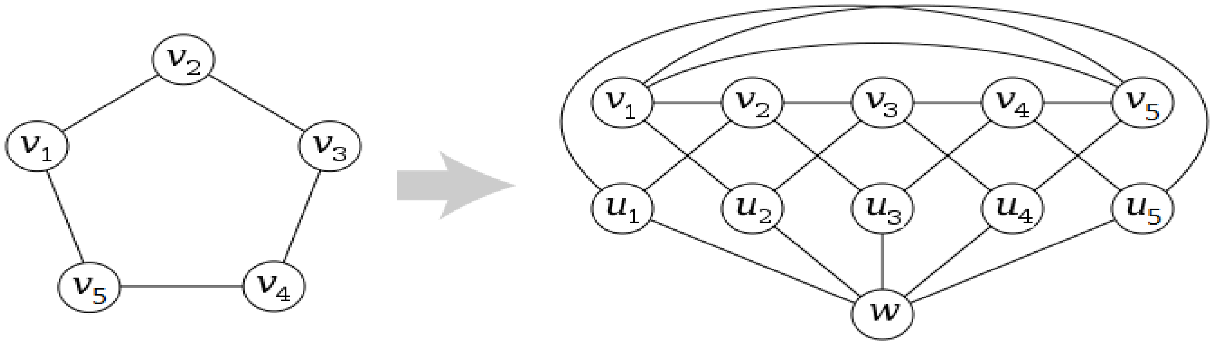
Special Case of Folkman Numbers

is just about graph chromatic number $\chi(G)$

Note: $G \rightarrow (2 \cdots_r 2)^v \iff \chi(G) \geq r+1$

For all $r \geq 1$, $F_v(2^r; 3)$ exists and it is equal to
the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1}; 3) \leq 2F_v(2^r; 3) + 1$, Mycielski construction, 1955



Small Cases of Chromatic Folkman Numbers

$F_v(2^2;3) = 5, \quad C_5$, Mycielskian, 1955

$F_v(2^3;3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4;3) = 22$, Jensen and Royle, 1995

$F_v(2^5;3) \leq 44$, Droogendijk, 2015

$32 \leq F_v(2^5;3) \leq 40$, Goedgebeur, 2017

Bounds for the smallest k-chromatic graphs of given girth,
Exoo, Goedgebeur, 2018

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Chromatic Folkman Graphs and Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$

Theorem (Nenov 2001, Lin 1972, others)

If $G \rightarrow (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \rightarrow (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

Definition. Chromatic vertex/edge Folkman graphs/numbers:

graphs

$$\mathcal{F}_v^{\chi}(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_v(a_1, \cdots, a_r; s), \text{ and } \chi(G) = m\},$$
$$\mathcal{F}_e^{\chi}(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_e(a_1, \cdots, a_r; s), \text{ and } \chi(G) = M\},$$

numbers

$$F_v^{\chi}(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v^{\chi}(a_1, \cdots, a_r; s)\},$$
$$F_e^{\chi}(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e^{\chi}(a_1, \cdots, a_r; s)\}.$$

Existence

Edge chromatic Folkman numbers exist. It follows from a construction by Nešetřil and Rödl, 1976. Their computation looks mostly hopeless.

Vertex chromatic Folkman numbers exist. It follows from some past and our present work. Their computation can be wrestled a little.

Warm-up

For all r, we have $F_v^{\chi}(2^r;3) = F_v(2^r;3)$

Let G be the unique witness to $F_v(3,4;5) \leq 13$,
we have $\chi(G) = 7$, $m = 6$, but

$$F_v^{\chi}(3,4;5) > 13 = F_v(3,4;5), \quad 17 \leq F_v(4,4;5) \leq 23$$

Problem and Conjecture

Problem

For which $s > a \geq 2$, it is true that $F_v^{\chi}(a, a; s) = F_v(a, a; s)$?

Conjecture

For $s \geq 2$, we have $F_v^{\chi}(s, s; s+1) = F_v(s, s; s+1)$.

For $s = 2$: $m = 3$, yes, C_5 is a witness
For $s = 3$: $m = 5$, yes, implied by PRU'1999
For $s = 4$: $m = 7$, open

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Main Theorem

Notation for r-color diagonals $F^{\chi}(r,s,t) = F^{\chi}_{v}(s^r;t)$.

