On Some Generalized Folkman Numbers

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Theory Canal
Henrietta, April 9, 2025

High $\chi(G)$ for H-free graphs G

small dense $H \in \{K_3, K_4 - e, K_4\}$

Let $J_k = K_k - e$.

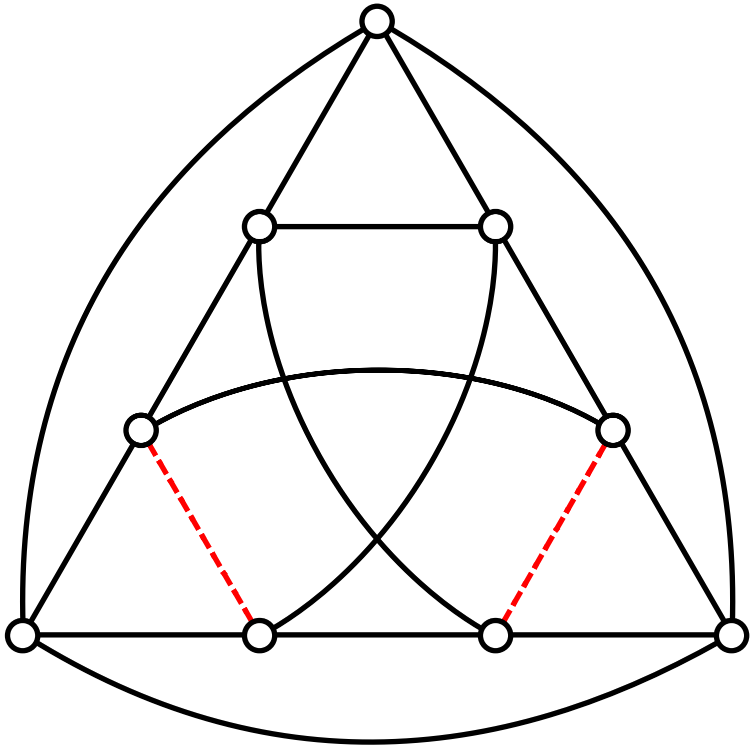

Figure: 4-chromatic J_4-free graphs G, 3 graphs with 16, 17, 18 edges.
$G \in \mathcal{F}_v(2,2,2;K_4 - e) = \mathcal{F}_v(2^3;J_4)$, $|\mathcal{F}_v(2,2,2;J_4;9)| = 3$.

High $\chi(G)$ for a J_4-free graph G

Much done by many authors for K_3-free and K_4-free graphs
We look specially in between at $J_4 = (K_4 - e)$-free graphs.

Previous slide: $G \in \mathcal{F}_v(2^3; J_4; 9)$

- Computations: there is no such graph on less than 9 vertices
- The smallest 4-chromatic J_4-free graph has 9 vertices
- Equivalently, $F_v(2^3; J_4) = 9$

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Folkman graphs and numbers

Def. For graph F and positive integers s, t

- $F \rightarrow (s,t)^v$ iff in every 2-vertex-coloring of $V(F)$ there is a monochromatic K_s in color 1 or K_t in color 2
- variants: coloring edges, more colors

Vertex Folkman graphs and numbers

$\mathcal{F}_v(s,t;k) = \{F \mid F \rightarrow (s,t)^v,\ K_k \not\subseteq F\}$
$F_v(s,t;k) =$ the smallest order of graphs in $\mathcal{F}_v(s,t;k)$

Theorem (Folkman 1970)
If $k > \max(s,t)$, then $F_e(s,t;k)$ and $F_v(s,t;k)$ exist

Folkman Problems

Arrowing and avoiding general graphs

$F_v(H_1, H_2; H) =$ smallest n for which there exists an H-free graph G of order n such that $G \rightarrow (H_1, H_2)^v$

$F_e(H_1, H_2; H) =$ smallest n for which there exists an H-free graph G of order n such that $G \rightarrow (H_1, H_2)^e$

- When H_1, H_2, H are complete graphs this is classics
- Some existence questions remain open
- $F_e(J_4, J_4; K_4) \leq 30193$, Lu 2008

  side effect of an attack on $F_e(3, 3; 4)$
- $F_v(K_3, K_3; J_4) \leq 135$, HJNXR'23, now in VHR'25 down to 45

Computing v/e arrowing

Folkman problems

Three levels of generality/difficulty

- Testing: $F \rightarrow (G, H)^{v/e}$ is Π_2^p-complete,
  in general it is coNP-complete for fixed G and H,
  can be P-time for special fixed G and H.
- Hunting: Use smart constructions to get small F.
  Use insight and heuristics.
- Exhausting: Prove nonexistence of constrained F.
  Can be very difficult.

