Some Ramsey Problems Involving Triangles - Computational Approach

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Outline - Triangles Everywhere

or avoiding K_3 in some/most colors

1. Ramsey Numbers - Two Colors
   - Some known and computed facts
   - $R(3, 10)$ is hard
   - Some things to do, computationally
2. Ramsey Numbers - More Colors
   - Some general bounds
   - $R(3, 3, 4)$, $R(3, 3, 3, 3)$ are hard
   - Things to do
3. Most Wanted Folkman Number
   - Edge-arrowing $(3, 3)$
   - K_4-free edge-arrowing $(3, 3)$
   - Things to do
4. So, what to do next?
   - Summary of things to do

Ramsey Numbers

- $R(G, H) = n$ iff
  n = least positive integer such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color

- $R(k, l) = R(K_k, K_l)$
- generalizes to r colors, $R(G_1, \cdots, G_r)$
- *2-edge-colorings* $\cong$ *graphs*

- Theorem (Ramsey 1930): Ramsey numbers exist

Values and Bounds on $R(k, l)$

two colors, avoiding cliques

l / k	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	73 115	92 149	97 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	125 316	143 442	159	185 848	209	235 1461	265
6				102 165	113 298	127 495	169 780	179 1171	253	262 2566	317	 5033	401
7					205 540	216 1031	233 1713	289 2826	405 4553	416 6954	511 10581	 15263	 22116
8						282 1870	317 3583	 6090	 10630	 16944	817 27490	 41525	861 63620
9							565 6588	580 12677	 22325	 39025	 64871	 89203	
10								798 23556		 81200			1265

[EIJC survey *Small Ramsey Numbers*]

#vertices / #graphs

3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592 $\approx 1.6 * 10^{11}$

———————too many to process———————

13	50502031367952 $\approx 5 * 10^{14}$
14	29054155657235488
15	31426485969804308768
16	64001015704527557894928
17	245935864153532932683719776
18	$\approx 2 * 10^{30}$

#vertices / #triangle-free graphs

3	3
4	7
5	14
6	38
7	107
8	410
9	1897
10	12172
11	105071
12	1262180
13	20797002
14	467871369
15	14232552452
16	581460254001 $\approx 6 * 10^{11}$

——————too many to process——————

17	$\approx 3 * 10^{12}$

Asymptotics

Ramsey numbers avoiding K_3

- Recursive construction yielding
  $R(3, 4k + 1) \geq 6R(3, k + 1) - 5$
  $\Omega(k^{\log 6/ \log 4}) = \Omega(k^{1.29})$
  Chung-Cleve-Dagum 1993

- Explicit $\Omega(k^{3/2})$ construction
  Alon 1994, Codenotti-Pudlák-Giovanni 2000

- Kim - 1995 (lower bound)
  Ajtai-Komlós-Szemerédi 1980 (upper bound)

