Failed power domination on graphs

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Abstract

Let G be a simple graph with vertex set V and edge set E, and let $S \subseteq V$. The *open neighborhood* of $v \in V$, $N(v)$, is the set of vertices adjacent to v; the *closed neighborhood* is given by $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of S, $N(S)$, is the union of the open neighborhoods of vertices in S, and the *closed neighborhood* of S is $N[S] = S \cup N(S)$.

The sets $\mathcal{P}^i(S), i \geq 0$, of vertices *monitored* by S at the ith step are given by $\mathcal{P}^0(S) = N[S]$ and $\mathcal{P}^{i+1}(S) = \mathcal{P}^i(S) \bigcup \{w : \{w\} = N[v] \backslash \mathcal{P}^i(S) \text{ for some } v \in \mathcal{P}^i(S)\}$. If there exists j such that $\mathcal{P}^j(S) = V$, then S is called a *power dominating set*, PDS, of G. Otherwise, G is a *failed power dominating set*.

The *power domination number* of a simple graph G, denoted $\gamma_p(G)$ gives the minimum number of PMUs required to observe a power network represented by G, and is the minimum cardinality of any PDS of G. The *failed power domination number* of G, $\bar{\gamma}_p(G)$ is the maximum cardinality of any FPDS of G, and represents the maximum number of PMUs that could be placed on a given power network represented by G, but fail to observe the full network. As a consequence, $\bar{\gamma}_p(G) + 1$ gives the minimum number of PMUs necessary to successfully observe the full network no matter where they are placed.

We prove that $\bar{\gamma}_p(G)$ is NP-hard to compute, determine graphs in which every vertex is a PDS, and compare $\bar{\gamma}_p(G)$ to similar parameters.

1 Introduction

This paper studies power domination on graphs, which arose because of applications to electric power networks [5, 16]. We denote by $G = (V, E)$ a graph with vertex set V and edge set E. In keeping with [11], which we refer to throughout the paper for basic graph theory definitions, we assume that the graph is finite and simple. In cases where the graph in question is ambiguous, we use $V(G)$ and $E(G)$. The *open neighborhood* of a vertex $v \in V$, denoted $N_G(v)$ or $N(v)$ when the graph is understood, is the set of neighbors of v, where u and v are referred to as *neighbors* of each other if uv is an edge of G. The *closed neighborhood* of v, $N[v]$, is $N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$, denoted $N(S)$, is the union of open neighborhoods of vertices in S, and the *closed neighborhood* of S, $N[S]$, is defined as $S \cup N(S)$. A vertex v is *dominated by* S if $v \in N[S]$. A set S is a *dominating set* if $N[S] = V$. The minimum cardinality of all dominating sets of G is the *domination number* $\gamma(G)$.

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Power domination differs from domination in that it contains a second step known as the *propagation step*. We use notation similar to that formalized in [1]. Let $i \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. If G is a graph and $S \subseteq V$, then the set of vertices *monitored* by S at Step i, denoted $\mathcal{P}^i(S)$, is defined as follows.

- $\mathcal{P}^0(S) = N[S]$,

- $\mathcal{P}^{i+1}(S) = \mathcal{P}^i(S) \bigcup \{w : \{w\} = N[v] \backslash \mathcal{P}^i(S) \text{ for some } v \in \mathcal{P}^i(S)\}$.

That is, Step 0 consists of finding the set of vertices dominated by S. For Step $i > 0$, if a vertex in $\mathcal{P}^i(S)$ has exactly one neighbor v outside of $\mathcal{P}^i(S)$, then we add v to $\mathcal{P}^{i+1}(S)$. The step corresponding to $i = 0$ is known as the *domination step* and those corresponding to $i > 0$ as the *propagation steps*. Note that for any $i \geq 0$, $\mathcal{P}^i(S) \subseteq \mathcal{P}^{i+1}(S)$. Also, if $\mathcal{P}^{i_0+1}(S) = \mathcal{P}^{i_0}(S)$ for some i_0, then $\mathcal{P}^j(S) = \mathcal{P}^{i_0}(S)$ for any $j \geq i_0$, and then we write $\mathcal{P}^\infty(S) = \mathcal{P}^{i_0}(S)$.

Definition 1.1.   *a. A* power dominating set *(PDS) of G is a set $S \subseteq V$ such that $\mathcal{P}^\infty(S) = V$.*

   b. *A* failed power dominating set *(FPDS) is a set $S \subseteq V$ such that S is not a PDS.*

   c. *A* stalled power dominating set *(SPDS) is a set $S \subseteq V$ such that $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$. That is, after the domination step, no propagation steps occur.*

   d. *The* power domination number *of G, denoted by $\gamma_p(G)$, is the minimum cardinality among all power dominating sets of G.*

   e. *The* failed power domination number *of G, denoted by $\bar{\gamma}_p(G)$, is the maximum cardinality among all failed power dominating sets of G.*

If S is an SPDS in G such that $S \cup \{u\}$ is a PDS for any vertex $u \in V \backslash S$, then we say that S is *maximally stalled*. To indicate that S is an SPDS and $\mathcal{P}^0(S) \subsetneq V$, we say that S is *properly stalled*. In [15], the authors defined a critical set in the context of directed graphs, which can be stated for undirected graphs as follows: a vertex set W is a *weakly critical set* of G if there is no vertex outside W that has exactly one neighbor in W. The results of this paper could be framed in terms of critical sets; however we use SPDS to connect these results with related work in zero forcing on undirected graphs [4, 13, 22]. Figures 1 and 2 show an example of a graph G with $\gamma_p(G) \leq 2$ but $\bar{\gamma}_p(G) \geq 20$.
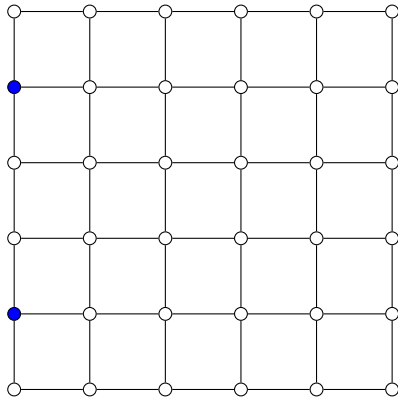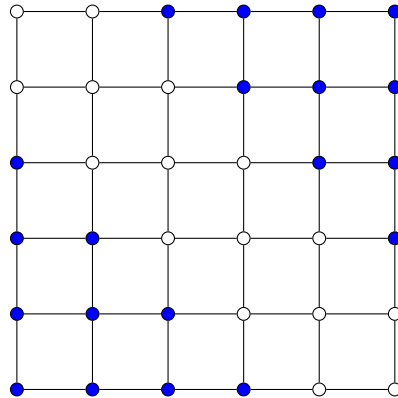

Figure 1: A PDS S in blue

Figure 2: An FPDS and SPDS S in blue

In this paper, we determine the computational complexity of the decision problem for failed power domination. We also study extremal values of $\bar{\gamma}_p(G)$ and find graphs that attain them. We present a list of graphs that have $\bar{\gamma}_p(G) = 0$, which is a particularly interesting case, since $\bar{\gamma}_p(G) = 0$ implies that any nonempty set of vertices in G is a PDS. We also discuss the relationship between $\bar{\gamma}_p(G)$ and some related parameters.

