On Some Generalized Vertex Folkman Numbers

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Abstract

For a graph G and integers $a_i \geq 2$, the expression $G \rightarrow (a_1, \dots, a_r)^v$ means that for any r-coloring of the vertices of G there exists a monochromatic a_i-clique in G for some color $i \in \{1, \cdots, r\}$. The vertex Folkman numbers are defined as $F_v(a_1, \dots, a_r; H) = \min\{|V(G)| : G \text{ is } H\text{-free and } G \rightarrow (a_1, \dots, a_r)^v\}$, where H is a graph. Such vertex Folkman numbers have been extensively studied for $H = K_s$ with $s > \max\{a_i\}_{1 \leq i \leq r}$. If $a_i = a$ for all i, then we use notation $F_v(a^r; H) = F_v(a_1, \dots, a_r; H)$.

Let J_k be the complete graph K_k missing one edge, i.e. $J_k = K_k - e$. In this work we focus on vertex Folkman numbers with $H = J_k$, in particular for $k = 4$ and $a_i \leq 3$. We prove that $F_v(3^r; J_4)$ is well defined for any

$r \geq 2$. The simplest but already intriguing case is that of $F_v(3,3;J_4)$, for which we establish the upper bound of 135. We obtain the exact values and bounds for a few other small cases of $F_v(a_1,\ldots,a_r;J_4)$ when $a_i \leq 3$ for all $1 \leq i \leq r$, including $F_v(2,3;J_4) = 14$, $F_v(2^4;J_4) = 15$, and $22 \leq F_v(2^5;J_4) \leq 25$. Note that $F_v(2^r;J_4)$ is the smallest number of vertices in any J_4-free graph with chromatic number $r+1$.

Keywords: Folkman number, vertex coloring
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1 Introduction

1.1 Notation and Background

All graphs considered in this paper are finite undirected simple graphs. The order of the largest independent set in graph G will be denoted by $\alpha(G)$, and the chromatic number of G by $\chi(G)$. Let J_k be the complete graph K_k missing one edge, i.e., $J_k = K_k - e$. Note that J_3 is the path on 3 vertices, and the diamond graph J_4 is formed by two triangles sharing an edge.

For a graph G and integers $a_1,\ldots,a_r$, such that $a_i \geq 2$ for $1 \leq i \leq r$, the expression $G \to (a_1,\ldots,a_r)^v$ means that for any r-coloring of the vertices of G there exists a monochromatic a_i-clique in G for some color $i \in \{1,\cdots,r\}$. In this paper, we call this property vertex-arrowing, or simply arrowing. It should be noted that an analogous edge-arrowing property $G \to (a_1,\ldots,a_r)^e$ is the basis of widely studied Ramsey edge-arrowing problems. In particular, the Ramsey number $R(a_1,\ldots,a_r)$ is defined as the smallest integer n such that $K_n \to (a_1,\ldots,a_r)^e$. All our results address vertex-arrowing, but in some places we will refer to edge-arrowing for comparison, context, or when used as a tool. The vertex Folkman numbers F_v are defined as

$$F_v(a_1,\ldots,a_r;H) = \min\{|V(G)| : G \text{ is } H\text{−free and } G \to (a_1,\ldots,a_r)^v\},$$

where H is a graph. If $a_i = a$ for all i, then we use a more compact notation, $F_v(a^r;H) = F_v(a_1,\ldots,a_r;H)$. The set of all H-free graphs satisfying the arrowing $G \to (a_1,\ldots,a_r)^v$ will be denoted by $\mathcal{F}_v(a_1,\ldots,a_r;H)$ and we will call it the set of Folkman graphs for the corresponding parameters. Further, $\mathcal{F}_v(a_1,\ldots,a_r;H;n)$ will denote a subset of the latter when restricted to graphs on n vertices.

Let us set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$. We will often use the following lemma by Nenov [18] stating a simple necessary condition for vertex arrowing to hold.

Lemma 1. *(Nenov 1980 [18]) If $G \to (a_1,\ldots,a_r)^v$, then $\chi(G) \geq m$.*

Observe that if $a_i = 2$ for all i, $1 \leq i \leq r$, then the converse is also true. In the terms of Folkman numbers, this means that $F_v(2^r;H)$ is equal to the smallest number of vertices in any H-free graph G with $\chi(G) = r + 1 = m$.

The vertex Folkman numbers have been extensively studied when the avoided graph is complete, i.e., when $H = K_s$ (see [10, 17, 15, 7, 28, 19, 20]). They are well defined when $s > \max\{a_i \mid 1 \le i \le r\}$, since it is known that for such s the minimum in the definition ranges over a nonempty set of graphs. The situation is easy for $s \ge m$. Moreover, much is known about vertex Folkman numbers and the corresponding Folkman graphs when s is close to, but less than, m. However, even some of the basic questions become difficult for small s, such as $s = 3$ or $s = 4$. One of the famous problems which can be stated in these terms is the task of finding the smallest triangle-free graph with given chromatic number r, which is equal to $F_v(2^{r-1}; K_3)$. See the following subsection for references and more details about this problem in relation to our current work. A recent Ph.D. thesis by Bikov [1] presents a variety of Folkman problems, focusing on a computational approach together with the known values and bounds for Folkman numbers.

For graphs F and G, the Ramsey number $R(F, G)$ is the smallest integer n such that if the edges of K_n are 2-colored, say red and blue, then necessarily this coloring includes a copy of red-colored F or a copy of blue-colored G. If F and G are complete graphs, then we write $R(s, t)$ instead of $R(K_s, K_t)$. A regularly updated survey *Small Ramsey Numbers* [26] contains the known bounds and values for a variety of Ramsey numbers.

