Multi-Deletion Reconstruction Numbers of Small Graphs

or: $\{\forall, \exists\}\{v, e\}rn_{k\in\{1,2,3\}}$ for $|V(G)| \leq 9$

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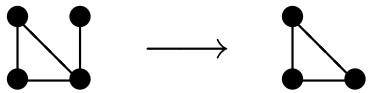
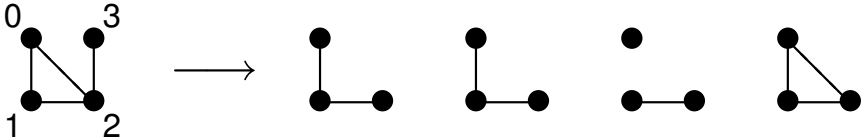
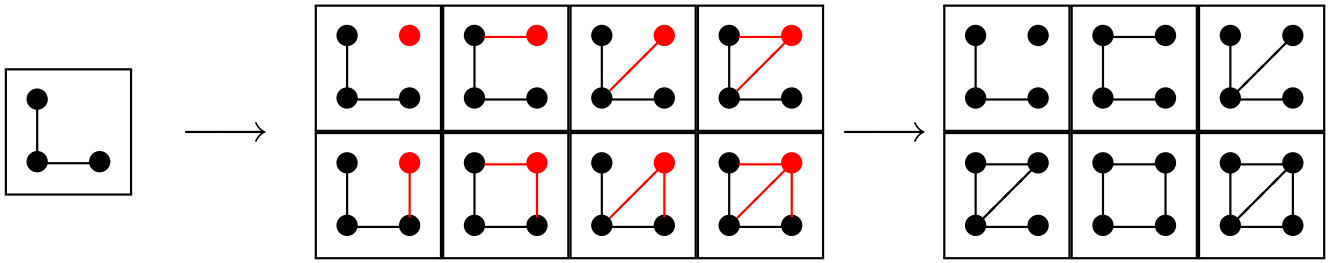
- $Card_1$ of G: subgraph with one vertex removed
  - 
- $Deck_1(G)$: multiset of all $Card$s of G
  - 

- $Extensions_1(G)$: all graphs on |V(G)|+1 vertices with induced G
  - 
- k-vertex-deleted versions: $Card_k$, $Deck_k$, $Extensions_k$
- k-edge-deleted versions: $\mathcal{E}Card_k$, $\mathcal{E}Deck_k$, $\mathcal{E}Extensions_k$

- Any graph on 3 or more vertices can be uniquely identified by its 1-vertex-deleted subgraphs (Kelly, Ulam - 1941)
- Any graph on 4 or more edges can be uniquely identified by its 1-edge-deleted subgraphs (Harary - 1964)
- Reconstruction Numbers: (Harary, Plantholt - 1985)