2 Motivation and related parameters

The idea of power domination on graphs is motivated by the need to monitor electric power networks. In [5], the authors describe the problem of observing a power system while minimizing the number of measurement devices known as phasor measurement units (PMUs) on the network. A PMU measures the voltage and phase angle, and allows for synchronization [20], which is one strategy described in [17] for making the power grid more robust. If a PMU measures the voltage and phase angle of vertex v (or edge e), then v (or e) is said to be *observed*. The vertex on which a PMU is placed is observed, as are its incident edges and adjacent vertices. In addition, any vertex that is incident to an observed edge is observed; any edge joining two observed vertices is observed; finally, from Kirchhoff's Law, given an observed vertex v with k incident edges, if $k-1$ of the edges are observed, then all k are observed. In [16] the authors formulate and investigate this problem as a graph theoretic problem. Later, Brueni and Heath [9] and Kneis et al. [19] independently showed that the problem can be simplified to omit any reference to edges. The formal set definition of $\mathcal{P}^i(S)$ was introduced in [1].

Under the model described in [16], the power domination number $\gamma_p(G)$ gives the minimum number of PMUs required to observe a power network represented by graph G. The power domination number has been studied for multiple families of graphs [8, 12, 24], as has the complexity of $\gamma_p(G)$ [16]. On the other hand, the failed power domination number $\bar{\gamma}_p(G)$ that we defined above gives a worst case scenario: what is the maximum number of PMUs that we could use on a given network represented by G, but fail to observe the full network? In addition, $\bar{\gamma}_p(G) + 1$ gives us the minimum number of PMUs necessary to successfully observe the full network no matter where we place the PMUs.

The concept of zero forcing, while related to power domination, was introduced in 2008 in the context of minimum rank problems [2, 6] as well as quantum networks in 2007 [10]. Zero forcing acts like power domination, but without the domination step. That is, given a set S, $\mathcal{Q}^0(S) = S$, and for $i \in \mathbb{N}_0$, $\mathcal{Q}^{i+1}(S) = \mathcal{Q}^i(S) \bigcup \{N[v] : v \in \mathcal{Q}^i(S)$ and $|N[v] \backslash \mathcal{Q}^i(S)| = 1\}$. Note that there exists an i_0 such that for all $j > i_0$, $\mathcal{Q}^j(S) = \mathcal{Q}^{i_0}(S)$, so we write $\mathcal{Q}^\infty(S) = \mathcal{Q}^{i_0}(S)$. If $\mathcal{Q}^\infty(S) = V$, then S is a *zero forcing set*. Otherwise, S is a *failed zero forcing set*. The smallest cardinality of any zero forcing set in G is the *zero forcing number* $\mathrm{Z}(G)$, and the largest cardinality of any failed zero forcing set is the *failed zero forcing number* $\mathrm{F}(G)$ [4, 13]. Complexity results for failed zero forcing were established in [22].

Lemma 2.1. *For a graph $G = (V, E)$, suppose $S \subseteq S' \subseteq V$. Then,*

1. *$\mathcal{Q}^\infty(S) \subseteq \mathcal{Q}^\infty(S')$, and*

2. *$\mathcal{P}^\infty(S) \subseteq \mathcal{P}^\infty(S')$.*

Proof. Suppose $S \subseteq S' \subseteq V$. Then $\mathcal{Q}^0(S) \subseteq \mathcal{Q}^0(S')$. Assume $\mathcal{Q}^k(S) \subseteq \mathcal{Q}^k(S')$. If $u \in \mathcal{Q}^{k+1}(S)$, then either $u \in \mathcal{Q}^k(S)$, implying $u \in \mathcal{Q}^{k+1}(S')$, or there exists $v \in \mathcal{Q}^k(S)$ such that $N[v] \backslash \mathcal{Q}^k(S) = \{u\}$ giving us $N[v] \backslash \{u\} \subseteq \mathcal{Q}^k(S')$. Thus $u \in \mathcal{Q}^{k+1}(S')$. Hence $\mathcal{Q}^i(S) \subseteq \mathcal{Q}^i(S')$ for any $i \in \mathbb{N}_0$, giving us $\mathcal{Q}^\infty(S) \subseteq \mathcal{Q}^\infty(S')$.

To prove 2, let $u \in \mathcal{P}^0(S)$. Then $u \in N[v]$ for some $v \in S$. Since $S \subseteq S'$, we have $u \in \mathcal{P}^0(S')$. Thus, $\mathcal{P}^0(S) \subseteq \mathcal{P}^0(S')$. The remainder of the proof is identical to the proof of 1. $\square$

Since any set $S \subseteq V$ is a subset of the set of vertices it dominates, we have the following observation.

Observation 2.2. $\bar{\gamma}_p(G) \leq F(G)$.

3   Complexity

In this section, we show that it is NP-hard to determine whether G has a failed power dominating set of cardinality at least k. We use a similar technique to the one used in [22] to show NP-completeness of failed zero forcing parameters.

We introduce two definitions that we use in this section and the sections that follow. A graph G is *connected* if G contains a path from u to v for every $u, v \in V(G)$. Note that the trivial graph is connected. For $S \subseteq V(G)$ where S is nonempty, the *subgraph of G induced by S*, denoted $G[S]$ has S as its vertex set, and two vertices $u, v \in S$ are neighbors in $G[S]$ if and only if they are neighbors in G.

FAILED POWER DOMINATING SET (FPDS), (G, m)
Instance: Graph $G = (V, E)$ and a positive integer m
Question: Does G have a proper stalled subset of cardinality at least m?

To prove that FPDS is NP-hard, we construct a polynomial reduction from the well-known NP-complete problem, INDEPENDENT SET, which remains NP-complete when restricted to connected graphs [22].

INDEPENDENT SET, (G, k)
Instance: Connected graph $G = (V, E)$ and a non-negative integer k.
Question: Does G contain an independent set of cardinality k?

The domination number of a path on k vertices, $\gamma(P_k)$, is known to be $\lceil k/3 \rceil$ [14].