In this work we focus on the vertex Folkman numbers for graphs avoiding $H = J_k$, in particular for $k = 4$ and $2 \le a_i \le 3$. Note that this special case of J_4-free graphs admits some triangles, but not too many, since in J_4-free graphs each edge can belong to at most one triangle. Note also that avoiding J_4 falls in-between the two extensively studied classical cases of avoiding K_3 and K_4.

Instantiating previous comments for $H = J_4$, we see that $F_v(2^r; J_4)$ is the smallest number of vertices in any J_4-free graph with chromatic number $r + 1$. These numbers are clearly well defined since any K_3-free graph is also J_4-free, and $F_v(2^r; K_3)$ is well defined for every $r \ge 1$. However, while the classical results for multicolor Folkman numbers by Nešetřil and Rödl [21, 22] guarantee the existence of $F_v(3^r; K_s)$ for $s \ge 4$ and of $F_v(3, 3; J_4)$, they do not so for $F_v(3^r; J_4)$ with $r > 2$.

1.2 Summary of New Results

This subsection summarizes our new results. The first result in our Theorem 2 is theoretical, the results in the following theorems are computational.

Theorem 2. *$F_v(3^r; J_4)$ is well defined for all $r \ge 1$.*

The J_4-free graphs satisfying the required arrowing property in Theorem 2 quickly become very large as r grows. The simplest, but already intriguing case, is that of $F_v(3, 3; J_4)$, for which we establish the upper bound of 135 in Theorem 7. Note that, by monotonicity, Theorem 2 implies the existence of Folkman numbers of the form $F_v(a_1, \dots, a_r; J_4)$ with $a_i \le 3$ for all $1 \le i \le r$.

A J_4-free graph G is called *maximal J_4-free* if the addition of any edge creates a J_4 in G. A graph G for which $G \to (a_1, \dots, a_r)$ is called *minimal* if after the deletion of any edge this arrowing does not hold. If G is maximal and minimal, it is referred to as *bicritical*. Using computational methods, we obtain the exact values and bounds for several small cases, as stated in Theorems 3–7 below.

Theorem 3. *$F_v(2^3; J_4) = 9$, and there are exactly 3 graphs in $\mathcal{F}_v(2^3; J_4; 9)$, of which 1 is maximal and 1 is minimal.*

Theorem 4. *$F_v(2^4; J_4) = 15$, and there are exactly 5 graphs in $\mathcal{F}_v(2^4; J_4; 15)$, of which 1 is maximal and 2 are minimal.*

Theorem 5. *$22 \le F_v(2^5; J_4) \le 25$.*

Theorem 6. *$F_v(2, 3; J_4) = 14$ and there are exactly 212 graphs in $\mathcal{F}_v(2, 3; J_4; 14)$, of which 24 are maximal, 26 are minimal, and 1 is bicritical.*

Theorem 7. *$F_v(3, 3; J_4) \le 135$.*

For the context and comparison with the cases involving K_3 and K_4 instead of J_4, we collect the values and bounds from Theorems 3–5 in Table 1. Observe that since $K_3 \subset J_4 \subset K_4$, we must have $F_v(2^r; K_3) \ge F_v(2^r; J_4) \ge F_v(2^r; K_4)$, for each r.

r	K_3	ref.	J_4	K_4	ref.
2	5	C_5	3	3	K_3
3	11	[4]	**9**	6	W_6
4	22	[11]	**15**	11	[17]
5	32–40	[9]	**22 − 25**	16	[14]

Table 1: Known values and bounds for $F_v(2^r; H)$, for $r \le 5$ and $H \in \{K_3, J_4, K_4\}$. The bold entries in the J_4 column were obtained in this work. For easy entries we give the upper bound witness graph. The unique witness for $F_v(2^3; K_4) = 6$ is the wheel graph $W_6 = K_1 + C_5 = K_6 - C_5$.

The following sections contain the proof of Theorem 2 and the description of computations leading to Theorems 3–7. The closing section states some open problems and it contains a few remarks for parameters beyond those studied in this paper. The witness graphs for Theorems 3–7, as well as the code implementing algorithm **A** in Section 3.1 are available at `https://www.cs.rochester.edu/~dnarvaez/folkmanj4/`.

We found a few discrepancies between our results and those claimed in the paper [12]. We investigated all such differences, and we arrived to the conclusion that the computations reported in [12] were incomplete. The computational results reported in this paper were obtained by two independent implementations which agreed on the final and intermediate claims.

2 The Existence of $F_v(3^r; J_4)$ and $F_e(3, 3; H)$

Two seminal papers by Nešetřil and Rödl [21, 22] lay the foundation for our reasoning in this section: the first one from 1976 implies that the edge Folkman numbers $F_e(3^r; K_4)$ are well defined for all $r \geq 1$, and the second paper from 1981 shows that $F_v(3, 3; J_4)$ is well defined. These, together with a technique developed by Dudek and Rödl in 2008 [6], permit us to give a rather elementary proof that $F_v(3^r; J_4)$ is well defined for all $r \geq 1$.

For graph $G = (V_G, E_G)$, we define the graph $F = DR(G)$ as in the construction by Dudek-Rödl [6], as follows:

$$DR(G) = F = (E_G, E_F),$$

where the set of vertices of F consists of the edges of G, and the edge set E_F contains the edges $\{ef, fg, eg\}$ for every edge-triangle efg in E_G. Note that each pair of edges from triangle $\{e, f, g\} \subseteq E_G$ spans the same three vertices in G, and thus the same three vertices in the corresponding vertex-triangle in F. Such triangles in F will be called *images* of triangles from G, other triangles in F will be called *spurious*. Note that for any edge ef in E_F, the edges e and f in E_G must share one vertex. It is easy to observe (see the proof of Lemma 8(2) below) that two image triangles may share vertices but no edges.