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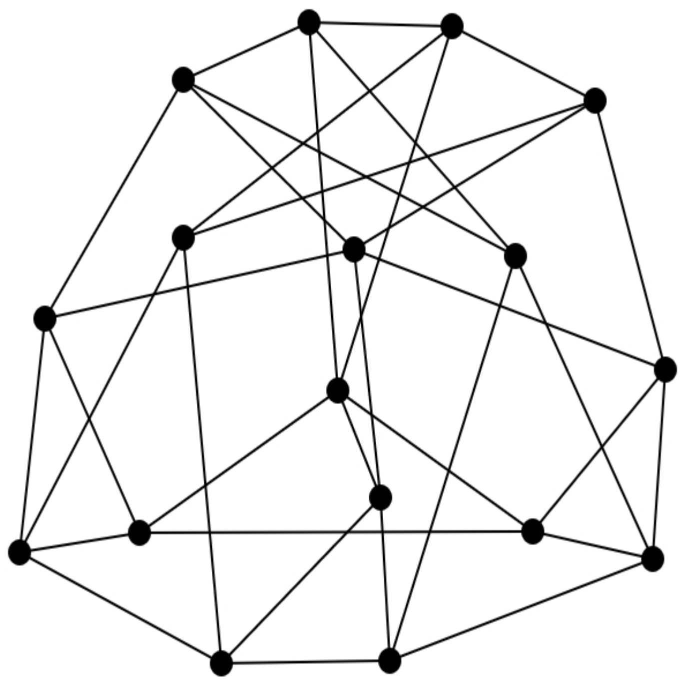
$F_v(2, 3; C_4) = 17$



unique $\mathcal{F}_v(2, 3; C_4; 17)$-graph

House of Graphs (HoG@G) id 51178

Arrowing and chromatic numbers

useful in constructions and computations

Observation

$$G \to (\underbrace{2, \cdots, 2}_{r})^v \iff \chi(G) \geq r+1$$

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$ $M = R(a_1, \cdots, a_r)$

Theorem (Nenov 2001, Lin 1972, others)

If $G \to (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \to (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

A pearl of vertex Folkman numbers

the smallest K_r-free graph G with $\chi(G) = r + 1$

Theorem (ancient folklore, ŁRU 2001)
$F_v(\underbrace{2, \cdots, 2}_{r}; r) = F_v(2^r; r) = r + 5$ for $r \geq 5$

Sketch of the proof

For the upper bound take $G = K_{r-5} + C_5 + C_5$,
$n(G) = r + 5$, $cl(G) = r - 1$, $\chi(G) = r + 1$.

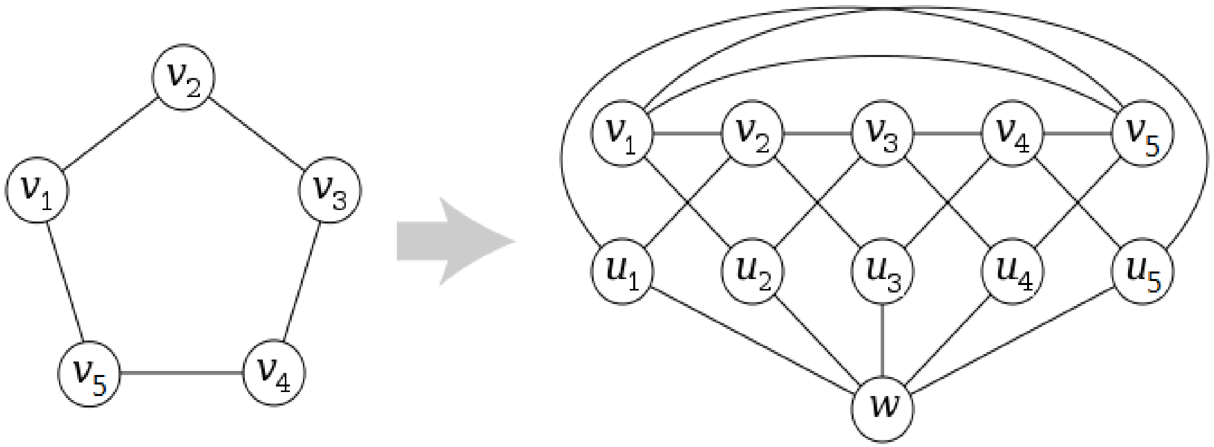
For the lower bound take any K_r–free graph H with $n(H) = r + 4$,
then use maximum matching M in $\overline{H}$ to show that $\chi(H) \leq r$.
It is easy for $|M| \geq 4$. It is trivial for $|M| \leq 2$.
For $|M| = 3$ some work is needed. $\diamond$