Lemma 3.1. *Let G be a graph that contains an induced subgraph P_k, where $k \geq 3$, all internal vertices of P_k have degree 2 in G, one end vertex of P_k has degree 1 in G, and the other end vertex, v, has degree at least 3 in G. If S is an SPDS containing at least one vertex of P_k, then $|S \cap P_k| \geq \gamma(P_k) = \lceil k/3 \rceil$. If S is maximally stalled and contains at least one vertex of P_k, then $|S \cap P_k| \geq k - 1$.*

Proof. Note that if there are at least two adjacent vertices in $\mathcal{P}^0(S) \cap P_k$, then for some $i \geq 0$, $V(P_k) \subseteq \mathcal{P}^i(S)$. If there is a vertex in $S \cap P_k$, then after the domination step, there are at least two adjacent vertices from P_k in $\mathcal{P}^0(S)$. Thus, if S is stalled, it must be that at least $\gamma(P_k)$ vertices on the path are in S; otherwise, $\mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$ is nonempty.

Since at least $\gamma(P_k)$ vertices on the path P_k are in S, it follows that $\mathcal{P}^0(S)$ contains all vertices in P_k. Thus, if S is maximally stalled and contains at least one vertex of P_k, it must contain all vertices other than v. That is, $|S \cap P_k| \geq k - 1$. $\square$

To prove the following lemma, we construct a polynomial reduction from INDEPENDENT SET. An example reduction instance is shown in Figures 3 and 4.
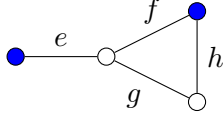
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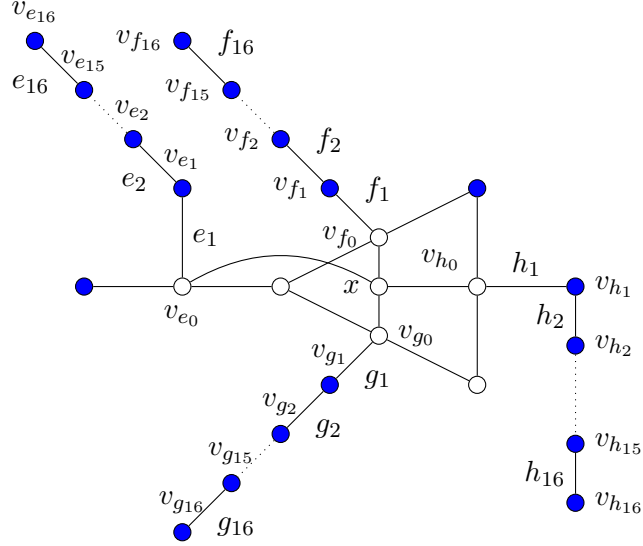Figure 3: A graph G with independent set of cardinality 2 shown in blue

Figure 4: The graph G' with FPDS S in blue, $|S| = 66$

Lemma 3.2. *FAILED POWER DOMINATING SET is NP-hard.*

Proof. Suppose (G, k) with $n = |V| \geq 3$, $k \geq 2$ is an instance of INDEPENDENT SET. We construct from it an instance (G', m) of FPDS for $m = n^2|E| + k$. Let U be an independent set of G. Then $U' = U \cup V_1' \cup V_2' \cup \cdots \cup V_{n^2}'$ is an SPDS of cardinality $n^2|E| + k$ in G', where $G' = (V', E')$ is constructed as follows.

1. $V \subseteq V'$.

2. Subdivide every edge of G. That is, for each $e = \{u, v\} \in E$, add a vertex v_{e_0} to V', and let $\{u, v_{e_0}\}, \{v_{e_0}, v\} \in E'$. Let V_0' denote these added vertices, and E_0' the added edges.

3. For each $e = \{u, v\} \in E$, add vertices v_{e_1} through $v_{e_{n^2}}$ to V'. For each $i = 1, 2, \ldots, n^2$, let V_i' denote all $v_i \in V'$, and add edge $e_i = \{v_{e_{i-1}}, v_{e_i}\}$ to E'. Let $P(e)$ denote the path from v_{e_0} to $v_{e_{n^2}}$. Let the set of all such paths be denoted ρ.

4. Add a vertex x to V'. For each vertex $v_{e_0} \in V_0'$, add $\{x, v_{e_0}\}$ to E'.

To see that U' is an SPDS in G', note that $\mathcal{P}^0(U') = U \cup V_0' \cup V_1' \cup V_2' \cup \cdots \cup V_{n^2}'$.

If U' is not an SPDS, then $\mathcal{P}^1(U') \backslash \mathcal{P}^0(U')$ is nonempty. The only vertices in $V' \backslash \mathcal{P}^0(U')$ are x and the vertices from $V \backslash U$. We know that $N_{G'}(x) = V_0'$, but each vertex in V_0' has at least one other neighbor in $V \backslash (U \cup \mathcal{P}^0(S))$ (since U is an independent set in G). Hence, $x \notin \mathcal{P}^1(U')$. Similarly, for any vertex $v \in V \backslash U$, the neighborhood $N_{G'}(v)$ is contained in V_0'. But for each $v_{e_0} \in V_0'$, $N_{G'}(v_{e_0})$ includes x and one vertex from $V \backslash U$. Hence if $v \in V \backslash U$, then $v \notin \mathcal{P}^1(U')$, and U' is stalled.

Suppose that $S \subseteq V(G')$ is maximally stalled with $|S| \geq n^2|E| + 2$. We will show that for each path $P(e) \in \rho$, $|S \cap P(e)| \geq n^2$. Since $|V'| = (n^2 + 1)|E| + n + 1$, there are at most $|E| + n - 1$ vertices in $V' \backslash S$. Each path $P(e)$ has $n^2 + 1$ vertices. Note that $n^2 + 1 - (|E| + n - 1) \geq n^2 - n + 2 - \frac{n(n-1)}{2} > 1$, and thus, $P(e)$ contains at least one vertex in S. By Lemma 3.1, then, $|S \cap P(e)| \geq n^2$, implying that $\cup_{i=0}^{n^2} V_i' \subseteq \mathcal{P}^0(S)$.