$$R(3, k) = \Theta\left(\frac{k^2}{\log k}\right)$$

Small $R(3, k)$ cases

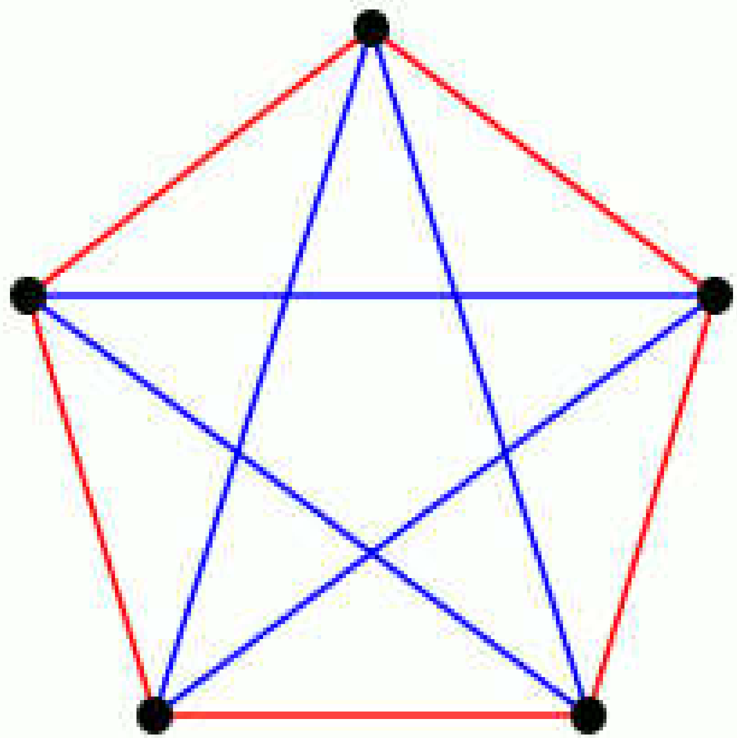
k	$R(3,k)$	year	reference [lower/upper]
3	6	1953	Putnam Competition
4	9	1955	Greenwood-Gleason
5	14	1955	Greenwood-Gleason
6	18	1964	Kéry
7	23	1966 / 1968	Kalbfleisch / Graver-Yackel
8	28	1982 / 1992	Grinstead-Roberts / McKay-Zhang
9	36	1966 / 1982	Kalbfleisch / Grinstead-Roberts

Known values of $R(3, k)$

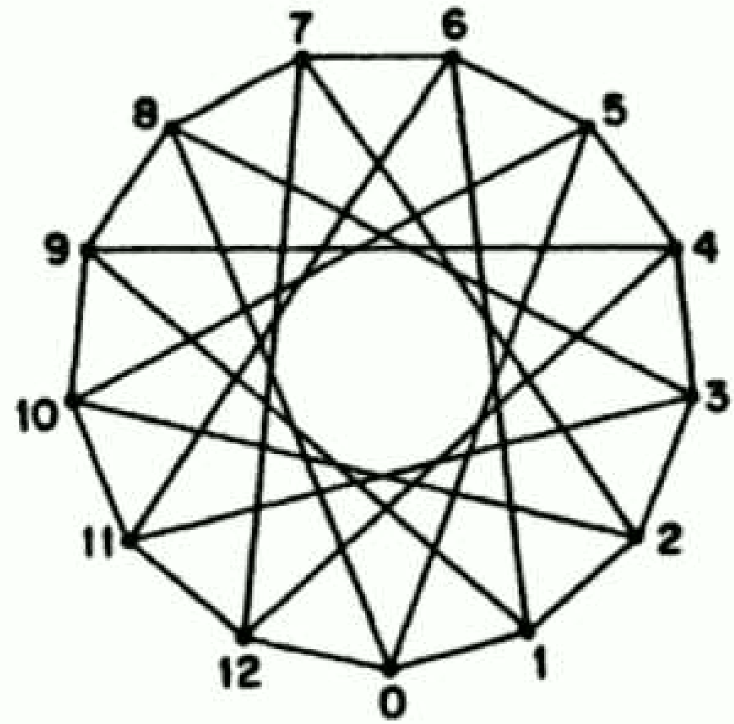
Questions (Erdős-Sós 1980) about
$3 \le \Delta_k = R(3, k) - R(3, k-1) \le k$:

$$\Delta_k \xrightarrow{k} \infty \,? \qquad \Delta_k/k \xrightarrow{k} 0 \,?$$

Unavoidable classics



$R(3,3) > 5$

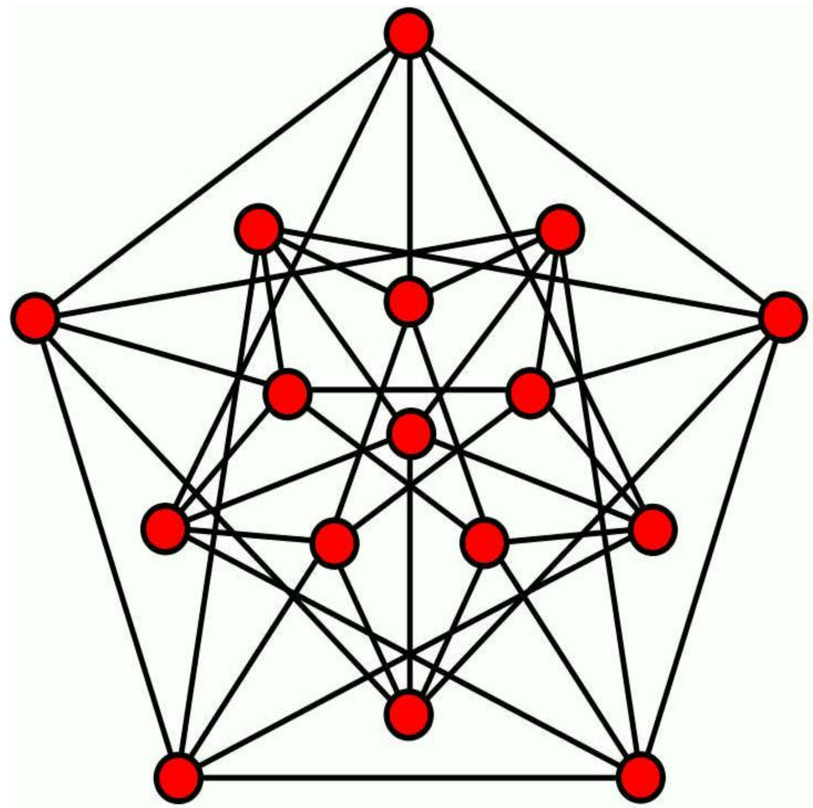


$R(3,5) > 13$ [GRS'90]

Clebsch $(3, 6; 16)$-graph on $GF(2^4)$

$(x, y) \in E$ iff $x - y = \alpha^3$

[Wikipedia]



Alfred Clebsch (1833-1872)

Larger Cases

K_3 versus $K_k - e$ or K_k

$$R(3, K_7 - e) = 21 \qquad R(3, K_8 - e) = 25 \qquad R(3, K_9 - e) = 31$$
$$R(3, 7) = 23 \qquad R(3, 8) = 28 \qquad R(3, 9) = 36$$

All $R(3, K_k - e)$ critical graphs are known for $k \leq 8$
All $R(3, K_k)$ critical graphs are known for $k \leq 7$

First open cases:

$$37 \leq R(K_3, K_{10} - e) \leq 38, \quad 42 \leq R(K_3, K_{11} - e) \leq 47$$
$$40 \leq R(K_3, K_{10}) \leq 43, \quad 46 \leq R(K_3, K_{11}) \leq 51$$

Counting edges

computing $R(3, 10)$ is difficult

Def: $e(k, n)$ = min # edges in n-vertex Δ-free k-graphs

- Very good lower bounds on $e(k-1, n-d)$ give good lower bounds on $e(k, n)$
- For any graph $G \in R(k, n, e)$

$$ne - \sum_{i=0}^{k-1} n_i(e(k-1, n-i-1) + i^2) \geq 0$$

- $e(9, n)$ not known for $27 \leq n \leq 35$ seem needed before improving on $e(10, n)$ for $n > 37$
- known $e(8, n)$-graphs not sufficient to improve on $e(9, n)$

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$R(K_3, G)$

general non-asymptotic results

- $R(nK_3, mK_3) = 2n + 3m$, for $n \geq m \geq 1, n \geq 2$,
  Burr-Erdős-Spencer 1975
- $R(C_3, C_n) = R(K_3, W_n) = 2n - 1$
  Faudree-Schelp 1974, Burr-Erdős 1983
  all critical colorings, R-Jin 1994
- $R(K_3, G) = 2n(G) - 1$, for connected G
  $e(G) \leq 17(n(G) + 1)/15$, $n(G) \geq 4$
  Burr-Erdős-Faudree-Rousseau-Schelp 1980
- $R(K_3, G) \leq 2e(G) + 1$, isolate-free G
  $R(K_3, G) \leq n(G) + e(G)$, a conjecture for all G
  Sidorenko 1992-3, Goddard-Kleitman 1994
- $R(K_3, G)$ for all connected G, $n(G) \leq 9$
  Brandt-Brinkmann-Harmuth 1998-2000