Special case of Folkman numbers

chromatic number $\chi(G)$ of K_3-free graphs

For all $r \geq 1$, $F_v(2^r; 3)$ exists and it is equal to the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1}; 3) \leq 2F_v(2^r; 3) + 1$, Mycielski construction, 1955



Special case of Folkman numbers

is just about graph chromatic number $\chi(G)$

$$G \to (\underbrace{2, \cdots, 2}_{r})^v \iff \chi(G) \geq r + 1$$

small cases

$F_v(2^2; 3) = 5$, C_5, Mycielskian, 1955

$F_v(2^3; 3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4; 3) = 22$, Jensen and Royle, 1995

$32 \leq F_v(2^5; 3) \leq 40$, Goedgebeur, 2020

$F_v(2^r; J_4) \leq F_v(2^r; K_3) = F_v(2^r; 3)$.

Problem

Compute $F_v(2^r; J_4)$ for small r.

Existence

old results

Based on a result by Nešetřil and Rödl (1981)

Theorem

For every integer $k \geq 3$,

(a) *the edge Folkman number $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ exists, and*

(b) *the vertex Folkman number $F_v(K_k, K_k; J_{k+1})$ exists.*

Asymptotics. Dudek-Rödl (2010): For any positive integer r there exists a constant $C = C(r)$ such that for every $s \geq 2$ it holds that

$$F_v(s^r; s+1) \leq Cs^2 \log^4 s$$

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Avoiding $J_k = K_k - e$

Problem.
Compute the following:

$F_v(K_3, K_4; J_5)$, perhaps doable

$F_e(K_3, K_3; J_5)$, perhaps hopeless

$F_e(K_3, K_3; K_4)$, hopeless, 50+ years of work by many

(if time permits at the end, there are 3 extra slides)

Existence of $F_v(3^r; J_4)$

also implied by work of Nešetřil-Rödl

Dudek-Rödl construction, 2008

For $G = (V_G, E_G)$, let

$$H = DR(G) = (E_G, E_H),$$

for every edge-triangle efg in G, make $\{ef, fg, eg\} \subset E_H$, $V_H = E_G$.

(edge-triangle $\{e, f, g\} \subset E_G$ becomes a vertex triangle in H)

Theorem

For every integer $r \geq 2$, and any graph G,
$G \in \mathcal{F}_e(3^r; K_4)$ if and only if $DR(G) \in \mathcal{F}_v(3^r; J_4)$.

It is known that $\mathcal{F}_e(3^r; K_4)$ is nonempty for every $r \geq 2$.

Existence

constructive upper bounds

Theorem

$F_v(a_1, a_2, \ldots, a_r; J_4)$ *exist for all* $a_i \in \{2, 3\}$*.*

Even if the conjecture that $G_{127} \to (3,3)^e$ holds, together with the DR-construction, it would imply only a weak bound $F_v(3,3;J_4) \leq 2667$.

Now we got $F_v(3,3;J_4) \leq 135$, then ≤ 43.

Problem

Obtain any reasonable upper bound on $F_v(3,3,3;J_4)$

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Summary of computational results

By monotonicity we have for each r

$$F_v(2^r; K_3) \geq F_v(2^r; J_4) \geq F_v(2^r; K_4)$$

r	K_3	ref.	J_4	K_4	ref.
2	5	C_5	3	3	K_3
3	11	Chv74	**9**	6	W_6
4	22	JeRo95	**15**	11	Nen84
5	32–40	Goe20	**22 – 25**	16	LR11/Nen07

Table: Known bounds for $F_v(2^r; H)$, for $r \leq 5$ and $H \in \{K_3, J_4, K_4\}$. The results of this work are marked in bold. The unique witness for $F_v(2^3; K_4) = 6$ is the wheel graph $W_6 = K_6 - C_5$.

