Generation, Verification, and Attacks on Elliptic Curves and their Applications in Signal Protocol

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Outline

1. Elliptic curves
2. Special types of elliptic curves
3. Arithmetic of EC, ECDLP
4. Usage in cryptography
5. What makes a secure curve?
6. Signal protocol
7. X3DH

Goal of the thesis

- G1: Generate curves based on the security specifications.
- G2: Verify the security of the generated curves.
- G3: Demonstrate attacks on weak curves.
- G4: Test the usability of the generated curves in protocols.
- G5: Can they be Quantum safe?

Weierstrass elliptic curves

- $a, b \in \mathbb{Z}_p$ such that $4a^3 + 27b^2 \neq 0$
- non-singular EC is a set E of solutions $(x, y) \in \mathbb{Z}_p$ to equation
$$y^2 = x^3 + ax + b \text{ (mod p)},$$
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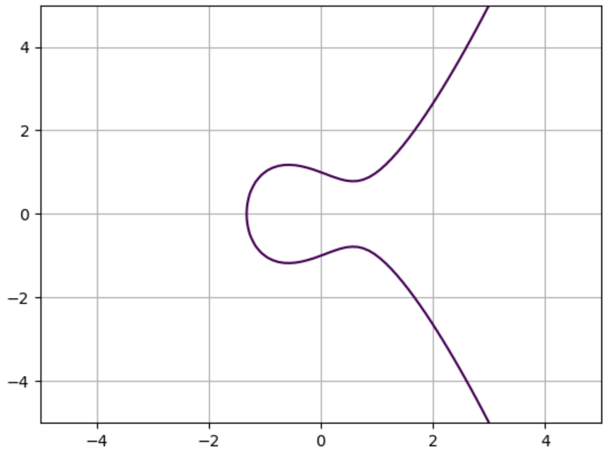
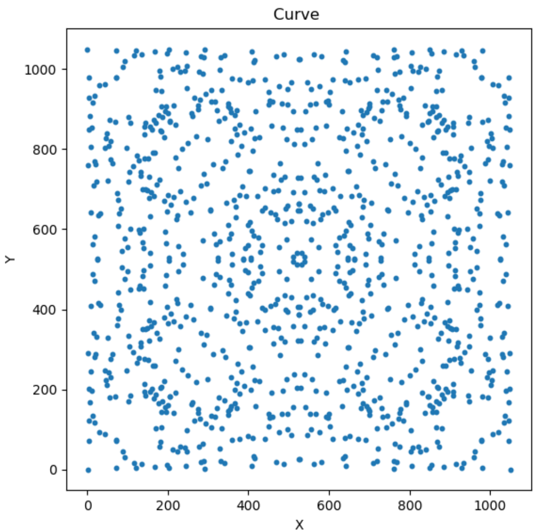
EC over reals:



Special types of elliptic curves

Edwards curve,

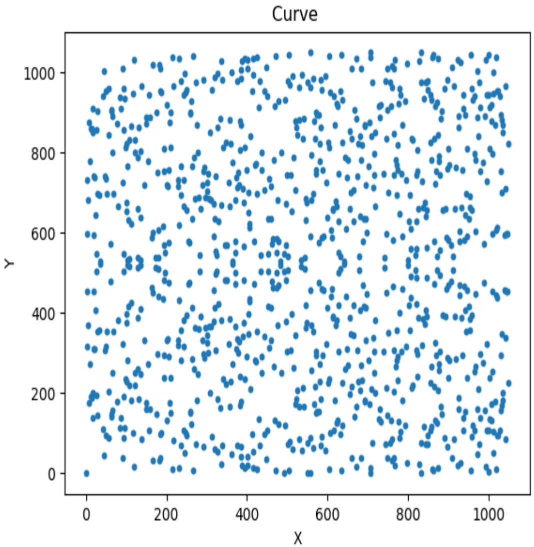
$x^2 + y^2 = 1 + dx^2y^2$



$x^2 + y^2 = 1 + 2x^2y^2 \pmod{1051}$

Montgomery curve,

$By^2 = x^3 + Ax^2 + x$



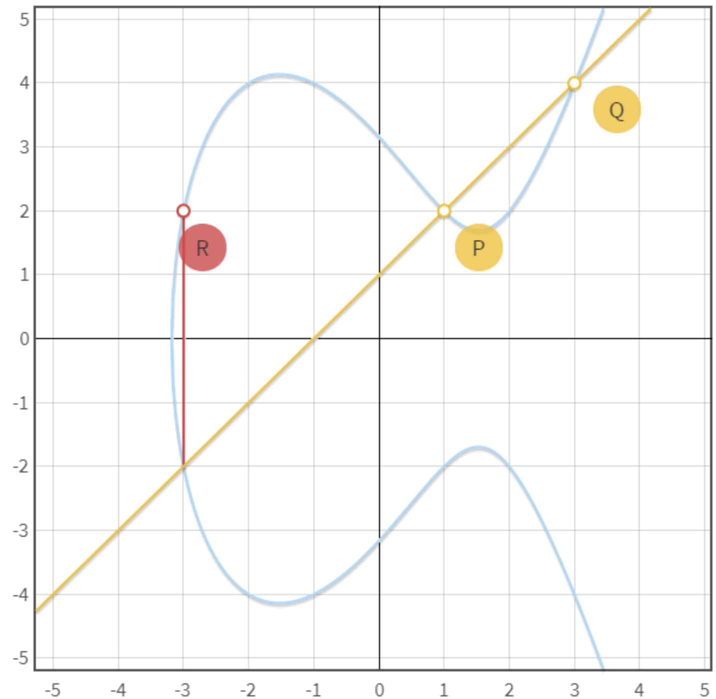
$y^2 = x^3 + 10x^2 + x \pmod{1051}$

Point addition

$P = (x_1, y_1)$
$Q = (x_2, y_2)$, where $x_1 \neq x_2$

$P + Q = R = (x_3, y_3)$,

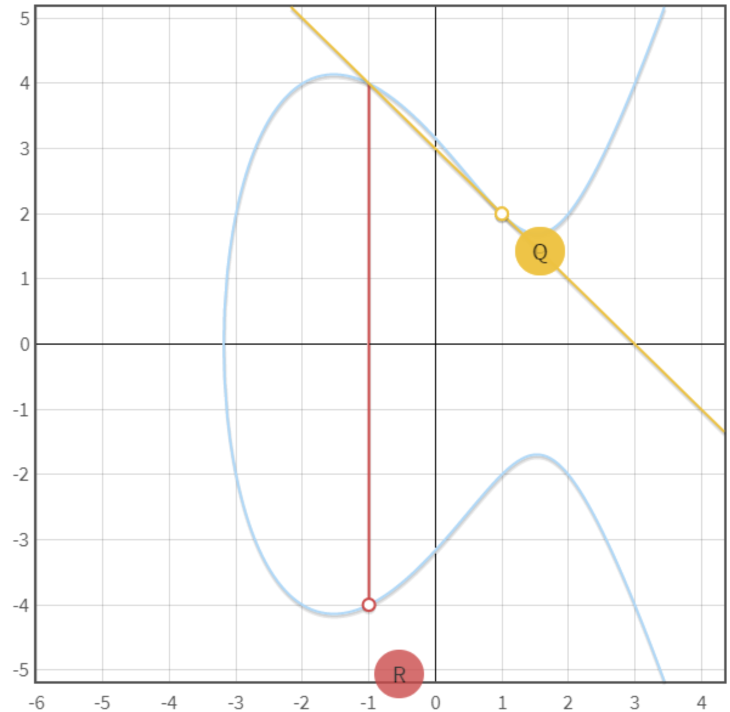
$x_3 = \lambda^2 - x_1 - x_2$,
$y_3 = \lambda(x_1 - x_3) - y_1$,
$where\ \lambda = (y_2 - y_1)(x_2 - x_1)^{-1}$.



Point doubling

$$Q + Q = 2Q = R = (x_3, y_3),$$

$$x_3 = \lambda^2 - x_1 - x_2,$$
$$y_3 = \lambda(x_1 - x_3) - y_1,$$
$$where\ \lambda = (3x_1^2 + a)(2y_1)^{-1}.$$



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ECDLP is a good candidate for one-way function.

- State of art DL solver: up to 750 bits.
- State of art ECDL solver: up to 110 bits.

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- Provide smaller keys compared to `RSA`.
- Key exchange: Elliptic Curve Diffie Hellman (ECDH)
- Digital signatures algorithms:
  Elliptic Curve Digital Signature Algorithm (ECDSA)
  Edwards Curve Digital Signature (EDDSA)

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(G1,G2,G3,G5)

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  Point compression and fast base field arithmetic.

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- Implementation dependent security.

  Resistance against side channel attacks.

  Microsoft Curveball vulnerability.

Signal curves

(G1)

`Signal` protocol is an end-to-end encryption protocol used in

- `WhatsApp`
- `Facebook Messenger`
- `Signal Messenger`

Two elliptic curves are used in `Signal`:

- Curve25519, Montgomery curve, Dan Bernstein, 2005
  128 bit security.

- Curve448, Edwards curve, Mike Hamburg, 2015
  224 bit security.

`Signal` uses extended triple Diffie-Hellman.