Things to do for two colors

- Enumerate all critical $(3, 8; 27)$-graphs
  430K+ known already
- Enumerate all critical $(3, 9; 35)$-graphs
  only one is known!
- Finish off $37 \leq R(3, K_{10} - e) \leq 38$
- $R(3, 10) \leq 43$, get it down first to 42
  ($R(3, 10) \geq 40$, don't even try to do better)

More colors

upper bound

$$R(k_1, \ldots, k_r) \leq 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$$

with strict $<$ if the RHS is even and sum has en even term
Greenwood-Gleason 1955

Only two known multicolor cases, (3,3,4) and (3,3,3,3), where the RHS is improved. Likely this bound is never tight, except for (3,3,3).

More colors

some results

- Xu-Xie-Exoo-R 2004
  - for $k_1 \geq 5$ and $k_i \geq 2$
    $R(k_1, 2k_2 - 1, k_3, \cdots, k_r) \geq 4R(k_1 - 1, k_2, k_3, \cdots, k_r)$
  - using $k_1 = l, k_2 = 2, k_3 = k$ in the above
    $R(3, k, l) \geq 4R(k, l - 1) - 3$
  - use $k = 3$
    $R(3, 3, l) \geq 4R(3, l - 1) - 3$
- $R(3, 3, k) = \Theta(k^3 \text{poly-log}\, k)$
  Alon-Rődl 2005

$R_r(3) = R(3, 3, \cdots, 3)$

- Much work on Schur numbers $s(r)$
  via sum-free partitions and cyclic colorings
  $s(r) > 89^{r/4 - c\log r} > 3.07^r$ [except small r]
  Abbott+ 1965+

- $s(r) + 2 \leq R_r(3)$
  $s(r) = 1, 4, 13, 44, \geq 160, \geq 536$

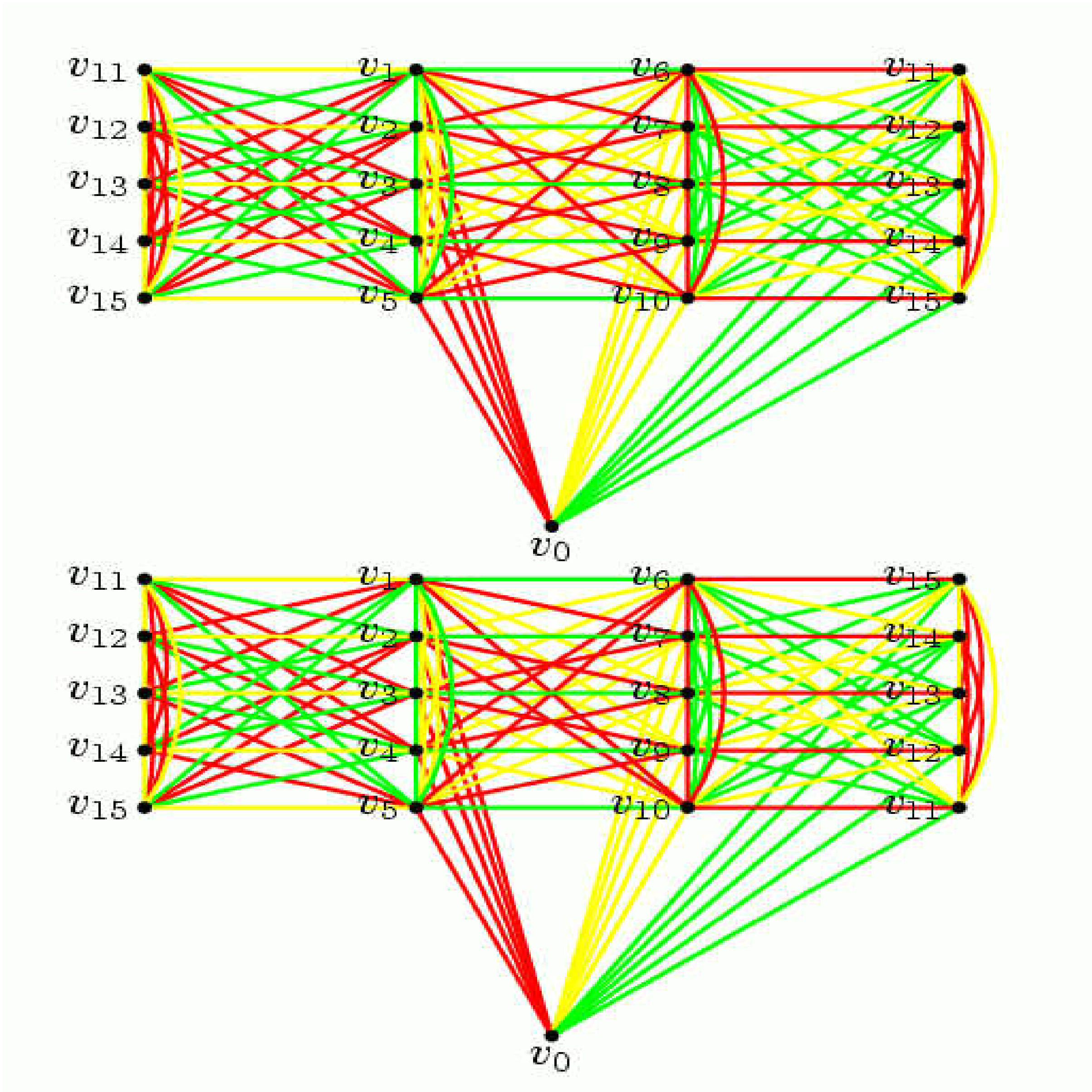
- $R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$
  Chung 1973

