Some Ramsey Problems Involving Triangles - Computational Approach

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Problems suggested during the talk at the Ramsey Theory Workshop
DIMACS, Rutgers University, May 28, 2009

Definition 1.

For graphs G and H, $R(G, H) = n$ if and only if n is the least positive integer such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color. $\diamondsuit$

We will write simply $R(k, l) = R(K_k, K_l)$ if the avoided graphs are complete. 2-colorings of the edges of K_n are often seen as graphs consisting of the edges in the first color, while their complements correspond to the edges in the second color. The $(k, l; n)$-graphs are n-vertex graphs lower-bounding $R(k, l)$, i.e. 2-colorings of the edges of K_n proving $n < R(k, l)$. The (k, l)-graphs will stand for $(k, l; n)$-graphs for some n. If $n = R(k, l) - 1$ then $(k, l; n)$-graphs are called critical. These concepts naturally generalize to r colors, r graphs, and the multicolor Ramsey numbers $R(G_1, \cdots, G_r)$.

Computational problems for Ramsey numbers, two colors

For detailed references to the subproblems listed in this and the next section please see the dynamic survey *Small Ramsey Numbers* in the *Electronic Journal of Combinatorics* [2]. Many historical comments and background information can be found in *The Mathematical Coloring Book* by Alexander Soifer [4].

The first open case of a Ramsey number of the form $R(3, k)$ is $40 \leq R(3, 10) \leq 43$. It seems that in order to determine the largest $(3, 10)$-graph we need to know more about $(3, 9; n \leq 35)$-graphs, which in turn requires the knowledge of $(3, 8; n \leq 27)$, which in turn requires the knowledge of $(3, 7; n \leq 22)$. All $(3, 6)$-graphs and all critical $(3, 7; 22)$-graphs are known, and there are 761692 and 191 of them, respectively. Thus the sequence of smaller, but still difficult, tasks towards solving $R(3, 10)$ could be as follows.

(a) Enumerate more $(3,7)$-graphs.

Enumerating all graphs in $(3,7;21)$ should be easy, more difficult for $(3,7;20)$ and perhaps feasible for $(3,7;19)$. More complete enumeration of $(3,7)$ can make it easier to progress on the further steps below.

(b) Enumerate all critical $(3,8;27)$-graphs.

More than 430K such graphs are already known, but there may be more of them. Full enumeration of $(3,8;26)$ seems to be very difficult, but it could likely be done for some well defined part, like graphs with at most 78 edges.

(c) Enumerate all critical $(3,9;35)$-graphs.

There is only one $(3,9;35)$-graph known, but there might be more of them. Finding all $(3,9;34)$-graphs also could be feasible.

(d) Finish off $37 \leq R(3, K_{10} - e) \leq 38$.

This number is between $R(3,9) = 36$ and $R(3,10)$, and the type of computations needed to decide the existence of $(3, K_{10} - e; 37)$-graphs is similar to what is needed in (c) and (e). (d) may possibly be easier, hence attacking it first is a good choice.

(e) Attack $R(3,10)$.

We know that $40 \leq R(3,10) \leq 43$. The author feels that 40 is likely the correct value. First, try to prove computationally that $R(3,10) \leq 42$. The results from (a) through (d) should help.

Computational problems for Ramsey numbers, multiple colors

The computational tasks related to the smallest and most studied open cases for multicolor Ramsey numbers are as follows.

(f) Improve on $45 \leq R(3,3,5) \leq 57$.

The task of just improving the inequality should not be too hard. We are not aware of any published dedicated attack on this number. The exact evaluation of $R(3,3,5)$ is a different matter, apparently well beyond what we can currently do.

(g) Finish off $30 \leq R(3,3,4) \leq 31$.

This is perhaps the only open case of a classical multicolor Ramsey number, for which we can anticipate exact evaluation in the not too distance future. Complete solution is likely feasible with a large scale computational effort similar to that in [PR1, PR2] as referenced in [2].

(h) Improve on $51 \leq R_4(3) \leq 62$.

This is the most studied and intriguing open multicolor case. We believe the lower bound to be close, if not equal, to the actual value. Improving the upper bound, while difficult, should be feasible with large scale computational effort, for example by extending on work [FKR] referenced in [2]. We are not aware of any heuristic approaches which would come even close to the lower bound 51. This could be used as an interesting novel test of strength of general heuristic search techniques. As of now, we do not seem to understand well why known heuristics are inadequate for this task.