Example. For $G = K_4$, $DR(G)$ has 6 vertices, 12 edges (it is 4-regular), and 8 triangles. These 8 triangles are split into 4 images of triangles from G and 4 spurious triangles.

Lemma 8. *Let G be any K_4-free graph, and let F denote the graph $DR(G)$. Then we have that:*

1. *F has no spurious triangles,*
2. *F is J_4-free, and*
3. *$G \to (3^r)^e$ if and only if $F \to (3^r)^v$, for every $r \geq 1$.*

Proof. First, we will show that the graph $F = DR(G)$ has no spurious triangles. For contradiction, suppose that efg is a spurious triangle in F, where $e = \{A, B\}$ and $f = \{A, C\}$ for some vertices $A, B, C \in V_G$. Since edge g is incident to both e and f, but efg is not an image of a triangle in G, we must have $g = \{A, D\}$ for another vertex $D \in V_G$. This implies that ABC, ABD, ACD are triangles, and thus also BCD, in G. Hence $ABCD$ forms a K_4 in G, contradicting the assumption. This shows part (1).

If F contains J_4 with the vertices $\{e, f, g, h\}$ and formed by two triangles $\{efg\}$ and $\{efh\}$, then we claim that at most one of them is an image triangle. In order to see this, let $e = \{A, B\}$, $f = \{A, C\}$ and note that the unique image triangle in F containing e and f is the one implied by the triangle ABC in G. Hence, at least one of $\{efg\}$ and $\{efh\}$ must be spurious, but by (1) this is impossible in $F = DR(G)$ obtained from a K_4-free graph G. Thus (2) follows.

For (3), consider the natural bijection between all r-vertex-colorings of F and r-edge-colorings of G. This bijection preserves the number of colors used in any edge-triangle in G when mapped to its image triangle in F. Hence, because of (1) and (2), we can conclude (3). $\square$

Using Lemma 8, we can give a simple proof that for every r there exists a J_4-free graph such that in any r-coloring of its vertices there must be a monochromatic triangle, or equivalently, $\mathcal{F}_v(3^r; J_4) \neq \emptyset$.

Proof of Theorem 2. A general result by Nešetřil and Rödl [21] implies that the sets $\mathcal{F}_e(3^r; K_4)$ are nonempty for all $r \geq 1$. This, together with the claim that $\mathcal{F}_v(3^2; J_4) \neq \emptyset$, was also discussed in [29]. For general r, consider any graph $G \in \mathcal{F}_e(3^r; K_4)$. Then by Lemma 8(3), we have that $DR(G) \in \mathcal{F}_v(3^r; J_4)$, and thus the numbers of the form $F_v(3^r; J_4)$ are well defined for all $r \geq 1$. $\square$

Note that, by monotonicity, Theorem 2 implies the existence of Folkman numbers of the form $F_v(a_1, \ldots, a_r; J_4)$ with $1 \leq a_i \leq 3$ for all $1 \leq i \leq r$. The orders of graphs in $\mathcal{F}_e(3^r; K_4)$ and $\mathcal{F}_v(3^r; J_4)$ can be expected to be quite large, even for small r. In the trivial case for $r = 1$ we have $F_e(3; K_4) = F_v(3; J_4) = 3$, but both problems become very difficult already for $r = 2$. For the edge problem, the best known bounds are $21 \leq F_e(3, 3; K_4) \leq 786$ [2, 13], while for the vertex problem we establish the bound $F_v(3, 3; J_4) \leq 135$ in Theorem 7. We are not aware of any reasonable bounds for $r \geq 3$ in either case.

We wish to point to a study of the existence of edge Folkman numbers for some small parameters [29]. While a simple argument easily shows that $F_e(3, 3; J_4)$ does not exist, for other cases with $|V(H)| \leq 5$ one can prove or disprove the existence of $F_e(3, 3; H)$ with some work, leaving only two open cases. Namely, the following is known: the sets $\mathcal{F}_e(3, 3; H)$ are nonempty for all connected graphs H containing K_4, and for some graphs not containing K_4. If H is any connected K_4-free graph on 5-vertices containing K_3, then $\mathcal{F}_e(3, 3; H) = \emptyset$ except for two possible cases: the wheel graph $W_5 = K_1 + C_4$ and its subgraph $\overline{P_2 \cup P_3} \subset W_5$. The latter two cases remain open.

3 Computational Proofs

3.1 Overview and Algorithms

In this section we describe the computations which were performed to obtain the proofs of Theorems 3–7 stated in Section 1.2. First, we give an overview of the algorithms that were used or developed for this work, including some details of their implementation. In the following subsections we summarize the results of our computations. We present graphs establishing the upper bounds in Theorems 3–6, and give counts for several intermediate graph families which were obtained. All graphs involved in the computations were J_4-free. The target sets of graphs had additional constraints consisting of the number of vertices, independence number, chromatic number, and the desired parameters of arrowing, $\{a_1, \ldots, a_r\}$, where $2 \leq a_i \leq 3$ for $1 \leq i \leq r$.

The basis of our software framework consisted of the package `nauty` developed by McKay [16], which includes a powerful graph generator `geng`, tools to remove graph isomorphs, and several other utilities for graph manipulation. In the following, we will list some of the graphs in their g6-format, a compact string representation of graphs in `nauty`. These graphs are also available at `https://www.cs.rochester.edu/~dnarvaez/folkmanj4/`.

