Some Folkman Problems

existence and non-existence of
generalized Folkman numbers,
computational challenges

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Erdős and Hajnal

Research Problem 2-5, JCT 2, p. 105, 1967

Construct a graph G which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color.

done by R.L. Graham, 1968

The proposers expect that for every cardinal m there is a graph G which contains no complete quadrilateral such that for every coloring of the edges by m colors there is a triangle all of whose edges have the same color.

proved for $m = 2$ by Folkman, 1970
generalized by Nešetřil and Rödl, 1976

50 Years of the Most Wanted Folkman Number

What is the smallest order n of a K_4-free graph which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	$- 10^{10}$?	Erdős offers \$100 for proof
1986	$- 8 \times 10^{11}$	Frankl-Rödl, almost won
1988	$- 3 \times 10^{9}$	Spencer, probabilistic, won \$100
1999	16 –	Piwakowski-R-Urbański, implicit
2007	19 –	R-Xu
2008	– 9697	Lu, eigenvalues
2008	– 941	Dudek-Rödl, maxcut-SDP
2012	– 100?	Graham offers \$100 for proof
2014	– 786	Lange-R-Xu, maxcut-SDP
2016	20 –	Bikov-Nenov

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \rightarrow (s,t)^e$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \rightarrow (G,H)^e$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2
- variants: coloring vertices, arrowing general graphs, more colors

Edge Folkman graphs

$\mathcal{F}_e(s,t;k) = \{F \mid F \rightarrow (s,t)^e,\ K_k \not\subseteq F\}$

Edge Folkman numbers

$F_e(s,t;k) =$ the smallest order of graphs in $\mathcal{F}_e(s,t;k)$

on the previous slide we discussed $F_e(3,3;4)$

Theorem (Folkman 1970)

If $k > \max(s,t)$, then $F_e(s,t;k)$ and $F_v(s,t;k)$ exist.

Bounds from Chromatic Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$.

Theorem (Nenov 2001, Lin 1972, others)

If $G \to (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \to (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

Special Case of Folkman Numbers

is just about graph chromatic number $\chi(G)$

Note: $G \rightarrow (2 \cdots_r 2)^v \iff \chi(G) \geq r+1$

For all $r \geq 1$, $F_v(2^r; 3)$ exists and it is equal to
the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1}; 3) \leq 2F_v(2^r; 3) + 1$, Mycielski construction, 1955

small cases

$F_v(2^2; 3) = 5, \quad C_5$, Mycielskian, 1955

$F_v(2^3; 3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4; 3) = 22$, Jensen and Royle, 1995

$F_v(2^5; 3) \leq 44$, Droogendijk, 2015

$32 \leq F_v(2^5; 3) \leq 40$, Goedgebeur, 2017

Generalized Folkman Problems

Arrowing and avoiding general graphs

$F_v(H_1, H_2; H)$ = smallest n for which there exists an H-free graph G of order n such that $G \to (H_1, H_2)^v$

$F_e(H_1, H_2; H)$ = smallest n for which there exists an H-free graph G of order n such that $G \to (H_1, H_2)^e$

- When H_1, H_2, H are complete graphs this is classics
- Some existence questions are discussed in the following
- Some other existence questions seem very difficult
- $F_e(K_4 - e, K_4 - e; K_4) \leq 30193$, Lu 2008

  side effect of an attack on $F_e(3, 3; 4)$

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Avoiding $K_k - e$

Notation: $J_k = K_k - e$

Theorem

For every integer $k \geq 3$*,*

(a) *the edge Folkman number* $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ *exists, and*

(b) *the vertex Folkman number* $F_v(K_k, K_k; J_{k+1})$ *exists.*

Proof:
Based on a result by Nešetřil and Rödl (1981), and on our lemma.

Challenge: compute the following

$F_v(K_3, K_4; J_5)$, perhaps doable
$F_e(K_3, K_3; J_5)$, almost hopeless
$F_e(K_3, K_3; K_4)$, 50 years intro slide, hopeless

Avoiding Books

Notation: $B_k = K_1 + K_{1,k}$, hence also $B_2 = J_4$

Theorem

The edge Folkman number $F_e(K_3, K_3; B_3)$ does not exist.

Problem

Does the edge Folkman number $F_e(K_3, K_3; B_4)$ exist?