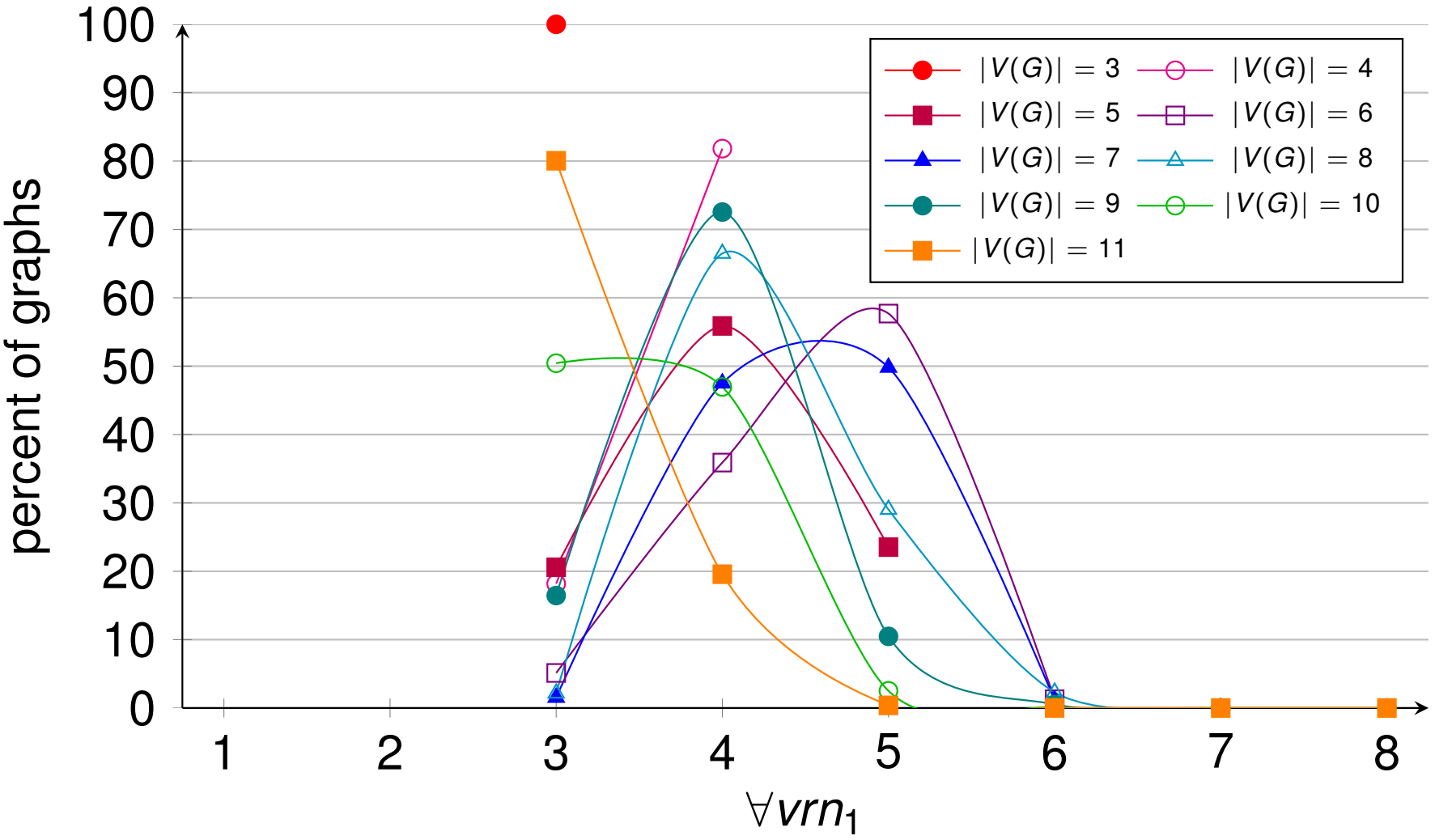
  $\exists rn(G)$ the smallest number of *Card*s required to reconstruct G

  $\forall rn(G)$ the minimum number such that any set of $\forall rn$ *Card*s can reconstruct G

  - Concept extends naturally to edge-reconstruction and k-deletion.

- $\exists vrnG = \exists vrn(\overline{G})$, $\forall vrnG = \forall vrn(\overline{G})$ (Harary, Plantholt - 1985)
- Almost every graph has $\forall vrn_1 = \exists vrn_1 = 3$ (Myrvold - 1988; Bollobás - 1990)
- Almost every graph has $\forall ern_1 = \exists ern_1 = 2$ (Lauri - 1992)
- $\forall k \geq 1$ there exists a graph on $2k$ vertices which is not k-vertex-reconstructible (Nýdl - 1992)
- Families of graphs with $\forall vrn_1 = 2\lfloor \frac{1}{3}(n-1) \rfloor + 1$ (Bowler, Brown, Fenner - 2009)
- Values $\exists vrn_1$, $\forall vrn_1$, $\exists ern_1$, and $\forall ern_1$ computed for all graphs up to 11 vertices (Rivshin, Radziszowski - 2009)