The template of our main extension algorithm **A** is presented and commented on below. We also implemented filters for extracting graphs with specified chromatic number, graph which are maximal J_4-free, those which arrow $(2,3)^v$ and $(3,3)^v$, and other utilities. Observe that by Lemma 1 the test for arrowing $(2^r)^v$ is the same as for chromatic number.

The graph families pointed to in Table 2 were obtained by using `geng` with filters for J_4-free graphs and for graphs with given chromatic number. For graph families on 13 or more vertices, we used mainly algorithm **A** together with other utilities, as described in the notes to Table 3.

Our custom filter for graphs with specified chromatic number range was tuned to process large number of graphs with small $\chi(G)$. For a given graph G, first we find all maximal independent sets, and then determine $\chi(G)$ as the minimum number of these independent sets which cover V_G. The custom filter for maximal J_4-free graphs has two modes: a full test detecting graphs for which addition of any edge forms a J_4, and a partial test for graphs being constructed within algorithm **A** which cannot be maximal J_4-free after **A** terminates. The latter permitted to significantly prune the output of **A**, which was then filtered through the full test.

Testing whether $G \to (2,3)^v$ was applied only to graphs with $\chi(G) \geq 4$, since by Lemma 1 this arrowing does not hold if $\chi(G) \leq 3$. This test was done by checking that for every maximal independent set $I \subset V_G$ the set of vertices $(V_G \setminus I)$ does not induce any triangle. The test for $G \to (2,2,3)^v$ was accomplished similarly by checking that the set of vertices $(V_G \setminus (I_1 \cup I_2))$ does not induce any triangle, for every pair of maximal independent sets I_1 and I_2. The test for $G \to (3,3)^v$ was accomplished with a totally distinct approach involving SAT-solvers as described in Section 3.4. This was necessary since for arrowing $(3,3)^v$ we were processing a large number of graphs on 100 to 200 vertices, for which an effective handling of their chromatic number and enumeration of maximal triangle-free sets would be computationally very expensive.

Algorithm $\mathbf{A}(\mathcal{G}, n, q, \chi, \delta)$

Input: $\mathcal{G}$ - a set of J_4-free graphs, each on n-vertices, q - extension degree, χ - target chromatic number, δ - minimum cone size.

Output: $\mathcal{H}$ - a set graphs which are extensions of graphs from $\mathcal{G}$. $H \in \mathcal{H}$ if and only if H is a q-vertex extension of any graph $G \in \mathcal{G}$, q new vertices in H form an independent set, $|V_H| = n + q$, new vertices have degree $\geq \delta$, and such that H is maximal J_4-free and $\chi(H) \geq \chi$.

$\mathcal{H} = \emptyset$
for every graph $G \in \mathcal{G}$ **do**
  Compute and store $\mathcal{C}_G$, the set of feasible cones in G of size at least δ
  Compute and store the values of $\tau(C, D)$, for all $C, D \in \mathcal{C}_G$
  for $k = 3$ to q **do**
    Using known $(k-1)$-tuples, make all k-tuples (k-multisets) of feasible cones $\{C_1, \ldots, C_k\}$ such that $\tau(C_i, C_j)$ is true for all $1 \leq i, j \leq q$. If $k = q$, then for each such q-tuple make graph H from G and $\{C_1, \ldots, C_q\}$. If H is maximal J_4-free, then add H to $\mathcal{H}$.
  end for
end for
Remove isomorphs from $\mathcal{H}$
Remove from $\mathcal{H}$ graphs H with $\chi(H) < \chi$

Extension Algorithm A. The inputs are: a family of graphs $\mathcal{G}$ consisting of n-vertex J_4-free graphs, an integer q, which is the extension degree, the target chromatic number χ, and the minimum degree δ of new vertices. For each $G \in \mathcal{G}$, algorithm $\mathbf{A}$ outputs all maximal J_4-free graphs H with $\chi(H) \geq \chi$ such that they can be obtained from G by adding an independent set $I = \{v_1, \ldots, v_q\}$ and some edges between I and V_G. New vertices have degree at least δ. These output graphs H will be called q-vertex extensions of the input graph G.

The new edges of H are defined by q cones $\{C_1, \ldots, C_q\}$, $C_i \subseteq V_G$, where the set of edges connecting v_i to V_G is $\{\{v_i, u\} \mid u \in C_i\}$. First, we precompute the set of all feasible cones $\mathcal{C}_G$ such that for each $C \in \mathcal{C}_G$ the 1-vertex extension of G using C is J_4-free and $|C| \geq \delta$. We also precompute a binary predicate $\tau(C, D)$ on pairs of feasible cones which is false if $C \cap D = \emptyset$, $C \cap D = \{x\}$ and there is no vertex $y \in (C \setminus D) \cup (D \setminus C)$ connected to x, or $C \cap D$ induces an edge in G. Otherwise, $\tau(C, D)$ is set to true. One can easily see that if $\tau(C, D)$ is false and both cones C and D are used in the extension, then H is not maximal J_4-free. This test significantly prunes the search space. Next, we assemble q-tuples of feasible cones such that each pair of cones used passes the τ-test. Each such q-tuple defines one graph H. Finally, the isomorphic copies of graphs are removed, and the remaining graphs H are tested for $\chi(H)$. □

Clearly, larger values of δ in $\mathbf{A}$ for the minimum cone size produce fewer cones and allow for faster computation. Maximal J_4-free graphs H with $\delta = 1$ are easy to characterize. These graphs have $\chi = 3$ and need not be generated using algorithm **A**. Thus, for most of our computations we set $\delta = 2$, but we also observe that when constructing graph families which are known to be χ-vertex-critical, it is sufficient to set $\delta = \chi - 1$.