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We show that $V_0' \cap S = \emptyset$. Without loss of generality suppose $v_{e_0} \in S$. Then $N_{G'}(v_{e_0}) = \{u, v, x, v_{e_1}\}$ where $e = \{u, v\}$, so $\{u, v, x\} \subseteq \mathcal{P}^0(S)$. Since G is connected, G' is also connected. Thus, there is a path in G' from u to any vertex in V. Since S is properly stalled and $(V' \backslash V) \subseteq \mathcal{P}^0(S)$, it follows that there must be some vertex $y \in V$ such that $y \notin \mathcal{P}^0(S)$. Then, on the path from u to y, there exists $\hat{e} = \{w, z\} \in E$ with $w \in \mathcal{P}^0(S)$ and $z \notin \mathcal{P}^0(S)$. Consider the vertex $v_{\hat{e}_0} \in V_0'$. The set $N_{G'}[v_{\hat{e}_0}] \backslash \mathcal{P}^0(S)$ consists only of the vertex z (since we just noted that $x \in \mathcal{P}^0(S)$), so $z \in \mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$, a contradiction of S being stalled. Hence, $V_0' \cap S = \emptyset$. Since we know that for each path $P(e) \in \rho$, $|P(e) \cap S| \geq n^2$, it follows that $\cup_{i=1}^{n^2} V_i' \subseteq S$.

Now, we show that $x \notin S$. Suppose $x \in S$. Note that $V \cap S$ is nonempty, because we assumed that $|S| \geq n^2 |E| + 2$. Also, $V \backslash S$ is nonempty since we showed that $\cup_{i=0}^{n^2} V_i' \subseteq \mathcal{P}^0(S)$. Since we're assuming that $x \in S$, if $V \subseteq S$, then $\mathcal{P}^0(S) = V'$, contradicting the assumption that S is properly stalled. Hence, there exists an edge $e = \{u, v\} \in E$ with $u \in S$ and $v \notin S$. Then the vertex v_{e_0} has $N_{G'}[v_{e_0}] \backslash \mathcal{P}^0(S) = \{v\}$, and $\mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$ is nonempty, a contradiction of S being stalled. Hence, $x \notin S$.

Finally, we will show that $S \cap V$ is an independent set of G. Suppose there exists an edge $e = \{u, v\} \in E$ for some $u, v \in S$. Then v_{e_0} has $N_{G'}[v_{e_0}] \backslash \mathcal{P}^0(S) = \{x\}$, and $x \in \mathcal{P}^1(S) \backslash \mathcal{P}^0(S)$, a contradiction of S being stalled. Hence, $S \cap V$ is an independent set in G.

This gives us that for any maximal properly stalled subset S of V',

$$|S| = n^2 |E| + t,$$

where t is the order of independent set $S \cap V$. Thus G' has an SPDS of order $m = n^2 |E| + k$ if and only if G has an independent set of order k. The construction of G' is polynomial and thus this completes our proof that FPDS is NP-hard. $\qquad \square$

For a graph G, positive integer k, and $S \subseteq V$ with $|S| \leq k$, it is verifiable in polynomial time whether or not S is a PDS [16]. Thus it is verifiable in polynomial time whether S is an FPDS, completing the proof of the following theorem.

Theorem 3.3. *FAILED POWER DOMINATING SET is NP-complete.*

4   Extreme values

In this section, we characterize n-vertex graphs G with $\bar{\gamma}_p(G) \geq n - 3$. We also give some results for the case $\bar{\gamma}_p(G) = 0$. The next observation follows from the definition of PDS.

Observation 4.1. *If S is a PDS of G, then $\mathcal{P}^0(S) \backslash S$ is a zero forcing set of $G[V \backslash S]$.*

Theorem 4.2. *Let $G = (V, E)$ be a simple graph with n vertices. Then we have the following characterization of graphs with high values of $\bar{\gamma}_p(G)$.*

1. *$\bar{\gamma}_p(G) = n - 1$ if and only if G has an isolated vertex.*

2. *$\bar{\gamma}_p(G) = n - 2$ if and only if G contains an isolated edge as a component, and no isolated vertices.*

3. *$\bar{\gamma}_p(G) = n - 3$ if and only if G contains no components that are isolated vertices or isolated edges and G contains as an induced subgraph one of the following:*

    - *P_3, where only the vertex of degree 2 in P_3 may be adjacent to other vertices in V, or*

    - *K_3, where at most one of the vertices may be adjacent to other vertices in V.*

Proof. If G has an isolated vertex v, let $S = V \backslash \{v\}$. Then S is an FPDS, and $\bar{\gamma}_p(G) = n - 1$. Conversely, let $\bar{\gamma}_p(G) = n - 1$, and let S be an FPDS. If the single vertex $v \in V \backslash S$ has an edge to any vertex $u \in S$, then $v \in \mathcal{P}^0(S)$. Hence, v is isolated, completing the proof of 1.

If G contains no isolated vertices, and one component is K_2 with vertices u, v, then let $S = V \backslash \{u, v\}$. Then S is an FPDS, and $\bar{\gamma}_p(G) = n - 2$.

Conversely, suppose $\bar{\gamma}_p(G) = n - 2$. We know G contains no isolated vertices. Let S be an FPDS with $|S| = n - 2$. Let u, v be the two vertices in $V \backslash S$. If u is adjacent to some vertex $w \in S$, then $u \in \mathcal{P}^0(S)$, giving us that all vertices except possibly v are in $\mathcal{P}^0(S)$. But then, $v \in \mathcal{P}^1(S)$, implying that S is a PDS. Therefore, neither u nor v is adjacent to any vertex in S, but since there are no isolated vertices, uv forms a copy of K_2, completing the proof of 2.

If G does not contain any isolated vertex or component that is K_2, then $\bar{\gamma}_p(G) \leq n - 3$. If G contains an induced copy of $P_3 = \{u, v, w\}$ with edges uv, vw, note that only v may be adjacent to other vertices in V. Let $S = V \backslash \{u, v, w\}$. Then it is possible that $v \in \mathcal{P}^0(S)$, but $u, w \notin \mathcal{P}^i(S)$ for any $i \geq 0$ since $N(u) = N(w) = \{v\}$. The same holds if $G[\{u, v, w\}]$ forms a copy of K_3.