		graph order								
		3	4	5	6	7	8	9	10	11
unique graphs		4	11	34	156	1044	12346	274668	12005168	1018997864
$\exists vrn_1$	3	4	8	34	150	1044	12334	274666	12005156	1018997864
	4		3		4		8		6	
	5				2		2	2	4	
	6						2			
	7								2	
$\forall vrn_1$	3	4	2	7	8	16	266	45186	6054148	815604300
	4		9	19	56	496	8208	199247	5637886	199382868
	5			8	90	520	3584	28781	301530	3922130
	6				2	12	284	1434	10686	83730
	7						4	20	914	4824
	8								4	12
$\exists vrn_1$	min	3	3	3	3	3	3	3	3	3
	max	3	4	3	5	3	6	5	7	3
	mean	3.00	3.27	3.00	3.05	3.00	3.00	3.00	3.00	3.00
	std dev	0.00	0.45	0.00	0.27	0.00	0.05	0.01	0.00	0.00
$\forall vrn_1$	min	3	3	3	3	3	3	3	3	3
	max	3	4	5	6	6	7	7	8	8
	mean	3.00	3.82	4.03	4.55	4.51	4.32	3.95	3.52	3.20
	std dev	0.00	0.39	0.66	0.61	0.55	0.55	0.54	0.55	0.41
% $\exists vrn_1 = 3$		100	73	100	96.2	100.0	99.903	99.9993	99.9999	100.0000
% $\forall vrn_1 = 3$		100	18	21	5.1	1.53	2.155	16.4511	50.4295	80.0398


100
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percent of graphs
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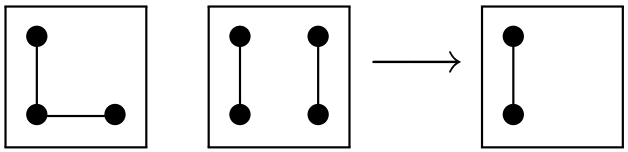
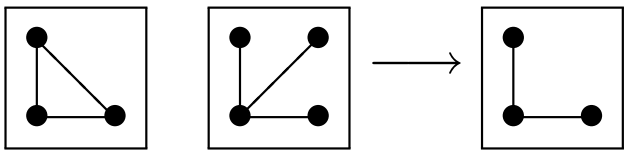
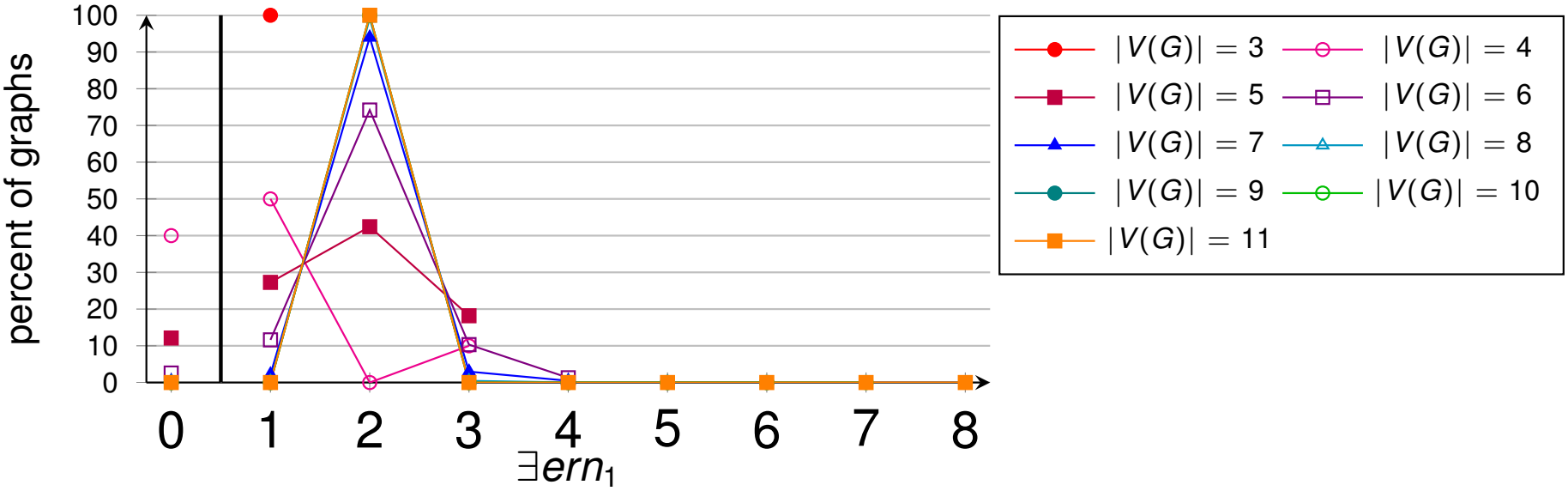
$\forall vrn_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$