The values of the Ramsey numbers of the form $R(J_4, K_q)$ are known for all $q \leq 6$ (cf. [26]). In particular, the values of importance to our computations are $R(J_4, K_4) = 11$, $R(J_4, K_5) = 16$ and $R(J_4, K_6) = 21$. We use these values to determine the value of parameter q for algorithm **A** by applying an observation that any J_4-free graph H must have $\alpha(H) \geq q$, provided $|V_H| = |V_G| + q \geq R(J_4, K_q)$.

Sections 3.2 and 3.3 list several sets of parameters for **A** which were used: $\mathcal{G}$ is taken from the cases reported in Table 2, or equal to $\mathcal{F}_v(2, 3; J_4; 14)$ or $\mathcal{F}_v(2^4; J_4; 15)$, and the ranges of other parameters are $1 \leq q \leq 7$, $4 \leq \chi \leq 6$, and $2 \leq \delta \leq \chi - 1$.

3.2 Enumerations for Small Cases

n	all graphs	J_4-free	$\chi = 2$	J_4-free, $\chi = 3$	J_4-free, $\chi = 4$
6	156	69	34	34	0
7	1044	255	87	167	0
8	12346	1301	302	998	0
9	274668	9297	1118	8175	3
10	12005168	97919	5478	92379	61
11	1018997864	1519456	32302	1484866	2287
12	165091172592	34270158	251134	33888537	130486

Table 2: The number of nonisomorphic graphs G by their type and the number of vertices n, $6 \leq n \leq 12$. The corresponding sets of graphs were obtained by using graph generator `geng` of `nauty` with tests for J_4-free graphs and chromatic number $\chi(G)$.

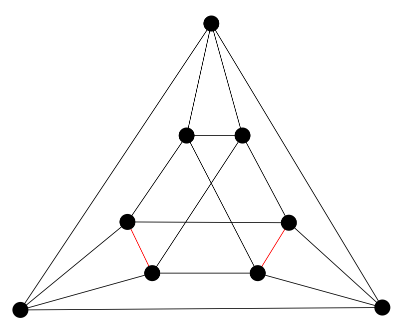

Figure 1: $|\mathcal{F}_v(2, 2, 2; J_4; 9)| = 3$. The 18-edge graph, which is `H{‘Ypgj` in g6-format, is formed by all depicted edges, the other two graphs with 17 and 16 edges are obtained by deleting the edges marked in red, in any order.

The results of our computations for small cases are summarized in Tables 2 and 3. The special graphs, which are witnesses for the exact values in Theorems 3, 4 and 6, are presented in Figures 1–4.

Notes to Table 2

- In each row n, the sum of entries for $\chi = 2, 3, 4$ is one less than the count of J_4-free graphs (because the only missed graph has $\chi = 1$). Note that $\chi(G) = 2$ implies that G is J_4-free.
- The entries 0 and 3 in rows 8 and 9, respectively, of the last column show that $F_v(2, 2, 2; J_4) = 9$ and $|\mathcal{F}_v(2, 2, 2; J_4; 9)| = 3$; the three witnesses have 16, 17 and 18 edges, respectively, and they are presented in Figure 1. This part proves Theorem 3.
- Obtaining the next row of Table 2 (for $n = 13$) by the same approach is doable but only at an extraordinary computational cost. Thus, for $n \geq 13$ we first targeted only maximal J_4-free graphs. The results are reported in Table 3.

type of graphs	$n = 13$	$n = 14$	$n = 15$
maximal J_4-free, $\chi = 2$	5	6	6
maximal J_4-free, $\chi = 3$	15684		
maximal J_4-free, $\chi = 4$	4750	74738	
maximal J_4-free, $\chi = 5$	0	0	1

Table 3: Counts of nonisomorphic maximal J_4-free graphs G by their chromatic number $\chi = \chi(G)$ and number of vertices n, for $13 \leq n \leq 15$. The results for $n \geq 14$ and $\chi \geq 4$ required significant computational resources.

The graph families for $\chi \geq 3$ summarized in Table 3 were constructed using algorithm **A**. The following lemma was used to determine the initial family of graphs $\mathcal{G}$. More details of how each entry with $\chi \geq 3$ was computed are listed in the notes to Table 3.

Lemma 9. *Let G be any graph with $\chi(G) \geq k$, and let $I \subseteq V(G)$ be any independent set in G. Then for $G' = G[V(G) \setminus I]$ we have $\chi(G') \geq k - 1$.*

Proof. Assume there exists an $I \subseteq V(G)$ such that the graph G' induced in G on $V(G) \setminus I$ has $\chi(G') \leq k - 2$. If $V(G')$ is colored with $k - 2$ colors, then all vertices in I can be colored with the same $(k - 1)$-st color, not used in the coloring of $V(G')$. This implies that $\chi(G) \leq k - 1$, which is a contradiction. $\square$

Notes to Table 3

- The row for $\chi = 2$ shows the number of complete bipartite graphs $K_{s,t}$ with $s + t = n$ and $s, t \geq 2$.

- *Graphs with* $n = 13, \chi \geq 3$. Using the fact that $R(J_4, K_4) = 11$, we can see that any J_4-free graph G of order at least 11 must have $\alpha(G) \geq 4$. The graphs with $n = 13$ were obtained by **A** in three different ways: via 3-, 2- and 1-vertex extensions of graphs with 10, 11 and 12 vertices, respectively. When $3 \leq \chi \leq 4$, we use $\delta = 2$. When $\chi = 5$, we set $\delta = \chi - 1 = 4$, since the target graphs are known to be χ-vertex-critical. This is because Lemma 9, $R(J_4, K_4) = 11$ and Table 2 imply that there is no J_4-free graph with $n = 12$ and $\chi = 5$.