- The limit $L = \lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ exists
  Chung-Grinstead 1983

  $(2s(r) + 1)^{\frac{1}{r}} = c_r \approx_{(r=6)} 3.199 < L$

$R(3,3,3) = 17$

two Kalbfleisch $(3,3,3;16)$-colorings, each color is a Clebsch graph



[Wikipedia]

Three colors - $R(3,3,4)$

the only (as of now) not hopeless case

- $30 \leq R(3,3,4)$, cyclic coloring, Kalbfleisch 1966
- $R(3,3,4) \leq 31$, computations, Piwakowski-R 1998

Theorem (Piwakowski-R 2001): $R(3,3,4) = 31$ iff there exists a $(3,3,4;30)$-coloring C in which every edge in 3-rd color has an endpoint x with degree 13. Furthermore, C has at least 25 vertices with color degree sequence $(8,8,13)$.

Proof: Gluing possible arrangements of color induced neighborhoods of v in a $(3,3,4;30)$-coloring:

$(3,4;s), (3,4;t), (3,3,3;u \geq 14)$ with $s+t+u=29$

too many $(3,3,3;13)$'s to proceed further $\diamond$

Four colors - $R_4(3)$

$51 \leq R(3,3,3,3) \leq 62$

year	reference	lower	upper
1955	Greenwood, Gleason	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert	46	
1973	Whitehead	50	65
1973	Chung, Porter	51	
1974	Folkman		65
1995	Sánchez-Flores		64
1995	Kramer (no computer)		62
2004	Fettes-Kramer-R (computer)		62

History of bounds on $R_4(3)$ [from FKR 2004]

Four colors - $R_4(3)$

color degree sequences for $(3,3,3,3;\geq 59)$-colorings

n	orders of $N_\eta(v)$	
65	[16, 16, 16, 16]	Whitehead, Folkman 1973-4
64	[16, 16, 16, 15]	Sánchez-Flores 1995
63	[16, 16, 16, 14]	
	[16, 16, 15, 15]	
62	[16, 16, 16, 13]	Kramer 1995+
	[16, 16, 15, 14]	–
	[16, 15, 15, 15]	Fettes-Kramer-R 2004
61	[16, 16, 16, 12]	
	[16, 16, 15, 13]	
	[16, 16, 14, 14]	
	[16, 15, 15, 14]	
	[15, 15, 15, 15]	
60	[16, 16, 16, 11]	guess: doable in 2015
	[16, 16, 15, 12]	
	[16, 16, 14, 13]	
	[16, 15, 15, 13]	
	[16, 15, 14, 14]	
	[15, 15, 15, 14]	
59	[16, 16, 16, 10]	
	[16, 16, 15, 11]	
	[16, 16, 14, 12]	
	[16, 16, 13, 13]	
	[16, 15, 15, 12]	
	[16, 15, 14, 13]	
	[15, 15, 15, 13]	
	[15, 15, 14, 14]	