Extension Algorithm A

Input:

$\mathcal{G}$ - set of J_4-free graphs on n-vertices

q - extension degree
χ - target chromatic number
δ - minimum cone size

Output:

$H \in \mathcal{H}$ - a set graphs which are extensions of graphs from $\mathcal{G}$

H is a q-vertex extension of $G \in \mathcal{G}$
q new vertices in H form an independent set, $|V_H| = n + q$
new vertices have degree $\geq \delta$
H is maximal J_4-free
$\chi(H) \geq \chi$

Exhaustive enumeration up to 12 vertices

n	all graphs	J_4-free	$\chi = 2$	J_4-free, $\chi = 3$	J_4-free, $\chi = 4$
6	156	69	34	34	0
7	1044	255	87	167	0
8	12346	1301	302	998	0
9	274668	9297	1118	8175	3
10	12005168	97919	5478	92379	61
11	1018997864	1519456	32302	1484866	2287
12	165091172592	34270158	251134	33888537	130486

Table: The number of nonisomorphic graphs G by their type and the number of vertices n, $6 \leq n \leq 12$. The corresponding sets of graphs were obtained by using graph generator `geng` of `nauty` with tests for J_4-free graphs and chromatic number $\chi(G)$.

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Algorithm A results for >12 vertices

tested also for <13 vertices

type of graphs	$n = 13$	$n = 14$	$n = 15$
maximal J_4-free, $\chi = 2$	5	6	6
maximal J_4-free, $\chi = 3$	15684		
maximal J_4-free, $\chi = 4$	4750	74738	
maximal J_4-free, $\chi = 5$	0	0	1

Table: Counts of nonisomorphic maximal J_4-free graphs G by their chromatic number $\chi = \chi(G)$ and number of vertices n, for $13 \leq n \leq 15$. The results for $n \geq 14$ and $\chi \geq 4$ required significant computational resources.

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$F_v(2, 3; J_4) = 14$

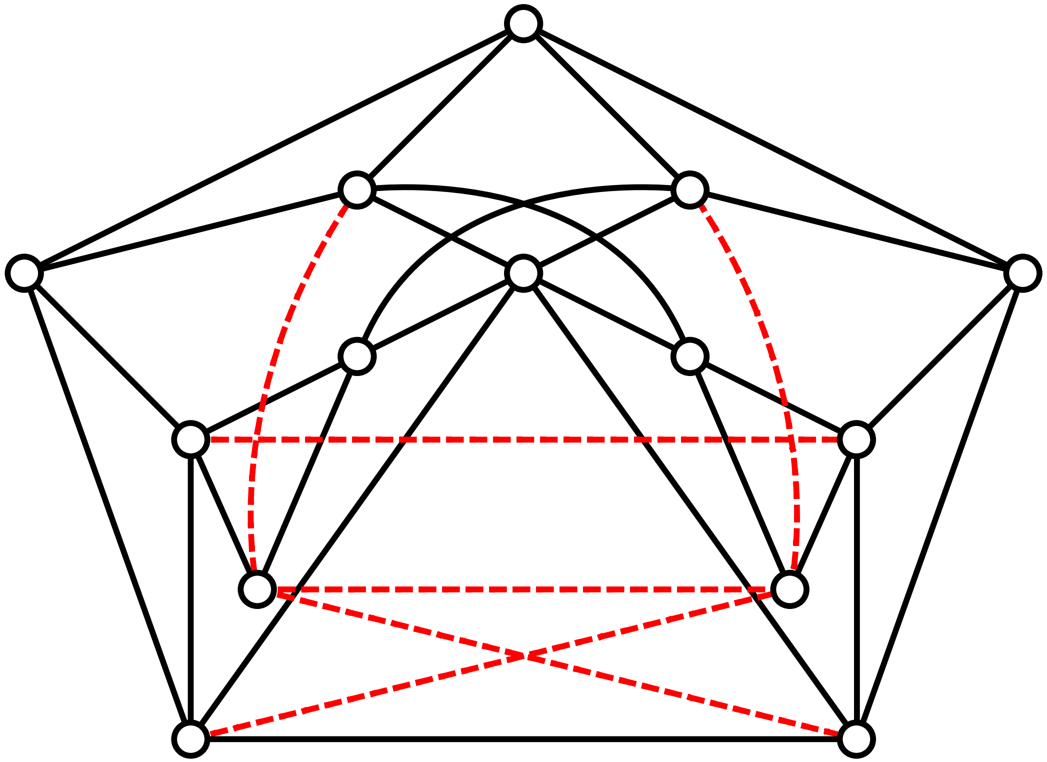

Figure: The unique bicritical graph $G_{14} \in \mathcal{F}_v(2, 3; J_4; 14)$. Edges marked in red do not belong to any triangle in G_{14}. The graph G_{14} has 33 edges, 9 triangles and just one non-trivial symmetry (left-right swap).