Theorem

For integers $r \geq 2$ and $s \geq 3$, let $b_i = i(s-1)+1$ for $i \in [r-1]$, and $B = \prod_{i=1}^{r-1} b_i$. Then $F^{\chi}(r,s,s+1)$ exists and

$$F^{\chi}(r,s,s+1) \leq 1 + s + \sum_{i=2}^{r-1} F^{\chi}(i,s,s+1) + B \cdot F^{\chi}(r,s-1,s).$$

In particular, for all $s \geq 3$, the chromatic vertex Folkman number $F^{\chi}(2,s,s+1)$ exists and $F^{\chi}(2,s,s+1) \leq 1 + s + sF^{\chi}(2,s-1,s)$.

Proof: double constructive induction.

Main Theorem

sketch of the proof, construction

- Construct the target graph $G(r,s) \in \mathcal{F}^{\chi}(r,s,s+1)$ given
  - $G_0 = K_1$, $G_1 = K_s$, $V_0 = V(G_0)$, $V_1 = V(G_1)$,
  - any graphs $G_i \in \mathcal{F}^{\chi}(i,s,s+1)$ for $2 \leq i \leq r-1$, G_i with vertices $|V_i| = F^{\chi}(i,s,s+1)$,
  - any graph H in $\mathcal{F}^{\chi}(r,s-1,s)$.
- $G(r,s)$ has vertices

$$V = V_0 \;\cup\; V_1 \;\cup\; \bigcup_{i=2}^{r-1} V_i \;\cup\; \bigcup_{(j_0,\cdots,j_{r-1})} V(H(j_0,\cdots,j_{r-1})),$$

where $1 \leq j_k \leq \chi(G_k)$ for $0 \leq k \leq r-1\}$, $\chi(G_i) = b_i = i(s-1)+1$, $B = \prod_{i=1}^{r-1} i(s-1)+1$ copies $H(j_0,\cdots,j_{r-1})$ of H are used.

- A proper corona of edges linking all parts is added.

Main Theorem

sketch of the proof, correctness

- Basis of induction is formed by the sets $\mathcal{F}^{\chi}(i, 2, 3)$.
- Prove the required properties of $G(r, s)$:
  - $cl(G(r, s)) < s + 1$,
  - $G(r, s) \rightarrow (K_s \cdots_r K_s)^v$, and
  - $\chi(G(r, s)) = m = r(s - 1) + 1$.
- The second part of the theorem is just an instantiation of the first part for two colors, $r = 2$.

Extension Theorem

Theorem

For any integers a, b and s such that $2 \leq a, b \leq s$, $F_v^{\chi}(a, b; s+1)$ exists and we have

$$F_v^{\chi}(a, b; s+1) \leq \frac{a+b-1}{2s-1} F_v^{\chi}(s, s; s+1).$$

Proof: constructive.

Another Sample Theorem

Theorem

For any integer $s \geq 2$, we have

$$F_v^{\chi}(2s, 2s; 2s+1) \leq (4s-1) F_v^{\chi}(s, s; s+1).$$

Proof: constructive.

$F_v(r, s, s+1) \leq C_r s^2 \log^2 s$, Hàn-Rödl-Szabó 2018

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Conjecture

Turán graph $T_{n,s}$ is a complete multipartite graph on n vertices whose s partite sets have sizes as equal as possible.

For any integer $s \geq 3$, let $n = F_v(s, s; s+1)$.

Conjecture (somewhat stronger than one on slide 12)

There exists an n-vertex K_{s+1}-free subgraph G of the Turán graph $T_{n,2s-1}$, such that $G \rightarrow (s, s)^v$.

If true, then it implies that

$$F_v^{\chi}(s, s; s+1) = F_v(s, s; s+1).$$

Some references

- Xiaodong Xu, Meilian Liang, SPR
  Chromatic Vertex Folkman Numbers
  `arXiv 1612.08136`, version 2, May 2018
- Xiaodong Xu, Meilian Liang, SPR
  On the Nonexistence of Some Generalized Folkman Numbers
  `arXiv 1705.06268`, May 2017
- Many papers by Bikov, Dudek, Erdős, Folkman, Graham, Li, Lin, Lu, Nenov, Nešetřil, Rödl, Ruciński, Soifer, Xu, and others ...

Thanks for listening!