Diffie-Hellman Key Exchange (G3,G4)

DH

Alice
choose $Pr_A = a \in \{2, 3, \ldots, p-2\}$

Bob
choose $Pr_B = b \in \{2, 3, \ldots, p-2\}$

$$\xrightarrow{Pub_A = g^a \pmod{p} = A}$$

$$\xleftarrow{Pub_B = g^b \pmod{p} = B}$$

computes $B^a \pmod{p}$

computes $A^b \pmod{p}$

Shared secret $B^a = (g^b)^a = g^{ab} = A^b$

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ECDH

Alice
choose $Pr_A = a \in \{2, 3, ..., \#E-1\}$

Bob
choose $Pr_B = b \in \{2, 3, ..., \#E-1\}$

$$\xrightarrow{Pub_A = aP = A = (x_A, y_A)}$$

$$\xleftarrow{Pub_B = bP = B = (x_B, y_B)}$$

compute $aB = T_{AB}$

compute $bA = T_{AB}$

Shared secret $T_{AB} = (x_{AB}, y_{AB})$

Curve25519

Equation of the curve is : $y^2 = x^3 + 486662x^2 + x$
Specified parameters as per RFC 7748:

Name	**Definition**
$field$	$2^{255} - 19$
$order$	2^{252} + 0x14def9dea2f79cd65812631a5cf5d3ed
$cofactor$	8

Base point $< P >$
(x(P), y(P)) = (9,14781619447589544791020593568409986887264606134616475288964881837755586237401).

Curve448

The equation of the curve is : $x^2 + y^2 = 1 - 39081x^2y^2$
Specified parameters as per RFC 7748:

Name	**Definition**
$field$	$2^{448} - 2^{224} - 1$
$order$	2^{446} - 0x8335dc163bb124b65129c96fde933d8d723a70aadc873d6d54a7bb0d
$cofactor$	4

Base point $< P >$
(x(P), y(P)) = (224580040295924300187604334099896036246789641632
564134246125461686950415467406032909029192869357953282578032075146446173674602635247710,
298819210078481492676017930443930673437544040154080242095928241372331506189835876003536878655418784733982303233503462500531545062832660)

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  IK_A - Alice's Identity Key.

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- Ephemeral keys
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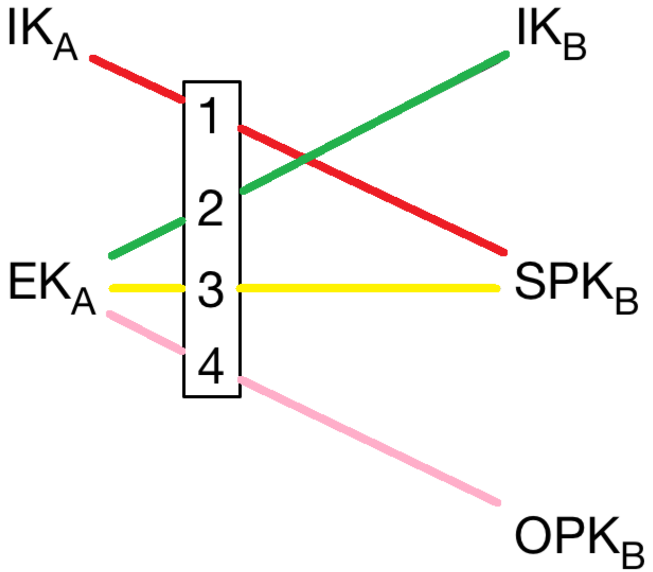
- Ephemeral keys
  EK_A - Alice's Ephemeral Key.
  EK_B - Bob's Ephemeral Key.
- Signed prekeys
  SPK_B - Bob's signed prekey.
  OPK_B - Bob's one time prekey.

X3DH

1. $DH1 = DH(IK_A, SPK_B)$
2. $DH2 = DH(EK_A, IK_B)$
3. $DH3 = DH(EK_A, SPK_B)$
4. If OPK_B exists,
   $DH4 = DH(EK_A, OPK_B)$



Secret Key $SK = KDF(DH1||DH2||DH3||DH4)$

Goals

- G1: Generate curves based on the security specifications. (March, 2020)
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- G4: Test the usability of the generated curves in protocols. (May, 2020)
- G5: Can they be Quantum safe? (May, 2020)

References

- J.-P. Flori, J. Plût, J.-R. Reinhard, and M. Ekerå, "Diversity and transparency for ecc", Cryptology ePrint Archive, no. 659, 2015.
- H. Edwards, A normal form for elliptic curves, Bulletin of the American Mathematical Society, `https://bit.ly/2NQylSy`, 2007.
- D. R. Stinson and M. B. Paterson, *Cryptography: Theory and Practice, Fourth Edition*. 2019.
- D. J. Bernstein, P. Birkner, M. Joye, T. Lange, and C. Peters, Twisted Edwards Curves, Cryptology ePrint Archive, `https://eprint.iacr.org/2008/013`.
- E. M. Barnard, *Tutorial of twisted edwards curves in elliptic curve cryptography*, UC SANTA BARBARA, CS 290G, FALL 2015, `https://koclab.cs.ucsb.edu/teaching/ecc/project/2015Projects/Barnard-Paper.pdf`, 2015.