$F_v(2^4; J_4) = 15$

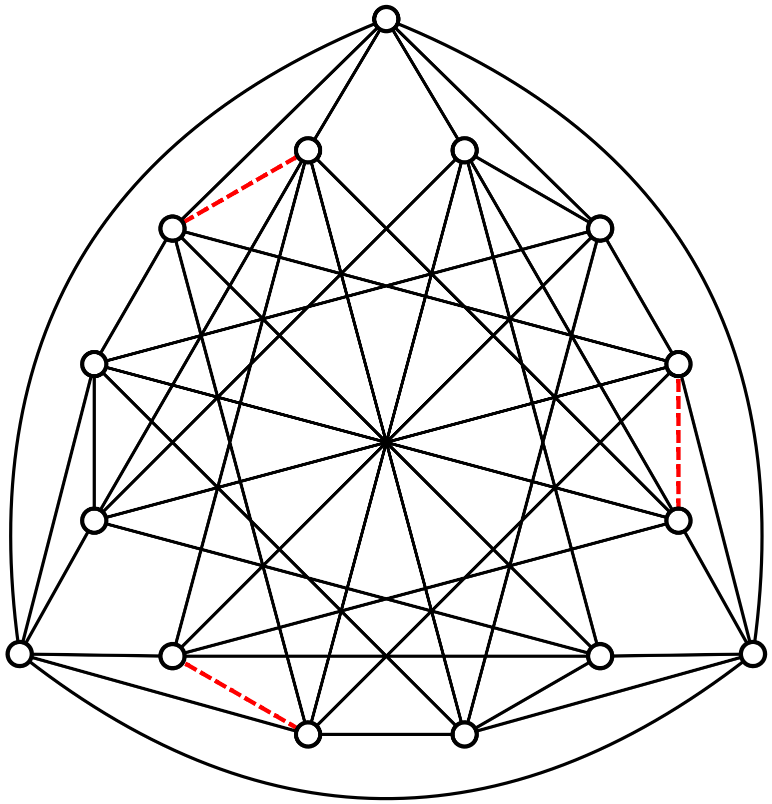

Figure: Graphs in $\mathcal{F}_v(2^4; J_4; 15)$. The maximal graph on 45 edges is 6-regular. Three edges marked in red can be removed, in any order, to give its subgraphs in $\mathcal{F}_v(2^4; J_4; 15)$.

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$F_v(2^4; J_4) = 15$

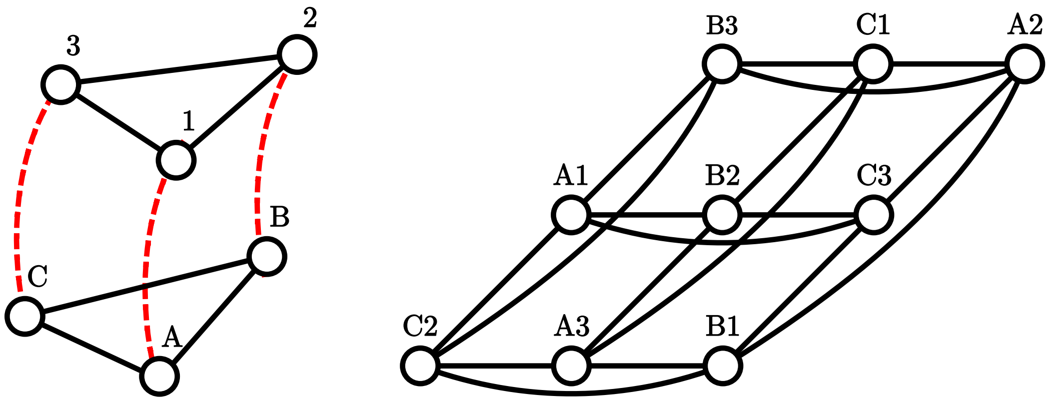
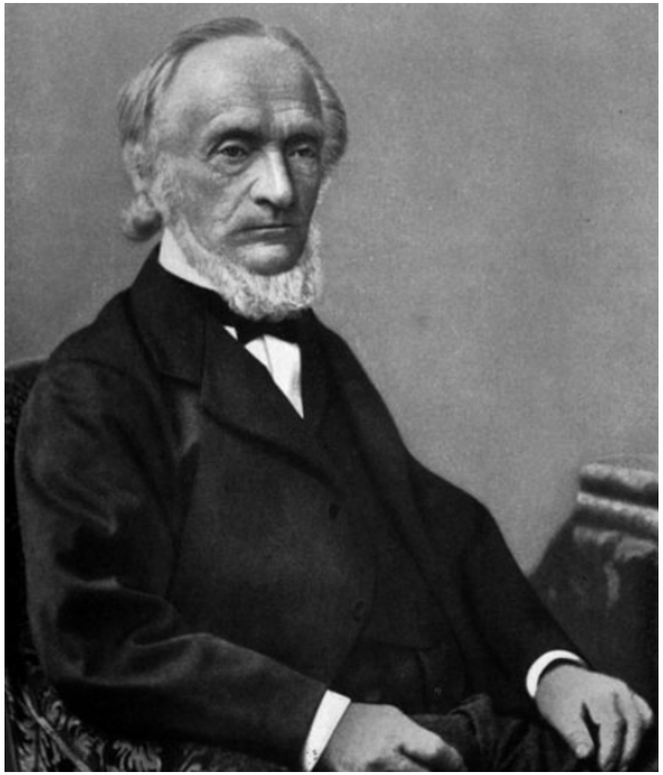