More colors - summary

k	value	or	bounds	reference(s)
2		6		[cf. Bush 1953]
3		17		Greenwood-Gleason 1955
4	51	–	62	Chung 1973 – Fettes-Kramer-R 2004
5	162	–	307	Exoo 1994 – easy
6	538	–	1838	Fredricksen-Sweet 2000 – easy
7	1682	–	12861	Fredricksen-Sweet 2000 – easy

Bounds and values of $R_k(K_3)$

Things to do

computational multicolor Ramsey numbers problems

- improve $45 \leq R(3, 3, 5) \leq 57$
- finish off $30 \leq R(3, 3, 4) \leq 31$
- understand why heuristics don't find $51 \leq R_4(3)$
- improve on $R_4(3) \leq 62$

More Arrowing

F, G, H - graphs, s, t, s_i - positive integers

Definitions

$F \rightarrow (s_1, ..., s_r)^e$ iff for every r-coloring of the edges
F contains a monochromatic copy of K_{s_i} in some color i.

$F \rightarrow (s_1, ..., s_r)^v$ iff for every r-coloring of the vertices
F contains a monochromatic copy of K_{s_i} in some color i.

$F \rightarrow (G, H)^e$ iff for every red/blue edge-coloring of F,
F contains a blue copy of G or a red copy of H.

Facts

$R(s, t) = \min\{n \mid K_n \rightarrow (s, t)^e\}$
$R(G, H) = \min\{n \mid K_n \rightarrow (G, H)^e\}$

Folkman problems

edge Folkman graphs
$\mathcal{F}_e(s, t; k) = \{G \to (s, t)^e : K_k \not\subseteq G\}$

edge Folkman numbers
$F_e(s, t; k)$ = the smallest n such that there exists an n-vertex graph G in $\mathcal{F}_e(s, t; k)$

vertex Folkman graphs/numbers
2-coloring vertices instead of edges

Theorem (Folkman 1970): For all $k > max(s, t)$, edge- and vertex Folkman numbers $F_e(s, t; k)$, $F_v(s, t; k)$ exist.

Two small cases

warming up

- $G = K_6$ has the smallest number of vertices among graphs which are not a union of two K_3-free graphs, or
  - $K_6 \rightarrow (K_3, K_3)^e$ and $K_5 \not\rightarrow (K_3, K_3)^e$

- What if we want G to be K_6-free?
  Graham (1968) proved that
  - $K_8 - C_5 = K_3 + C_5 \rightarrow (K_3, K_3)$
    $|V(H)| < 8 \wedge K_6 \not\subset H \Rightarrow H \not\rightarrow (K_3, K_3)$