		graph order								
		3	4	5	6	7	8	9	10	11
unique graphs		3	10	33	155	1043	12345	274667	12005167	1018997863
not reconstructible		0	4	4	4	4	4	4	4	4
$\exists ern_1$	min	1	1	1	1	1	1	1	1	1
	max	1	3	3	4	4	5	6	7	8
	mean	1.00	1.33	1.90	2.01	2.02	2.00	2.00	2.00	2.00
	std dev	0.00	0.75	0.71	0.53	0.27	0.10	0.03	0.01	0.00
$\forall ern_1$	min	1	1	1	1	1	1	1	1	1
	max	1	3	6	8	11	15	20	26	33
	mean	1.00	2.00	3.69	4.28	4.74	4.56	4.10	3.63	3.29
	std dev	0.00	1.00	1.39	1.44	1.31	1.18	0.95	0.75	0.54
% $\exists ern_1 = 2$		0	0	42	74.2	93.96	99.166	99.9476	99.9982	100.0000
% $\forall ern_1 = 2$		0	0	6	9.0	1.82	0.413	0.0553	0.0133	0.2434

- The graphs with $\exists ern_1 = \forall ern_1 = 1$ are: K_2, K_n, $K_n - e$
- The non-reconstructible graphs are:

P_3 and $2K_2$  x2

K_3 and $K_{1,3}$  x3


percent of graphs
100
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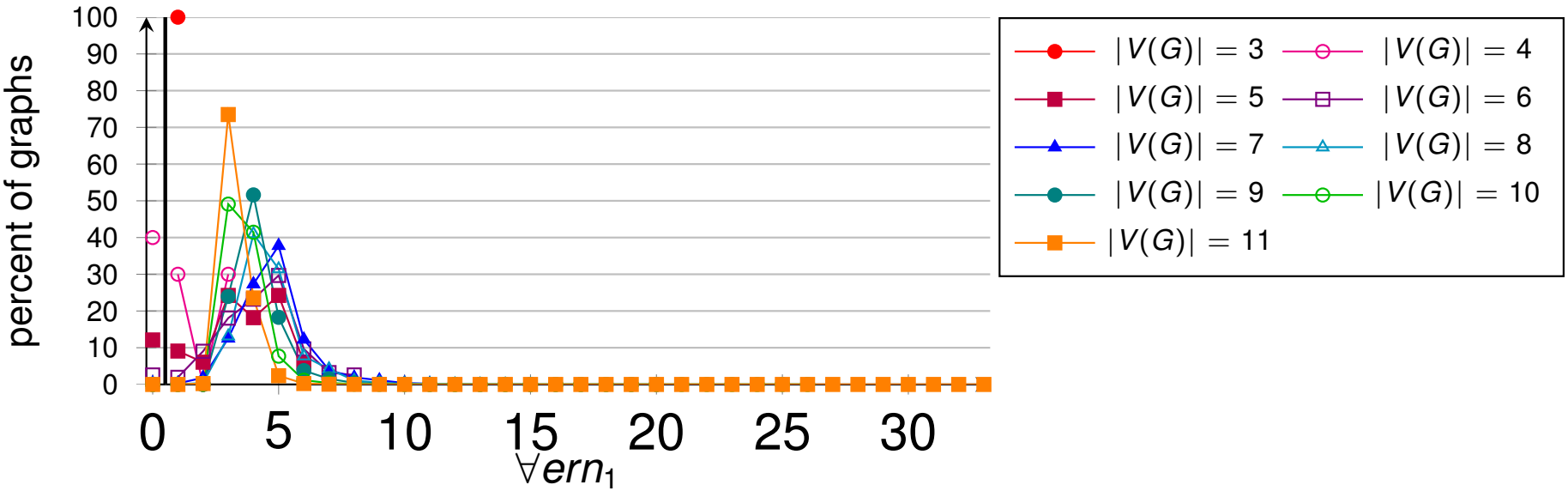
$\exists ern_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$


percent of graphs
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$\forall ern_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$