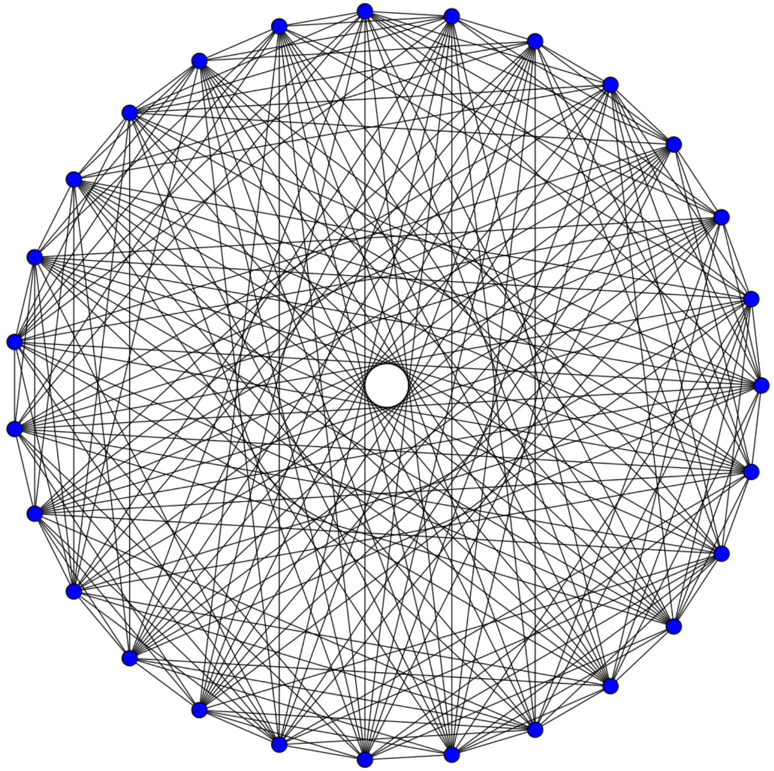
Figure: Set view of $\mathcal{F}_v(2^4; J_4; 15)$. The 9-vertex 3×3 grid on the right has 6 triangles and 6 independent sets of 3 vertices. It is a self-complementary graph. The vertices {A,B,C} and {1,2,3} on the left are connected to the grid by 18 edges as indicated by the labels. Equivalently, 9 vertices of the grid on the right connect to pairs of vertices (one from each of two triangles) on the left. The red edges can be dropped (in any order) yielding graphs with 44, 43 and 42 edges, respectively.

Schläfli $(J_7, J_4; 27)$-graph

building block of $(J_5, J_7; 64)$



[Wikipedia]



Ludwig Schläfli (1814-1895), Bern

$srg(27, 16, 10, 8)$

$51840 = 2^7 3^4 5$ automorphisms

Ramsey numbers $R(G, H)$, one missing edge

$J_n = K_n - e$, Table IIIa from *Small Ramsey Numbers*, SPR 2024

G \ H	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$	$K_{11}-e$
K_3-e	3	5	7	9	11	13	15	17	19
K_3	5	7	11	17	21	25	31	37	42 45
K_4-e	5	10	13	17	28	30 32	36 45	43 57	73
K_4	7	11	19	30	37 49	52 71	62 102	135	170
K_5-e	7	13	22	37	65	66 91	69 136	188	261
K_5	9	16	30 33	43 62	65 102	81 173	121 262	381	511
K_6-e	9	17	37	45 70	66 124	83 206	334	505	757
K_6	11	21	43 53	58 104	205	353	612	944	1346
K_7-e	11	28	65	66 124	247	432	761	1218	1964
K_7	13	28 29	65 82	80 184	370	716	1269	2119	3197

Schläfli graph is $(J_4, J_7; 27)$-good. It is a jewel!