Known values/bounds for $F_e(3,3;k)$

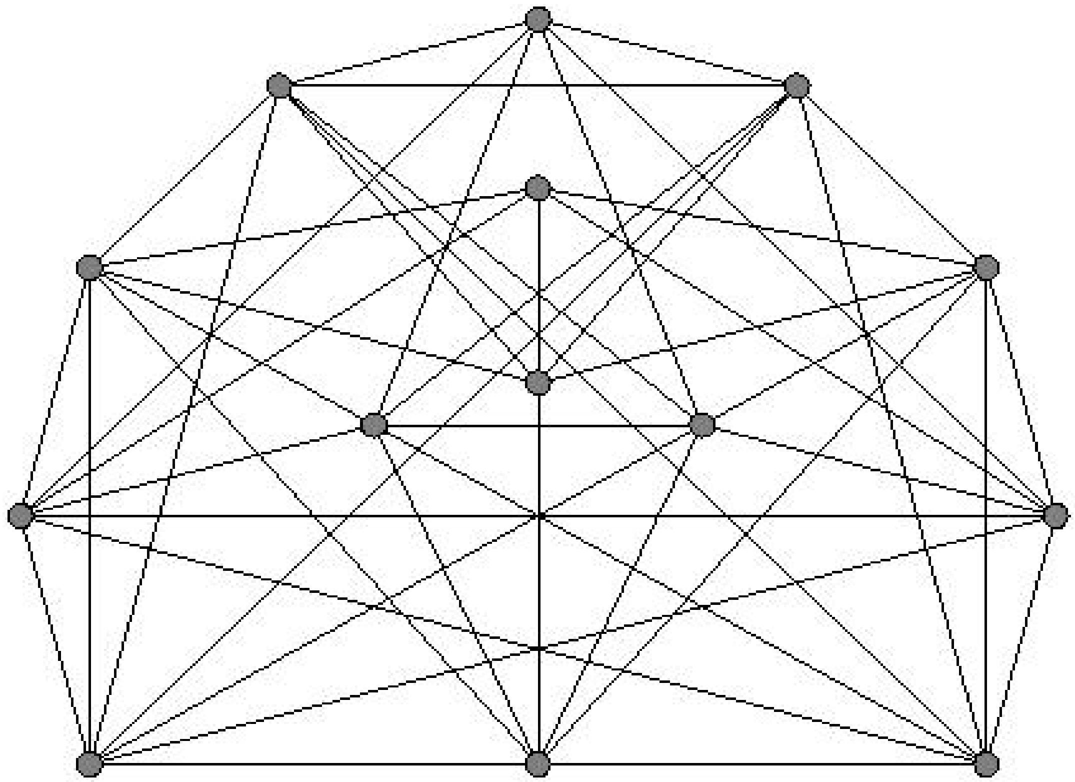
the challenge is to compute $F_e(3,3;4)$

$k > R(s,t) \Rightarrow F_e(s,t;k) = R(s,t)$
$k \leq R(s,t)$, very little known in general

k	$F_e(3,3;k)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham 1968
5	15	659 graphs	Piwakowski-Urbański-R 1999
4	≤ 941	$\alpha^5 \mod 941$	Dudek-Rődl 2008

$F_e(3,3;5) = 15$, and $F_v(3,3;4) = 14$

$G + x \rightarrow (3,3)^e$, and $G \rightarrow (3,3)^v$



unique 14-vertex bicritical $F_v(3,3;4)$-graph G [PRU 1999]

History of upper bounds on $F_e(3, 3; 4)$

- 1967 - Erdős, Hajnal state the problem
- 1970 - Folkman proves his theorem for 2 colors
  VERY large bound for $F_e(3, 3; 4)$.
- 1975 - Erdős offers $100 (or 300 Swiss francs)
  for deciding if $F_e(3, 3; 4) < 10^{10}$
- 1988 - Spencer, probabilistic proof for the bound 3×10^8
  (1989 - Hovey finds a mistake, bound up to 3×10^9)
- 2007 - Lu, ≤ 9697, spectral analysis of modular circulants
- 2008 - Dudek-Rődl, $F_e(3, 3; 4) \leq 941$
  circulant arc lengths $\alpha^5 \mod 941$

$F_e(3,3;4) \leq 941$

some details of the proof by Dudek-Rődl

- **Theorem:** If for every vertex $v \in V(G)$

$$Maxcut(G[N(v)]) < \frac{2}{3}|E(G[N(n)])|$$

then $G \rightarrow (3,3)^e$.

- Define graph H on vertices $E(G)$ with edges $\{(e,f) : e,f \in E(G), efg$ is a triangle in G for some $g\}$.

Maxcut approximation in H can imply $G \rightarrow (3,3)^e$.