- *Graphs with* $n = 14, \chi \geq 4$. These graphs were obtained by computing 4-vertex extensions of the graphs with $n = 10$ and $\chi \geq 3$ using $\delta = 2$. We set $\delta = 4$ when generating graphs with $\chi = 5$ since no graphs with $\chi = 5$ were found on 13 vertices.

- *Graphs with* $n = 15, \chi \geq 5$. The unique maximal graph with $n = 15$ and $\chi = 5$ was obtained by performing 3-vertex extensions of all graphs with $n = 12$ and $\chi \geq 4$, and independently by 4-vertex extensions of all graphs with $n = 11$ and $\chi \geq 4$. Since no graphs with $\chi = 5$ were found on 14 vertices, we set $\delta = 4$.

- Empty entries correspond to graphs whose full enumeration was not attempted. These would be difficult to obtain, and they are not relevant for this work. Still, many such graphs were obtained as side result of other computations.

- The entry requiring most CPU time (about one CPU-week if run on a single processor) was that for $\chi = 4, n = 14$. It was obtained by applying algorithm **A** to make J_4-maximal 4-vertex extensions of the 97918 graphs with $n = 10$ and $\chi \geq 2$ (though using $\chi \geq 3$ would suffice). Among the resulting 74738 graphs G, there are 24 of them for which $G \rightarrow (2,3)^v$. No graph reported in column 13 satisfies this arrowing (though, by Lemma 1, it would suffice to test only 4750 graphs with $\chi = 4$), and thus $F_v(2, 3; J_4) = 14$.

- The complete set $\mathcal{F}_v(2, 3; J_4; 14)$ was obtained from the above 24 maximal J_4-free graphs by repeatedly deleting the edges until they did not satisfy the arrowing. This set consists of 212 nonisomorphic graphs, with the number of edges ranging from 31 to 39, the number of triangles from 8 to 10, and the orders of their automorphism groups ranging from 1 to 8. 26 of these graphs are minimal. There exists a unique (up to isomorphisms) bicritical graph, namely the graph G_{14} shown in Figure 2, for which addition of any edge creates a J_4, and deletion of any edge e yields $G_{14} - e \not\rightarrow (2,3)^v$. This part proves Theorem 6.

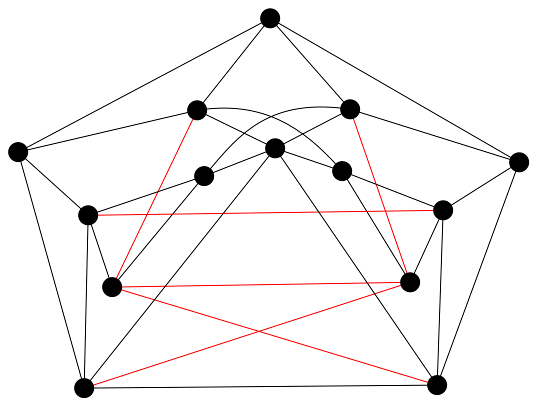

Figure 2: Unique bicritical graph $G_{14} \in \mathcal{F}_v(2,3;J_4;14)$. Edges marked in red do not belong to any triangle in G_{14}. The graph G_{14}, which is `M?K_iqg‘QDqQXBpw?` in g6-format, has 33 edges, 9 triangles and just one non-trivial symmetry (left-right swap of the figure).

- The second most-expensive-to-obtain entry was that for $n = 15, \chi = 5$. The entries in the last row of Table 3 prove that $F_v(2^4;J_4) = 15$. The unique maximal graph in $\mathcal{F}_v(2^4;J_4;15)$ was obtained in two ways: as a 4-extension and 3-extension of all graphs with $\chi = 4$ on 11 and 12 vertices, respectively.

- The complete set $\mathcal{F}_v(2^4;J_4;15)$ consists of 5 graphs with 45, 44, 43, 43 and 42 edges, respectively. Four of these graphs (except one on 43 edges) form a chain presented in Figure 3. The graph on 45 edges is formed by all edges, three red edges can be removed (in any order) to give its subgraphs which are also in $\mathcal{F}_v(2^4;J_4;15)$. The fifth graph is a subgraph of one on 44 edges. This part proves Theorem 4.

- Another view of the graphs in $\mathcal{F}_v(2^4;J_4;15)$ is presented in Figure 4. The 9-vertex grid on the right has 6 independent sets of 3 vertices. The vertices of triangles ABC and 123, on the left, are connected to the grid as indicated by the labels. The red edges can be dropped (in any order), yielding graphs with 44, 43, 42 edges, also in $\mathcal{F}_v(2^4;J_4;15)$. In this view we can easily see all 72 symmetries of the minimal graph.