Clearly, $F_e(K_3, K_3; B_k) = 6$ for all $k \geq 5$, hence we study B_k-free and K_4-free graphs arrowing $(3,3)^e$.

Existence of $F_e(K_3, K_3; H)$ for small $H \supset K_4$

Notation: Graphs $\widehat{K}_{4,i} \subset K_5$ for $i \in [4]$, where $\widehat{K}_{n,s}$ is the graph obtained by connecting a new vertex v to s vertices of a K_n.

Lemma

$15 = F_e(K_3, K_3; K_5) \leq F_e(K_3, K_3; J_5) \leq F_e(K_3, K_3; K_4) \leq 786.$

(observe that $\widehat{K}_{4,4} = K_5$, $\widehat{K}_{4,3} = J_5$)

Proof: by monotonicity.

Lemma

$F_e(K_3, K_3; \widehat{K}_{4,2}) = F_e(K_3, K_3; \widehat{K}_{4,1}) = F_e(K_3, K_3; K_4).$

Proof: short and cool.

Existence of $F_e(K_3, K_3; H)$ for small $H \not\supset K_4$

Theorem
The edge Folkman number $F_e(K_3, K_3; K_1 + P_4)$ does not exist.

Theorem
Let H be any connected K_4-free graph on 5-vertices containing K_3. Then the edge Folkman number $F_e(K_3, K_3; H)$ does not exist, except for two possible cases for H, namely W_5 and $\overline{P_2 \cup P_3}$.

Proofs: some work but not that hard.

Existence of $F_e(K_3, K_3; H)$ for small $H \not\supset K_4$

Problem

Prove or disprove the existence of

(a) *the edge Folkman number* $F_e(K_3, K_3; \overline{P_2 \cup P_3})$*, and*
(b) *the edge Folkman number* $F_e(K_3, K_3; K_1 + C_4)$*.*

$W_5 = K_1 + C_4 \subset J_5 = K_5 - e$, hence
if $F_e(K_3, K_3; W_5)$ exists, then $F_e(K_3, K_3; J_5) \leq F_e(K_3, K_3; W_5)$.

The analogous statement holds for $\overline{P_2 \cup P_3}$,
with an extra condition implied by $\overline{P_2 \cup P_3} \subset W_5$.

Future work:
Study $F_e(K_3, K_3; H)$ for graphs H on at least 6 vertices (beyond B_4),
Computational projects: existence and bounds.

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From Edge- to Vertex-Arrowing

Lemma

For $k \geq s \geq 2$, if graph G is H-free, $H \subset K_{k+1}$, and $G \rightarrow (K_s, K_k)^e$, then for every vertex $u \in V(G)$ and $s-1$ colors we have $G - u \rightarrow (K_k, \cdots, K_k)^v$.

Corollary

For $2 \leq s \leq k$ and graph $H \subset K_{k+1}$, if $F_e(K_s, K_k; H)$ exists, then $F_v^{s-1}(K_k; H)$ also exists and $F_e(K_s, K_k; H) \geq F_v^{s-1}(K_k; H) + 1$.

Special case with $s = 3$ and $H = K_{k+1}$ gives

$F_e(3, k; k+1) > F_v(k, k; k+1)$.

Back to $F_e(3,3;4)$

and how to earn \$100 from RL Graham

The best known bounds:

$$20 \leq F_e(3,3;4) \leq 786.$$

- Upper bound 786 from a modified residue graph via SDP.
- Ronald Graham Challenge for $\$100$ (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \rightarrow (3,3)^e$, moreover
  - Removing 33 vertices from G_{127} (3 indsets of 11) gives a G_{94} which still looks good for arrowing, if so, worth \$100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972), 16 (PUR 1999), 19 (RX 2007), 20 (Bikov-Nenov 2016).

Graph G_{127}

Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \rightarrow (3, 3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

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Some references

- Xiaodong Xu, Meilian Liang, SPR
  On the Nonexistence of Some Generalized Folkman Numbers
  `arXiv 1705.06268`, May 2017

- Xiaodong Xu, Meilian Liang, SPR
  Chromatic Vertex Folkman Numbers
  `arXiv 1612.08136`, December 2016

- Many papers by Bikov, Dudek, Erdős, Folkman, Graham, Li, Lin, Lu, Nenov, Nešetřil, Rödl, Ruciński, Soifer, Xu, and others ...

Thanks for listening!