- This works for the graph

$$G = (\mathcal{Z}_{941}, \{(i,j) : i - j = \alpha^5 \mod 941\})$$

History of lower bounds on $F_e(3,3;4)$

- $10 \leq F_e(3,3;4)$ Lin 1972

- $16 \leq F_e(3,3;4)$ Piwakowski-Urbański-R 1999
  since $F_e(3,3;5) = 15$, all graphs in $\mathcal{F}_e(3,3;5)$ on 15 vertices are known, and all of them contain K_4's

- $19 \leq F_e(3,3;4)$ R-Xu 2007
  $18 \leq F_e(3,3;4)$ proof "by hand"
  $19 \leq F_e(3,3;4)$ computations

- ANY proof technique improving on 19
  very likely will be of interest

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Testing arrowing is hard

theory/practice

- Testing whether $F \rightarrow (3,3)^e$ is **coNP**-complete
  Burr 1976
- Determining if $R(G,H) < m$ is **NP**-hard
  Burr 1984
- Testing whether $F \rightarrow (G,H)^e$ is Π_2^p-complete
  Schaefer 2001
- Implementing fast $F \rightarrow (3,3)^e$ is challenging

G_{127}

Hill-Irving 1982

$G_{127} = (\mathcal{Z}_{127}, E)$
$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$

Ramsey (4, 12)-graph, a color in (4, 4, 4; 127)
Exoo started to study if $G_{127} \rightarrow (3, 3)^e$

- 127 vertices, 2667 edges, 9779 triangles
- no K_4's, independence number 11, regular of degree 42
- vertex- and edge-transitive
- 5334 ($= 127 * 42$) automorphisms
- $(127, 42, 11, \{14, 16\})$ - regularity
- K_{127} can be partitioned into three G_{127}'s

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Reducing $\{G \mid G \not\rightarrow (3,3)^e\}$ to 3-SAT

edges in $G \longmapsto$ variables of ϕ_G
each (edge)-triangle xyz in $G \longmapsto$ add to ϕ_G

$$(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z})$$

Clearly,

$$G \not\rightarrow (3,3)^e \iff \phi_G \text{ is satisfiable}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 3-clauses,
2 for each of the 9779 triangles.

Note: By taking only the positive clauses, we obtain
a reduction to ϕ'_G in NAE-3-SAT with half of the clauses.

Use SAT-solvers

SAT 2007 Competition
3 medals in each of 9 categories

$$(\text{random, crafted, industrial}) \times (\text{SAT, UNSAT, ALL})$$

SATzilla CRAFTED (UBC) - winner of 2007 competition
in the category (crafted, UNSAT) - the one we need!

Rsat, Picosat, Minisat, March_KS
other recent leading SAT-solvers

$G_{127} \longrightarrow (3,3)^e$?

zChaff experiments on $\phi_{G_{127}}$

- Pick $H = G_{127}[S]$ on $m = |S|$ vertices.
  Use zChaff to split H:
  - $m \le 80$, H easily splittable
  - $m \approx 83$, phase transition ?
  - $m \ge 86$, splitting H is very difficult
- $\#(clauses)/\#(variables) = 7.483$ for G_{127}, far above conjectured phase transition ratio $r \approx 4.2$ for 3-SAT.
  It is known that
  $$3.52 \le r \le 4.596$$

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Folkman problems to work on

Is it true that $50 \leq F_e(3,3;4) \leq 100$?

- Decide whether $G_{127} \rightarrow (3,3)^e$
- Improve on $19 \leq F_e(3,3;4) \leq 941$
- Study $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$
- Study $F_e(K_4 - e, K_4 - e; K_4)$
- Don't study $F_e(K_3, K_3; K_4 - e)$
  it doesn't exist :–)

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So, what to do next?

computationally

Hard but potentially feasible tasks:

- Improve any of the Ramsey bounds
    - $40 \le R(3,10) \le 43$
    - $30 \le R(3,3,4) \le 31$
    - $51 \le R(3,3,3,3) \le 62$

- Folkman arrowing of K_3
    - Improve on $19 \le F_e(3,3;4) \le 941$
    - Study $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$

References

- Alexander Soifer
  The Mathematical Coloring Book, Springer 2009
- SPR's survey *Small Ramsey Numbers* at *ElJC*,
  revision #11, Aug 2006, here and on the web.
  All other references therein.
- Survey revision #12 in the Summer 2009 ...

Thanks for listening