Conversely, suppose $\bar{\gamma}_p(G) = n - 3$, and let S be an FPDS with $|S| = n - 3$. Let $\{u, v, w\} = V \backslash S$. Suppose $\{u, v\} \subseteq N(S)$. Then $w \notin N(S)$, because $w \in N(S)$ implies $\{u, v, w\} \subseteq \mathcal{P}^0(S)$. However, since w cannot be an isolated vertex, $w \in N(u)$ (without loss of generality) but then $w \in \mathcal{P}^1(S)$, implying that S is a PDS. Hence, only one of $\{u, v, w\}$ may be in $N(S)$. Without loss of generality, say it is v. Since G has no isolated vertices or K_2 component, and vertices u and w have no neighbors outside of $\{u, v, w\}$, then $G[\{u, v, w\}]$ is either K_3 or P_3. If it is K_3, we are done. If it is P_3, and v has any other neighbors in G, note that v must be the middle vertex. If not, $\{u, w\} \subseteq \mathcal{P}^2(S)$, implying S is not an FPDS. This completes the proof of 3. $\qquad \square$

4.1 Graphs in which every vertex is a PDS

In this section, we present some results on graphs that have $\bar{\gamma}_p(G) = 0$. Note that if $\bar{\gamma}_p(G) = 0$, then any single vertex is a PDS of G. We use the notation $\mathcal{P}^i_G(S)$ to indicate $\mathcal{P}^i(S)$ in G only when the graph in question is ambiguous. We also use the following graph operation. Given graphs G_1 and G_2, the *join* $G = G_1 \vee G_2$ of G_1 and G_2, has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{\{u, v\} : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Lemma 4.3. $\bar{\gamma}_p(G_1 \vee G_2 \vee \cdots \vee G_n) = 0$ *if and only if for each $i = 1, 2, \ldots, n$, where $n \geq 2$, either* $\bar{\gamma}_p(G_i) = 0$ *or* $G_i = \overline{K_2}$.

Proof. Let G_1 and G_2 be graphs, and let $v \in V(G_1)$. Then $\mathcal{P}^0_{G_1 \vee G_2}(\{v\}) = \mathcal{P}^0_{G_1}(\{v\}) \cup V(G_2)$, and as a result, $\mathcal{P}^i_{G_1 \vee G_2}(\{v\}) = \mathcal{P}^i_{G_1}(\{v\}) \cup V(G_2)$ for any $i \geq 0$, unless $G_1 = \overline{K_2}$, in which case $\mathcal{P}^1_{G_1 \vee G_2}(\{v\}) = V(G_1) \cup V(G_2)$. Hence, $\{v\}$ is a PDS in $G_1 \vee G_2$ if and only if $\{v\}$ is a PDS in G_1 or $G_1 = \overline{K_2}$, and similarly for G_2. That is, $\bar{\gamma}_p(G_1 \vee G_2) = 0$ if and only if $\bar{\gamma}_p(G_1) = 0$ or $G_1 = \overline{K_2}$, and $\bar{\gamma}_p(G_2) = 0$ or $G_2 = \overline{K_2}$. We can use the same argument if G_1 or G_2 is itself the join of two graphs. Hence, by induction, $\bar{\gamma}_p(G_1 \vee G_2 \vee \cdots \vee G_n) = 0$ if and only if $\bar{\gamma}_p(G_i) = 0$ or $G_i = \overline{K_2}$ for each $i = 1, 2, \ldots, n$. $\qquad \square$

In a poster [23], Tostado listed several families of graphs that have $\bar{\gamma}_p = 0$. We include this list here with proofs. For $n \geq 4$, a *wheel on n vertices*, W_n, is defined by $W_n = C_{n-1} \vee \{v\}$.

Theorem 4.4. *The following graphs have $\bar{\gamma}_p = 0$ [23].*

1. *a path on n vertices, P_n for $n \geq 1$,*

2. *a cycle on n vertices, C_n for $n \geq 3$,*

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3. *a complete graph on n vertices, K_n, for $n \geq 1$,*

4. *a wheel on n vertices, W_n for $n \geq 4$.*

Proof. If $G = P_n$ or $G = C_n$, and $S = \{v\}$ for any vertex $v \in V(G)$, then $\mathcal{P}^0(S)$ consists of at least two adjacent vertices, and S is a PDS, proving 1 and 2. For 3, note that $K_n = G_1 \vee G_2 \vee \cdots \vee G_n$ where G_i consists of a single vertex, and thus by Lemma 4.3, $\bar{\gamma}_p(K_n) = 0$. Finally, since $W_n = C_{n-1} \vee \{v\}$, also by Lemma 4.3, $\bar{\gamma}_p(W_n) = 0$. Thus 4 also holds. $\qquad\square$

We add several families of graphs to this list. An example for item 4 from Theorem 4.6 below is shown in Figure 5. In the proof of Theorem 4.6, we use a property that follows from the definitions of PDS and zero forcing sets:

Lemma 4.5. *Suppose G is a graph, and $S \subseteq V$. Suppose that for some $i \geq 0$, a subset S' of $\mathcal{P}^i(S)$ is a zero forcing set of the graph induced by $(V \backslash \mathcal{P}^i(S)) \cup S'$. Then S is a PDS of G.*

Proof. For each $j \geq 0$, $\mathcal{Q}^j(S') \subseteq \mathcal{P}^{i+j}(S)$. Thus, if $\mathcal{Q}^\infty(S') = V$, then $\mathcal{P}^\infty(S') = V$ as well. $\qquad\square$

    (a) S (b) $\mathcal{P}^0(S)$ (c) $\mathcal{P}^1(S)$ (d) $\mathcal{P}^2(S)$ (e) $\mathcal{P}^3(S)$
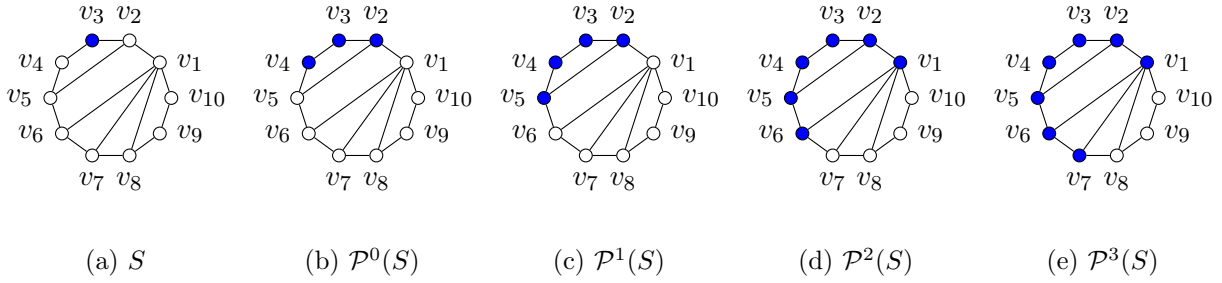
Figure 5: A graph G with $\bar{\gamma}_p(G) = 0$ as in Theorem 4.6, item 4. $S = \{v_3\}$ is shown in blue on the left, followed by $\mathcal{P}^0(S)$ through $\mathcal{P}^3(S)$. Continuing, $\mathcal{P}^4(S) = \{v_1, v_2, \ldots v_8\}$, and $\mathcal{P}^5(S) = V$.