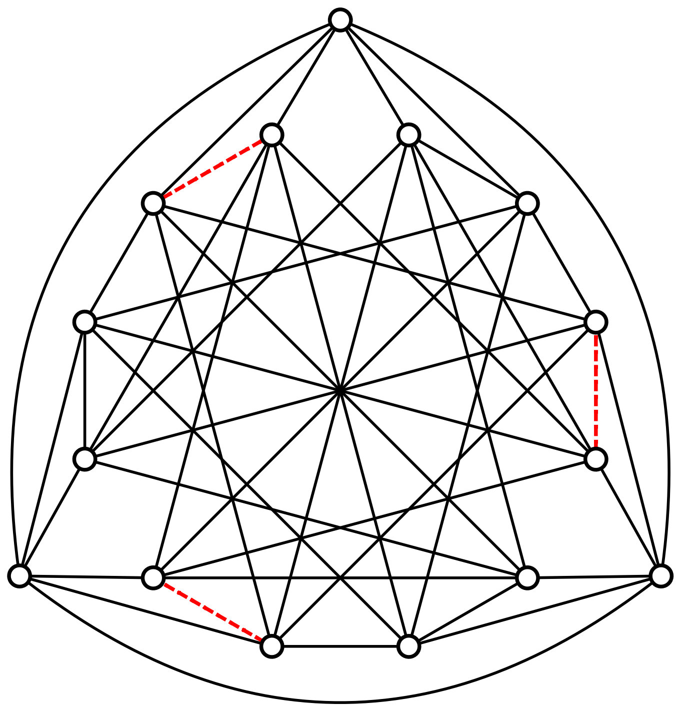

Figure 3: A view of graphs in $\mathcal{F}_v(2^4; J_4; 15)$. The maximal graph on 45 edges, which is `N{eCIhJSaWEfIuKqDeG` in g6-gormat, is formed by all depicted edges. It is 6-regular. Three vertices of the outer triangle form one orbit, 12 other vertices form the second orbit (the center of the figure is not a vertex). Three edges marked in red can be removed, in any order, to give its subgraphs in $\mathcal{F}_v(2^4; J_4; 15)$. The 5-th graph is a subgraph of one with 44 edges. The minimal graph on 42 edges (formed by the black edges) has 72 automorphisms, more than the other four graphs.

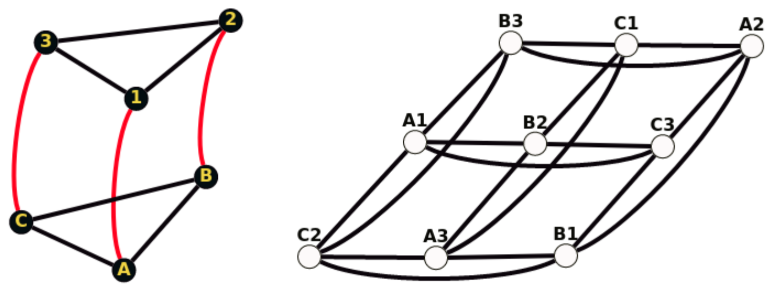

Figure 4: Set view of $\mathcal{F}_v(2^4; J_4; 15)$. The 9-vertex 3×3 grid on the right has 6 triangles and 6 independent sets of 3 vertices. It is a self-complementary graph. The vertices {A,B,C} and {1,2,3} on the left are connected to the grid by 18 edges as indicated by the labels. Equivalently, 9 vertices of the grid on the right connect to pairs of vertices (one from each of two triangles) on the left. The red edges can be dropped (in any order) yielding graphs with 44, 43 and 42 edges, respectively. This figure describes 4 graphs isomorphic to those in Figure 3, but presenting them in a very different way. The vertices of the middle row of the grid form the triangle corresponding to the outer triangle in Figure 3.

3.3 Bounds for $F_v(2^5; J_4)$ and $F_v(2, 2, 3; J_4)$

Easy bounds on the order of the smallest 6-chromatic J_4-free graph, $F_v(2^5; J_4)$, are implied in prior work by others on avoiding K_3 and K_4 (see Table 1 in Section 1.2), namely:

$$16 = F_v(2^5; K_4) \leq F_v(2^5; J_4) \leq F_v(2^5; K_3) \leq 40.$$

We obtain much better bounds stated in Theorem 5:

$$22 \leq F_v(2^5; J_4) \leq 25$$

These bounds were computed as follows. Take S to be the Schläfli graph on 27 vertices [27]: S is a strongly regular graph of degree 16. Its complement is J_4-free and it has $\chi(\overline{S}) = 6$. Removing from $\overline{S}$ any two adjacent vertices with all incident edges yields a 25-vertex witness to $F_v(2^5; J_4) \leq 25$ (the g6-format of this graph is `XIPA@CQA_KEBIIHKBHGicBXB_w}auURYbDu_maULkdQTseOfpp?`). Removing any further vertices reduces the chromatic number below 6.

For the lower bound, since $R(J_4, K_6) = 21$, any 21-vertex J_4-free graph must have an independent set of 6 vertices. Thus, any graph $G \in \mathcal{F}_v(2^5; J_4; 21)$ can be obtained by adding a 6-independent set (with 6 cones) to one of the 5 graphs in $\mathcal{F}_v(2^4; J_4; 15)$. This was verified with algorithm **A** and no suitable graph was found with $\chi \geq 6$. Thus $F_v(2^5; J_4) \geq 22$. This part proves Theorem 5.

The bounds we have for $F_v(2, 2, 3; J_4)$ are rather weak, namely

$$F_v(2, 3; J_4) + 6 = 20 \leq F_v(2, 2, 3; J_4) \leq F_v(3, 3; J_4) \leq 135.$$

The upper bound follows from Theorem 7 and by an easy observation that for any graph G, if $G \to (3,3)^v$, then $G \to (2,2,3)^v$. For the lower bound, suppose that $G \in \mathcal{F}_v(2,2,3;J_4;k)$. Note that by Lemma 1, we must have $\chi(G) \geq 5$. For $k = 19$, since $R(J_4, K_5) = 16$, we have $\alpha(G) \geq 5$, and thus G is a 5-vertex extension of at least one of the 212 graphs in $\mathcal{F}_v(2,3;J_4;14)$. Using again algorithm **A** we have found no suitable graph G, and hence $k > 19$. We attempted to use **A** and other ad-hoc methods to construct a witness G for $k \geq 20$, but all such searches failed. The complement of the Schläfli graph $\overline{S}$ does not arrow $(2,2,3)^v$. We also tested 8933 J_4-free graphs G on 20 vertices with $\chi(G) = \alpha(G) = 5$ [25] and found that none of them arrows $(2,2,3)^v$. For comparison with the cases for K_4-free graphs, we note that it is not hard to check that $F_v(2,3;K_4) = 7$ with the unique witness graph $K_7 - C_7$ (cf. Theorem 3 in [15]), and it is known that $F_v(2,2,3;K_4) = 14$ [5].