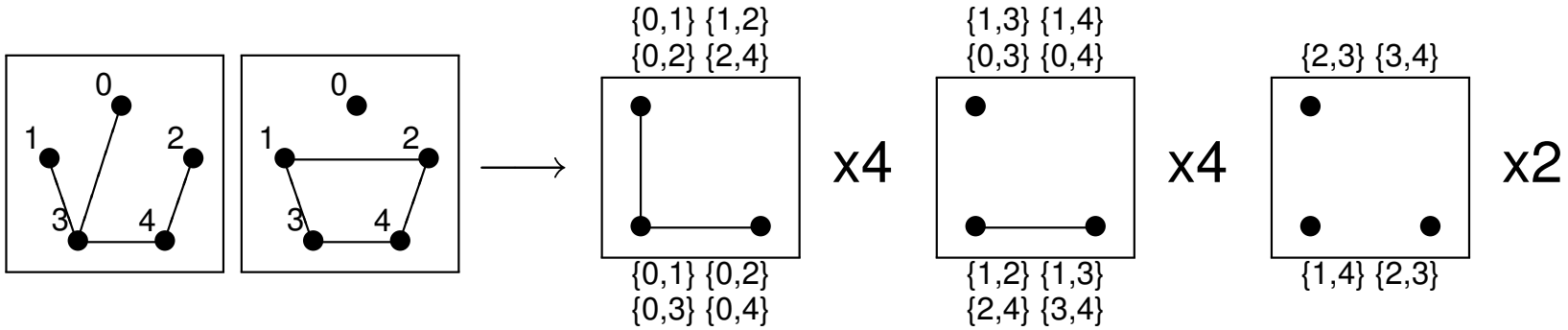
- Most computations on Opteron 248 (2.2GHz) CPUs
- Larger computations used RIT CASCI Cluster (94 1.4GHz P-III CPUs)

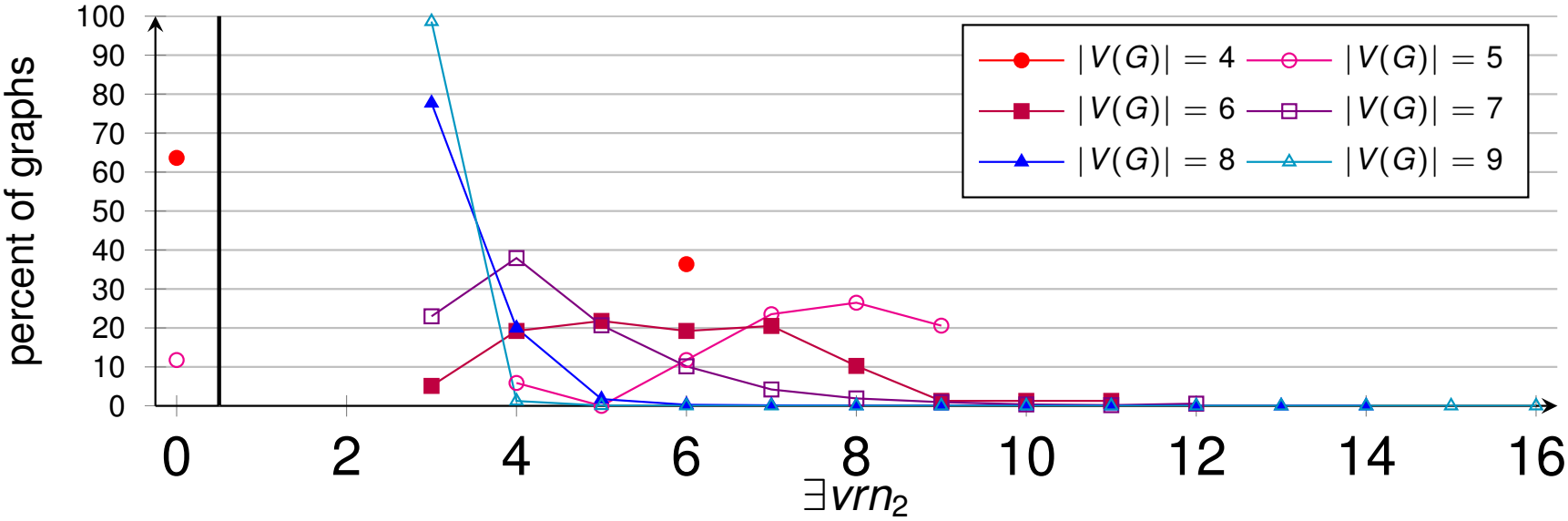
			vertex reconstruction		edge reconstruction	
$\lvert V\rvert$	total graphs	$\leq \frac{1}{2}\binom{\lvert V\rvert}{2}$ edges	total CPU time	ms per graph	total CPU time	ms per graph
6	156	78	0.02 seconds	0.26	0.04 seconds	0.25
7	1044	522	0.52 seconds	1.07	0.74 seconds	0.71
8	12346	6996	16.8 seconds	2.45	16.3 seconds	1.32
9	274668	154354	14.0 minutes	5.52	10.0 minutes	2.18
10	12005168	6002584	20.9 hours	12.6	10.7 hours	3.21
11	1018997864	509498932	174 days	29.5	47.4 days	4.02