Theorem 4.6. *If G is any of the following graphs, then $\bar{\gamma}_p(G) = 0$.*

1. $\overline{C_n}$ *for $n \geq 5$,*

2. $\overline{P_n}$ *for $n \geq 4$,*

3. $C_n = v_1 v_2 \ldots v_n v_1$ *with k chords: $\{v_1, v_i\}, \{v_1, v_{i+1}\}, \ldots, \{v_1, v_{i+k-1}\}$, where $i \geq 3$, $n \geq 4$, and $i + k \leq n - 1$.*

4. $C_n = v_1 v_2 \ldots v_n v_1$ *with $k+1$ chords: $\{v_1, v_i\}, \{v_1, v_{i+1}\}, \ldots, \{v_1, v_{i+k-1}\}$, and $\{v_2, v_{i-1}\}$ where $i \geq 5$, $n \geq 6$, and $i + k \leq n - 1$.*

5. $G \vee H$ *where $\bar{\gamma}_p(G) = 0$ and either $\bar{\gamma}_p(H) = 0$ or $H = \overline{K_2}$.*

Proof. To prove 1, let $G = \overline{C_n}$ with $n \geq 5$, and $S = \{v\}$ for any vertex $v \in V(G)$. Let the two neighbors of v in $C_n = \overline{G}$ be u and w. In $G = \overline{C_n}$, then $\mathcal{P}^0(S) = V(G) \backslash \{u, w\}$, and it follows easily that $\mathcal{P}^1(S) = V(G)$.

To prove 2, note that if $G = \overline{P_n}$ with $n \geq 4$, and $S = \{v\}$ for any vertex v with $\deg(v) = 2$ in P_n, then we can use the same argument as for $\overline{C_n}$. Otherwise, if $\deg(v) = 1$ in P_n, then

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$\mathcal{P}^0(S) = V(G)\backslash\{u\}$ where u is the unique neighbor of v in P_n. Since $n \geq 4$, we see that $\mathcal{P}^1(S) = V$, and $S = \{v\}$ is a PDS.

To prove 3, let P_1 and P_2 be the unique paths from v_1 to v_i and from v_1 to v_{i+k-1}, respectively, whose internal vertices all have degree 2. Note that $\{v_1, v_j, v_{j+1}\}$ is a zero forcing set of G for $2 \leq j \leq n-1$. If $S = \{v_1\}$, then $\mathcal{P}^0(S)$ consists of v_1, v_2, v_n, and v_i through v_{i+k-1}, which is a zero forcing set of G. Hence, $S = \{v_1\}$ is a PDS. Similarly, if $S = \{v_j\}$ where $i \leq j \leq i + k - 1$, then $\mathcal{P}^0(S)$ includes v_1, v_{j-1}, v_j, and v_{j+1}, a zero forcing set of G. Hence $S = \{v_j\}$ is a PDS. Finally, suppose $S = \{u\}$ for an internal vertex of P_1 or P_2. There exists j such that $v \in \mathcal{P}^j(S)$, and both neighbors of u on the cycle are also in $\mathcal{P}^j(S)$, so $\mathcal{P}^j(S)$ is a zero forcing set, and it follows that $\{u\}$ is a PDS.

To prove 4, note that $\{v_1, v_j, v_{j+1}\}$ is a zero forcing set of G for $2 \leq j \leq n-1$. Let P_1 denote the path with all internal vertices of degree 2 from v_2 to v_{i-1}, and P_2 the similar path from v_{i+k-1} to v_1. If $S = \{v_1\}$, then $\mathcal{P}^0(S)$ contains v_1 and $\{v_j, i \leq j \leq i+k-1\}$, which is a zero forcing set of G; hence $S = \{v_1\}$ is a PDS. Similarly, if $S = \{v_\ell\}$ for $\ell \in \{2, i, i+1, \ldots, i+k-1, n\}$, then $\mathcal{P}^0(S)$ contains $\{v_1, v_j, v_{j+1}\}$ for some j, a zero forcing set. Thus, $S = \{v_\ell\}$ for $\ell \in \{2, i, i+1, \ldots, i+k-1, n\}$ is a PDS. Suppose $S = \{v_{i-1}\}$. Then $\mathcal{P}^0(S) = \{v_{i-2}, v_{i-1}, v_i, v_2\}$. Since v_{i-2} has a unique neighbor v_{i-3} outside of $\mathcal{P}^0(S)$, the next vertex along P_1, $\mathcal{P}^1(S) = \{v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_2\}$. This continues for all internal vertices of P_1, giving us that for some ℓ, $\mathcal{P}^\ell(S) = V(P_1) \cup \{v_i\}$. Since v_1 is the only neighbor of v_2 outside of $\mathcal{P}^\ell(S)$, $v_1 \in \mathcal{P}^{\ell+1}(S)$, so $\mathcal{P}^{\ell+1}(S)$ includes at least two adjacent vertices in G as well as v_1, which is a zero forcing set. Hence, $S = \{v_{i-1}\}$ is a PDS. If $S = \{u\}$ where $u \in V(P_1)$ or $u \in V(P_2)$, there exists ℓ such that $\mathcal{P}^\ell(S)$ contains all vertices in P_1 and the vertex v_1, or all vertices in P_2 (which includes v_1). This is a zero forcing set; hence $S = \{u\}$ is a PDS for any $u \in V(P_1)$ or $V(P_2)$. For an example, see Figure 5.

The proof of 5 follows immediately from Lemma 4.3. $\qquad\square$

Note that if G is not connected, then the vertices of any single component form an FPDS of G, giving us the following observation.

Observation 4.7. *If $\bar{\gamma}_p(G) = 0$, then G is connected.*

The *path cover number* of a graph G, denoted $\mathrm{P}(G)$, is the minimum number of vertex disjoint paths, each of which is an induced subgraph of G, that contain all vertices of G. Hogben [18, Theorem 2.13] showed that $\mathrm{P}(G) \leq \mathrm{Z}(G)$, which easily leads to the next theorem.

Theorem 4.8. *Suppose $\bar{\gamma}_p(G) = 0$ and either G has a vertex of degree one or a cut-vertex. Then $G = P_n$ for some $n \geq 1$.*

Proof. Suppose G has a vertex v of degree one, and $\bar{\gamma}_p(G) = 0$. Then $\mathcal{P}^0(\{v\}) = \{u, v\}$ where u is the unique neighbor of v. By Observation 4.1, $\{u\}$ is a zero forcing set of $G[V\backslash\{v\}]$. Thus $\mathrm{Z}(G[V\backslash\{v\}]) = 1$, and by [18, Theorem 2.13], since $\mathrm{P}(G) \leq \mathrm{Z}(G)$, $\mathrm{P}(G[V\backslash\{v\}]) = 1$. Hence, $G[V\backslash\{v\}]$ is a path, and consequently, G is as well.