$R(J_5, J_7) = 65$, Goedgebeur-Van Overberghe (2022)

Bounds for $F_v(2^5; J_4)$

6-chromatic J_4-free

Easy bounds from prior results:

$$16 = F_v(2^5; K_4) \leq F_v(2^5; J_4) \leq F_v(2^5; K_3) \leq 40$$

We obtain much better bounds:

$$22 \leq F_v(2^5; J_4) \leq 25$$

- Upper bound 25: drop 2 vertices from the complement of the Schläfli graph, which is a $srg(27, 16, 10, 8)$
- Lower bound 22: implied by our computations

$20 \leq F_v(3,3;J_4) \leq 135$

Turn the arrowing property into a Boolean formula, then use SAT-solvers for J_4-free stream of graphs.

Theorem

$$F_v(3,3;J_4) \leq 135$$

Proof.

Computational. We provide the graph, the Boolean formula encoding its arrowing property, and a proof of unsatisfiability. ■

The lower bound we have seems weak:

$$F_v(2,3;J_4) + 6 = 20 \leq F_v(2,2,3;J_4) \leq F_v(3,3;J_4)$$

Reduction of arrowing K_3 to 3-SAT

upgrade works for J_4

Turn the arrowing property into a Boolean formula, then use SAT-solvers. For vertices (v_1, v_2, v_3), if they form K_3, we output

$$(\overline{v_1} \vee \overline{v_2} \vee \overline{v_3}) \wedge (v_1 \vee v_2 \vee v_3)$$

A satisfying assignment assigns at least one of the vertices in $\{v_1, v_2, v_3\}$ to FALSE and at least one of them to TRUE. Taking FALSE and TRUE to be colors, for a graph G the formula

$$\bigwedge_{\substack{(v_1, v_2, v_3) \in V(G) \\ \text{s.t. } G[\{v_1, v_2, v_3\}] \sim K_3}} (\overline{v_1} \vee \overline{v_2} \vee \overline{v_3}) \wedge (v_1 \vee v_2 \vee v_3)$$

is satisfiable iff there is a way to assign colors to $V(G)$ that avoids monochromatic K_3.

More Folkman Constructions

Van Overberghe-Hassan-SPR, 2024-2025

- `nauty` + `geng` + filters
- polycirculant graphs
- locally linear graphs
- modifying special graphs, use of projective planes

More Folkman Constructions

polycirculant graphs

- polycirculant graphs,
  they generalize circulants with one cyclic orbit of vertices
- all vertex orbits have the same size,
  graph can be partitioned to circulant blocks
- there are efficient exhaustive isomorph-free generation methods
  with assumed automorphism group

More Folkman Constructions

locally linear graphs

- locally linear graph has every edge in exactly one triangle (special maximal J_4-free graphs)
- nice recursive structure implying an efficient exhaustive isomorph-free generation
- Schläfli graph again!

New Constructions

Two colors

- $F_v(2,3;C_4) = 17$, using `geng` with filters
- $23 \leq F_v(3,3;J_4) \leq 45$, locally linear, transitive
- Progress from $F_e(K_3,K_3;J_5) \leq 136$ to $F_e(K_3,K_3;J_5) \leq 43$
  Schläfli graph in work!

Three colors

- $F_v(2,2,3;J_4) \leq 36$
- $F_v(3,3,3;J_5) \leq 46$
- $F_v(2,3,3,3;K_5) \leq 32$
- $F_v(3,3,3,3;K_6) \leq 30$
- other constructions and indirect bounds

All nice graphs live in the **House of Graphs**
(`houseofgraphs.org`)

Existence Question, Ghent 2017

small open edge case

The sets $\mathcal{F}_e(K_3, K_3; S)$ are nonempty for all connected graphs S containing K_4, and for some graphs not containing K_4.

If S is any connected K_4-free graph on 5-vertices containing K_3, then $\mathcal{F}_e(K_3, K_3; S) = \emptyset$ except for two possible cases: wheel $W_5 = K_1 + C_4$ and its subgraph $\overline{P_2 \cup P_3} \subset W_5$.

Problem

(a) $\mathcal{F}_e(K_3, K_3; K_1 + C_4) = \emptyset$?

(b) $\mathcal{F}_e(K_3, K_3; \overline{P_2 \cup P_3}) = \emptyset$?

YES answer to part (a) implies YES answer to part (b).

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New Construction

avoiding W_5

Observation: If G is a C_4-free graph, then $G + v$ is W_5-free.