Computational Folkman problems

The Folkman problems we are concerned with in this part can be expressed using the usual Ramsey arrowing operator restricted to graphs not containing K_m (or not containing some other graph). For detailed references to the background, history and problems similar to those listed below see *The Mathematical Coloring Book* by Alexander Soifer [4]. Many technical comments and further references can be found in [1] and [3].

Definition 2.

- $F \rightarrow (s_1, ..., s_r)^e$ if and only if for every r-coloring of the edges, the graph F contains a monochromatic copy of K_{s_i} in some color i, $1 \leq i \leq r$.
- $F \rightarrow (G, H)^e$ if and only if for every red/blue edge-coloring of F, the graph F contains a blue copy of G or a red copy of H.
- $\mathcal{F}_e(s, t; k) = \{G \rightarrow (s, t)^e : K_k \not\subseteq G\}$ is called the set of **edge Folkman graphs**.
- $F_e(s, t; k)$ is defined as the smallest integer n such that there exists an n-vertex graph G in $\mathcal{F}_e(s, t; k)$. These are called the **edge Folkman numbers**. $\diamondsuit$

Theorem (Folkman 1970).
For all $k > max(s, t)$ edge Folkman numbers $F_e(s, t; k)$ exist.

The most wanted edge Folkman number $F_e(3, 3; 4)$ involves the smallest parameters for which the problem is nontrivial, and quite surprisingly it is already extremely difficult to compute. Equivalently, $F_e(3, 3; 4)$ is equal to the order of the smallest K_4-free graph which is not a union of two triangle-free graphs. We know that $19 \leq F_e(3, 3; 4) \leq 941$, where the lower bound was established in [3] and the upper bound in [1]. Much of the history of work on such cases is reported in [3] and [4]. In particular, it seems that even the question if $50 \leq F_e(3, 3; 4) \leq 100$ could be very hard to answer.

Computational Folkman problems to work on

(i) Improve on $F_e(3,3;4) \leq 941$.

This bound was established by Dudek and Rödl in 2008 [1], after a few decades of colorful history reported in [4] and [3].

One of the possible options to proceed forwards is as follows. In 1982, Hill and Irving defined the graph $G_{127} = (Z_{127}, E)$, $E = \{(x,y) | x-y = \alpha^3 \pmod{127}\}$ in the context of Ramsey numbers. It is a $(4,12;127)$-graph, and also the monochromatic subgraph in each of three colors of a $(4,4,4;127)$ witness to the lower bound $128 \leq R(4,4,4)$. Exoo suggested to study if $G_{127} \rightarrow (3,3)^e$. If true it would prove that $F_e(3,3;4) \leq 127$.

(j) Improve on $19 \leq F_e(3,3;4)$.

No reasonable, even large scale, computation seems to be sufficient to improve on the lower bound of 19, which was established with significant computational effort in [3].

(k) Study $F_e(K_4 - e, K_4 - e; K_4)$.

We know that $19 \leq F_e(3,3;4) \leq F_e(K_4 - e, K_4 - e; K_4) \leq 30193$. The lower bound follows from monotonicity of $F_e()$, the upper bound, probably not a very strong one, was observed by Lu in his work on $F_e(3,3;4)$.

(l) Study $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$.

Similar to (k), but this time vary the forbidden graph while still considering arrowing triangles. We are not aware of any work related to these cases.

(m) Don't study $F_e(3,3;K_4 - e)$,

because after a moment of thought the reader can certainly discover that this number doesn't exist.

References

1. Andrzej Dudek and Vojtěch Rödl, On the Folkman Number $f(2,3,4)$, *Experimental Mathematics*, Vol. 17(1) (2008) 63-67. Upper bound $F_e(3,3;4) \leq 941$.

2. Stanisław Radziszowski, *Small Ramsey Numbers*, *Electronic Journal of Combinatorics*, DS1, revision #12, August 2009, 72 pages, `http://www.combinatorics.org/Surveys`. All other Ramsey numbers related references therein.

3. Stanisław Radziszowski and Xu Xiaodong, On the Most Wanted Folkman Graph, *Geombinatorics,* Vol. XVI (4) (2007) 367-381. History of problems related to $F_e(3,3;4)$, and the lower bound of 19.

4. Alexander Soifer, *The Mathematical Coloring Book*, Springer 2009. This book contains many further references to Ramsey and Folkman problems.