Finding any non-obvious bounds for $F_v(2,3,3;J_4)$ or $F_v(3,3,3;J_4)$ is an interesting challenge which we pose as a problem to work on.

3.4 The J_4-free process and $F_v(3,3;J_4)$

The triangle-free process begins with an empty graph of order n, and iteratively adds edges chosen uniformly at random, subject to the constraint that no triangle is formed. The triangle-free process has been used to prove that

$$R(3,t) \geq \left(\frac{1}{4} + o(1)\right) \frac{t^2}{\log t},$$

which currently is the best known lower bound for $R(3,t)$ obtained by Bohman and Keevash in 2013/2019 [3] and independently by Fiz Pontiveros, Griffiths and Morris in 2013/2020 [8].

Similarly to the triangle-free process, the J_4-free process begins with an empty graph of order n, and iteratively adds edges chosen uniformly at random, subject to the constraint that no J_4 is formed. The asymptotic properties of this process were analyzed in [23]. We implemented the J_4-free process in C++ and generated several graphs for which we then checked the arrowing property. The check was done by turning the arrowing property into a Boolean formula and then using Boolean satisfiability (SAT) solvers on the resulting formula. The formula is computed as follows: for every triple of vertices (v_1, v_2, v_3), if they form a triangle, we output the disjunctions

$$(\overline{v_1} \vee \overline{v_2} \vee \overline{v_3}) \wedge (v_1 \vee v_2 \vee v_3)$$

A satisfying assignment for this subformula will assign at least one of the vertices in $\{v_1, v_2, v_3\}$ to the value False and at least one of them to the value True. Taking False and True to be colors, it is clear that for a graph G the formula

$$\bigwedge_{\substack{(v_1,v_2,v_3) \in V_G \\ \text{s.t. } G[\{v_1,v_2,v_3\}] \sim K_3}} (\overline{v_1} \vee \overline{v_2} \vee \overline{v_3}) \wedge (v_1 \vee v_2 \vee v_3)$$

is satisfiable if and only if there is a way to assign colors to the vertices of G that avoids monochromatic triangles. We are thus searching for J_4-free graphs G that yield unsatisfiable instances, as these witness the bound $F_v(3,3;J_4) \leq |V_G|$. The smallest such graph we were able to find has 135 vertices, thus establishing that

$$F_v(3,3;J_4) \leq 135.$$

This part proves Theorem 7.

It is easy to see that $G \to (3,3)^v$ implies $K_1 + G \to (3,3)^e$, where the graph K_1+G is obtained from G by adding one new vertex connected to all of V_G. By applying this implication we can also see that $F_e(3,3;K_1+H) \leq F_v(3,3;H)+1$. The latter inequality is tight for $H = K_4$, as it was shown that $F_e(3,3;K_5) = 15$ and $F_v(3,3;K_4) = 14$ [24]. Now, using similar steps for $H = J_4$, by Theorem 7 we obtain $F_e(3,3;J_5) \leq 136$. We observe that by the monotonicity with respect to the avoided graph H we have $F_e(3,3;K_5) \leq F_e(3,3;J_5) \leq F_e(3,3;K_4)$.

4 Open Problems and Remarks

We close this paper by posing some related open problems.

Problem 1. *Give a general lower bound for $F_v(3^r;J_4)$, or any nonobvious lower bound for $F_v(3,3;J_4)$, which are not easily implied by known bounds for other more studied parameters.*

Similarly, we do not know much about the cases like $F_v(2^r;K_4)$, $F_v(3^r;K_4)$, or $F_v(4^r;J_5)$: no general methods are known to obtain good lower or upper bounds.

Problem 2. *Does there exist a J_4-free graph G such that every set of $|V_G|/2$ vertices induces a triangle?*

If not, this could help in the analysis of $F_v(3,3;J_4)$. We may consider this problem in a more general case. By using density arguments, it is known how to obtain upper bounds on $F_v(3^r;K_4)$, but not on $F_v(3,3;J_4)$ or $F_v(3^r;J_4)$.

Except for $F_e(3,4;K_5) \geq 22$ [30], we do not know of any other nonobvious bounds for: (a) $F_v(3,4;J_5)$, (b) $F_v(4,4;J_5)$, (c) $F_e(3,4;K_5)$, or (d) $F_e(4,4;J_5)$. Case (a) might be solvable with computational methods. We also think that case (b), while far from easy, is easier than (c) and much easier than (d). For a similar case of $F_v(4,4;K_5)$, the best known bounds were obtained in [1, 28]:

$$19 \leq F_v(2,2,2,4;K_5) \leq F_v(4,4;K_5) \leq 23.$$

Finally, we state a related existence problem, which was already raised in an earlier work [29], and which is also described at the end of Section 2.

Problem 3. (a) $\mathcal{F}_e(3,3;K_1+C_4)=\emptyset$? (b) $\mathcal{F}_e(3,3;\overline{P_2\cup P_3})=\emptyset$?

We note that the YES answer to part (a) implies YES answer to part (b), which eliminates one YES/NO combination of answers (out of four possible ones). This problem can be rephrased in some interesting ways. For example, part (a) is equivalent to the following question: *Does there exist any W_5-free graph which is not a union of two triangle-free graphs?*

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