- Based on graphs from projective planes one obtains $F_v(K_3, K_3; C_4) \leq 63$
- This yields $F_e(K_3, K_3; W_5) \leq 64$
- Half of the Ghent'2017 Problem is solved!

Remains open:

$$\mathcal{F}_e(K_3, K_3; \overline{P_2 \cup P_3}) = \emptyset?$$

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  Graphs and Combinatorics, 2018
- Zohair Hassan, Yu Jiang, David Narváez, Xiaodong Xu, SPR
  On Some Generalized Vertex Folkman Numbers
  Graphs and Combinatorics, 2023
- Steven Van Overberghe, Zohair Raza Hassan, SPR
  Some New Bounds on Folkman Numbers Arrowing K_2 or K_3
  in preparation, 2025

- Many papers by Bikov, Dudek, Erdős, Folkman, Graham, Li, Lin, Lu, Nenov, Nešetřil, Rödl, Ruciński, Soifer, Xu, and others ...
  1970 – ∞

Thanks for listening!

Most Wanted Folkman Number: $F_e(3,3;4)$

and how to earn \$100 from RL Graham

The best known bounds:

$$21 \leq F_e(3,3;4) \leq 786.$$

- Upper bound 786 from a modified residue graph via SDP.
- Ronald Graham Challenge for $\$100$ (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \rightarrow (3,3)^e$, moreover
  - removing 33 vertices from G_{127} gives graph G_{94},
    which still looks good for arrowing, if so, worth \$100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972),
  16 (PRU 1999), 19 (RX 2007), 20, 21 (Bikov-Nenov 2016, 2020).

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Graph G_{127}

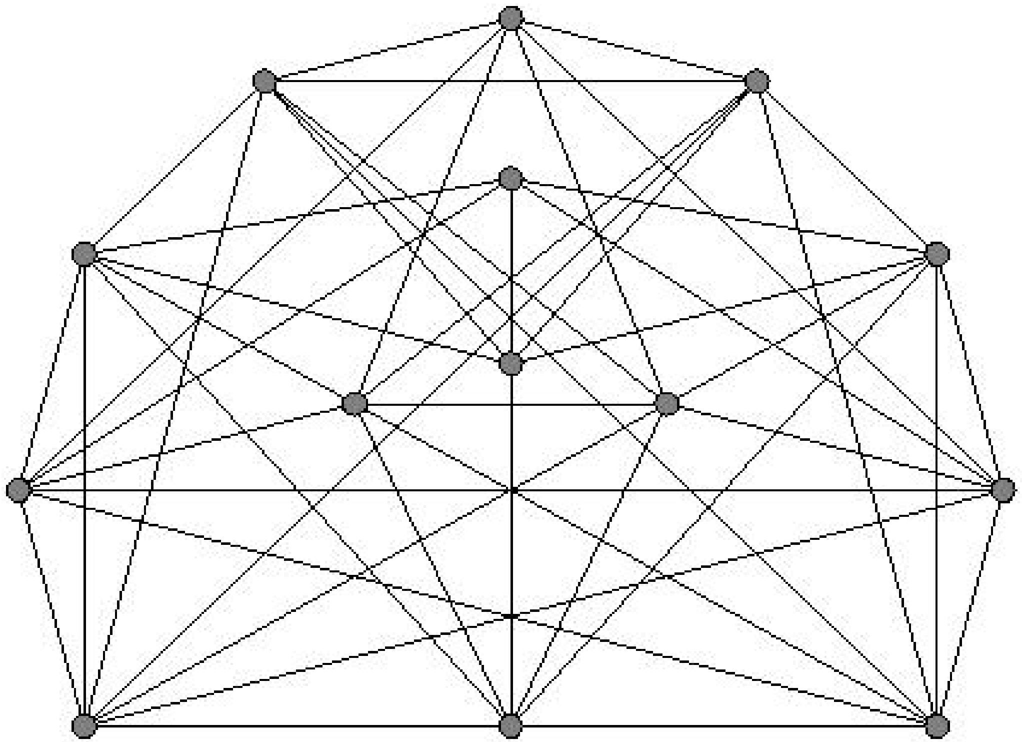
Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \to (3, 3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

$F_e(3,3;5) = 15$, and $F_v(3,3;4) = 14$



unique 14-vertex bicritical $F_v(3,3;4)$-graph G [PRU 1999]
$cl(G) = 3$, $\chi(G) = 5$, $|Aut(G)| = 2$ and $G \to (3,3)^v$, $G + x \to (3,3)^e$

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