Suppose G has a cut-vertex v. Let $u \in V$ with $u \neq v$, and let $\mathcal{K}_u$ be the component of $G[V\backslash\{v\}]$ containing u. Let vertex w be in a different component, $\mathcal{K}_w$ of $G[V\backslash\{v\}]$. By assumption, both $\{u\}$ and $\{w\}$ are PDS. Then there exists some j such that $v \in \mathcal{P}^j(\{u\})$, and all other vertices in $\mathcal{P}^j(\{u\})$ are in the component $\mathcal{K}_u$. Then $|N(v)\backslash V(\mathcal{K}_w)| = 1$, because we assumed that $|N(v)\backslash V(\mathcal{K}_w)| \geq 1$, and if $|N(v)\backslash V(\mathcal{K}_w)| \geq 2$, then $\{u\}$ is an FPDS. Since we can make the same argument using w instead of u, we know that v has exactly two neighbors: $u' \in V(\mathcal{K}_u)$ and $w' \in V(\mathcal{K}_w)$. The set $S' = \{v\}$ is a PDS by assumption. Then $\mathcal{P}^0(S') = \{u', v, w'\}$. $G[V\backslash\{v\}]$ consists of two components, $\mathcal{K}_u$ and $\mathcal{K}_w$, so $\{u'\}$ is a zero forcing set of $\mathcal{K}_u$ and $\{w'\}$ is a zero forcing set of $\mathcal{K}_w$. Thus, $\mathcal{K}_u$ is a path with end vertex u', and $\mathcal{K}_w$ is a path with end vertex w'. It follows that $G = P_n$. $\qquad\square$

5 Values of $\bar{\gamma}_p(G)$ for special graphs

In this section, we determine the value of $\bar{\gamma}_p(G)$ for some specific graph families.

Theorem 5.1. *The failed power domination number of the complete bipartite graph $K_{m,n}$ with $m \geq n \geq 1$ is given by*

$$\bar{\gamma}_p(K_{m,n}) = \begin{cases} m - 2 & \text{if } m \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $m = 1$, $G = K_2$, clearly resulting in $\bar{\gamma}_p(G) = 0$. If $m = n = 2$, $K_{m,n} = C_4$, so $\bar{\gamma}_p(K_{m,n}) = 0$ by Theorem 4.4, item 2. If $n = 1$ but $m \geq 2$, then $\bar{\gamma}_p(K_{m,n}) = m - 2$ by Theorem 4.2, item 3.

Assume $m \geq n \geq 2$ and let $S \subseteq V_1$ with $|S| = m - 2$. Let u, v be the vertices in $V_1 \setminus S$. Then $\mathcal{P}^0(S) = S \cup V_2$, and $V \setminus \mathcal{P}^0(S) = \{u, v\}$. Since $N(u) = N(v) = V_2$, $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$, S is stalled, and $\bar{\gamma}_p(G) \geq m - 2$. Note that if S consists of vertices in both V_1 and V_2, then $\mathcal{P}^0(S) = V$, so if S is an FPDS, $S \subseteq V_1$ or $S \subseteq V_2$. Then $|S| \leq m - 2$ because if $|S| > m - 2$, then $\mathcal{P}^1(S) = V$. Hence $\bar{\gamma}_p(G) = m - 2$. $\qquad\square$
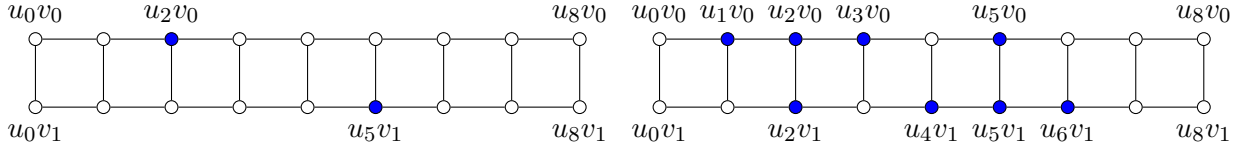

Figure 6: A ladder graph, $P_9 \square P_2$ with FPDS S in blue on the left and $\mathcal{P}^0(S)$ in blue on the right.

For graphs G and H, we denote by $G \square H$ the *Cartesian product* of G and H, where $V(G \square H) = V(G) \times V(H)$. If $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$, then (u_1, v_1) is adjacent to (u_2, v_2) in $G \square H$ if and only if $u_1 = u_2$ in G and $\{v_1, v_2\} \in E(H)$, or $\{u_1, u_2\} \in E(G)$ and $v_1 = v_2$ in H. Here, we write $u_i v_j$ for vertex (u_i, v_j) for brevity.

A *ladder graph* is the graph $P_n \square P_2$ for $n \geq 2$. Each copy of P_2 is called a *rung*.

Theorem 5.2. *For the ladder graph $G = P_k \square P_2$ with $k \geq 4$, $\bar{\gamma}_p(G) = \left\lceil \frac{k-4}{3} \right\rceil$.*

Proof. Let the vertices of P_k be denoted by u_i, $0 \leq i \leq k - 1$, and the vertices of P_2 by v_j, $j = 0$ or 1. Define $S \subseteq V(P_k \square P_2)$ by $u_i v_j \in S$ if and only if $i \equiv 2 \mod 3$ with $2 \leq i \leq k - 3$ and $j \equiv i \mod 2$, as in Figure 6. Then $|S| = \left\lceil \frac{k-4}{3} \right\rceil$. We show that S is an FPDS. Suppose that $u_i v_j \in \mathcal{P}^0(S)$. If $u_i v_j \in S$, then $N(u_i v_j) \subseteq \mathcal{P}^0(S)$. Otherwise, if $u_i v_j \in \mathcal{P}^0(S) \setminus S$, then $u_i v_j$ has exactly two neighbors in $V \setminus \mathcal{P}^0(S)$, namely $u_{i-1} v_j$ or $u_{i+1} v_j$, and $u_i v_{j'}$ where $j' \equiv (j + 1) \mod 2$. Thus $\mathcal{P}^\infty(S) = \mathcal{P}^0(S)$; that is, S is an SPDS, giving us $\bar{\gamma}_p(G) \geq \left\lceil \frac{k-4}{3} \right\rceil$.

To show that $\bar{\gamma}_p(G) \leq \left\lceil \frac{k-4}{3} \right\rceil$, note that if $\{u_0 v_0, u_0 v_1\} \subseteq Z$, then Z is a zero forcing set, and similarly for $\{u_{k-1} v_0, u_{k-1} v_1\}$. If $u_0 v_j \in S$, note that $\{u_0 v_0, u_0 v_1\} \subseteq \mathcal{P}^0(S)$, which implies that $\mathcal{P}^0(S)$ is a zero forcing set, and S is a PDS, and similarly for the case that $u_{k-1} v_j \in S$. Further, if $u_1 v_j \in S$, then $\{u_0 v_j, u_1 v_j\} \subseteq \mathcal{P}^0(S)$, and $\{u_0 v_0, u_0 v_1, u_1 v_j\} \subseteq \mathcal{P}^1(S)$, meaning that $\mathcal{P}^1(S)$ is a zero forcing set, and S is a PDS, and similarly for $u_{k-2} v_j \in S$. Thus, if S is an FPDS, $u_i v_j \notin S$ for $i \in \{0, 1, k-2, k-1\}$ and $j \in \{0, 1\}$. That is, no vertices from the first two or last two rungs of the ladder are in any FPDS.

Also, if $\{u_i v_0, u_i v_1, u_{i+1} v_0, u_{i+1} v_1\} \subseteq Z$ for any $i \leq k - 2$ (that is, if all vertices from two consecutive rungs are in Z), then Z is a zero forcing set. Thus, if S is an FPDS with $u_i v_j \in S$, then $u_{i-1} v_j, u_{i-1} v_{j'}, u_{i+1} v_j, u_{i+1} v_{j'} \notin S$, and further, $u_{i-2} v_{j'}, u_{i+2} v_{j'} \notin S$ for $j' \equiv j \mod 2$. Suppose

that $u_iv_j, u_{i+2}v_j \in S$ for $2 \leq i \leq k-5$ and $j = 1$ or 2. Then $\{u_iv_j, u_iv_{j'}, u_{i+1}v_j, u_{i+2}v_j, u_{i+2}v_{j'}\} \subseteq \mathcal{P}^0(S)$ where $j' = (j+1) \mod 2$. Since $u_{i+1}v_{j'}$ is the only neighbor of $u_{i+1}v_j$ outside of $\mathcal{P}^0(S)$, $u_{i+1}v_j \in \mathcal{P}^1(S)$, giving us that $\{u_iv_j, u_iv_{j'}, u_{i+1}v_j, u_{i+1}v_{j'}, u_{i+2}v_j, u_{i+2}v_{j'}\} \subseteq \mathcal{P}^1(S)$. This forms a zero forcing set of G; hence, S is a PDS. Thus, if S is an FPDS with $u_iv_j, u_{i'}v_{j'} \in S$, then $|i-i'| \geq 3$. That is, $\bar{\gamma}_p(G) \leq \lceil \frac{k-4}{3} \rceil$. $\qquad\square$

Theorem 5.3. *For the graph $G = K_k \square P_\ell$ with $k, \ell \geq 3$, $\bar{\gamma}_p(G) = (k-2)\lfloor \frac{\ell-1}{2} \rfloor$.*

Proof. Let the vertices of K_k be denoted w_i, $0 \leq i \leq k-1$, and the vertices of P_ℓ denoted x_i, $0 \leq i \leq \ell - 1$. Define $S \subseteq V(K_k \square P_\ell)$ by $w_ix_j \in S$ if and only if $i \leq k-3$ and j is odd with $j < \ell - 1$. Then $\mathcal{P}^0(S) = V(G \square H) \setminus (\{w_ix_j | i \geq k-2 \text{ and } j \text{ is even}\} \cup \{w_ix_{l-1}| \text{ for any } i \text{ if } l \text{ is even}\})$. If $w_ix_j \in S$ with j even, then w_ix_j is adjacent to $w_{k-1}x_j$ and w_kx_j; if $w_ix_j \in S$ with j odd and $i < k-2$, then $N(w_ix_j) \subseteq S$; if $w_ix_j \in S$ with j odd and $i \geq k-2$, then w_i, x_j is adjacent to vertices w_ix_{j-1} and w_ix_{j+1}, both of which are not in $\mathcal{P}^0(S)$. Hence $\mathcal{P}^1(S) = \mathcal{P}^0(S)$, and S is an FPDS with $|S| = (k-2)\lfloor \frac{\ell-1}{2} \rfloor$, giving us $\bar{\gamma}_p(G) \geq (k-2)\lfloor \frac{\ell-1}{2} \rfloor$.

Now, any set S' with a vertex w_ix_0 or $w_ix_\ell \in S'$ for any i is a PDS. Further, if $w_ix_j \in S'$ and $w_ix_{j+1} \in S'$ for any j, then S' is a PDS. Hence if S' is an FPDS with $|S'| > (k-2)\lfloor \frac{\ell-1}{2} \rfloor$, then there exists some t with $1 \leq t \leq \ell - 2$ and $w_ix_t \in S'$ for all i except at most one. If $w_ix_t \in S'$ for all i, then S' is a PDS, thus we must have $w_ix_t \notin S$ for some i, say $i = k-1$. Then $\mathcal{P}^0(S')$ includes w_ix_t for all i as well as w_ix_{t+1} and w_ix_{t-1} for all $i < k-1$. If $w_{k-1}x_{t-1}$ or $w_{k-1}x_{t+1} \in \mathcal{P}^m(S')$ for any m, then S' is a PDS. Hence $2 \leq t \leq \ell - 3$, and for every i, none of the following vertices are in S': $w_ix_{t-2}, w_ix_{t-1}, w_ix_{t+1}$, or w_ix_{t+2}. This gives us $|S'| \leq (k-1)\lceil \frac{\ell-4}{3} \rceil$. Note that $l \geq 5$ since $l < 5$ implies S' is a PDS. Since $k \geq 3$, $|S'| < (k-2)\lfloor \frac{\ell-1}{2} \rfloor$. Hence $\bar{\gamma}_p(G) = (k-2)\lfloor \frac{\ell-1}{2} \rfloor$. $\qquad\square$

6 Future work

While we were able to produce a list of graphs that have $\bar{\gamma}_p(G) = 0$ (where every single vertex is itself a PDS), a complete description of all such graphs is still open. The zero forcing number of trees has been related to other parameters such as the path cover number [2], and a technique for determining the zero forcing number of a graph with a cut-vertex was also described [21]. Achieving similar results for the failed power domination number of a graph is a feasible problem. Many parameters in zero forcing, especially related to minimum rank, are investigated for their adherence to a property known as the *Graph Complement Conjecture* which states that the sum of the parameter on G and on the complement graph $\overline{G}$ is bounded by $|V(G)|$ plus a small constant. For minimum rank, $\mathrm{mr}(G)$, the conjecture is: $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) = |V(G)| + 2$. Originally mentioned at an American Institute for Mathematics workshop [3], it formally appeared in [7]. It is natural, and likely challenging, to investigate whether there is any such relationship among power dominating numbers or failed power dominating numbers of graphs and their complements.

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