- There are 165,091,172,592 graphs on 12 vertices
  - Est. vertex-deletion time: $\approx$ 200 equivalent CPU-years
  - Est. edge-deletion time: $\approx$ 25 equivalent CPU-years

		graph order					
		4	5	6	7	8	9
unique graphs		11	34	156	1044	12346	274654
not reconstructible		7	4	0	0	0	0
$\exists vrn_2$	min	6	4	3	3	3	3
	max	6	9	11	12	14	16
	mean	6.00	7.43	5.81	4.51	3.26	3.02
	std dev	0.00	1.33	1.63	1.45	0.57	0.16
$\forall vrn_2$	min	6	8	10	12	14	16
	max	6	10	14	19	25	30
	mean	6.00	9.30	13.13	16.80	20.10	22.05
	std dev	0.00	0.78	0.84	0.99	1.26	1.69
% $\exists vrn_2 = 3$		0	0	5.1	22.99	77.693	98.6219

- The non-reconstructible graphs on 5 vertices:




percent of graphs
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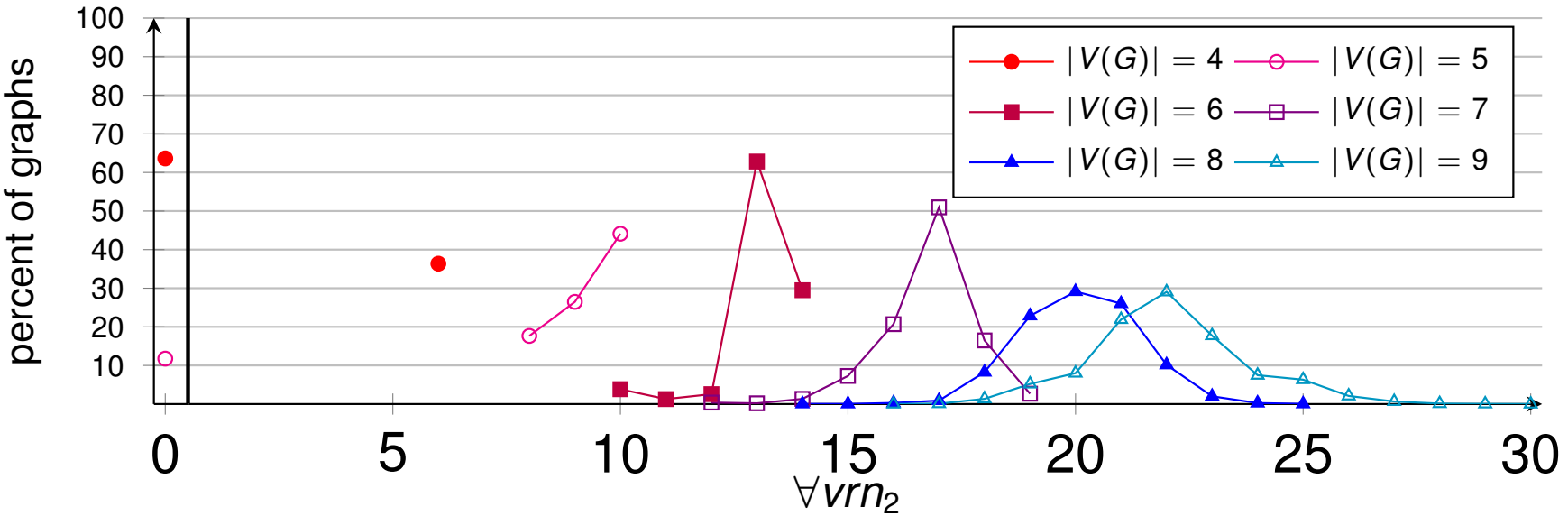
$\exists vrn_2$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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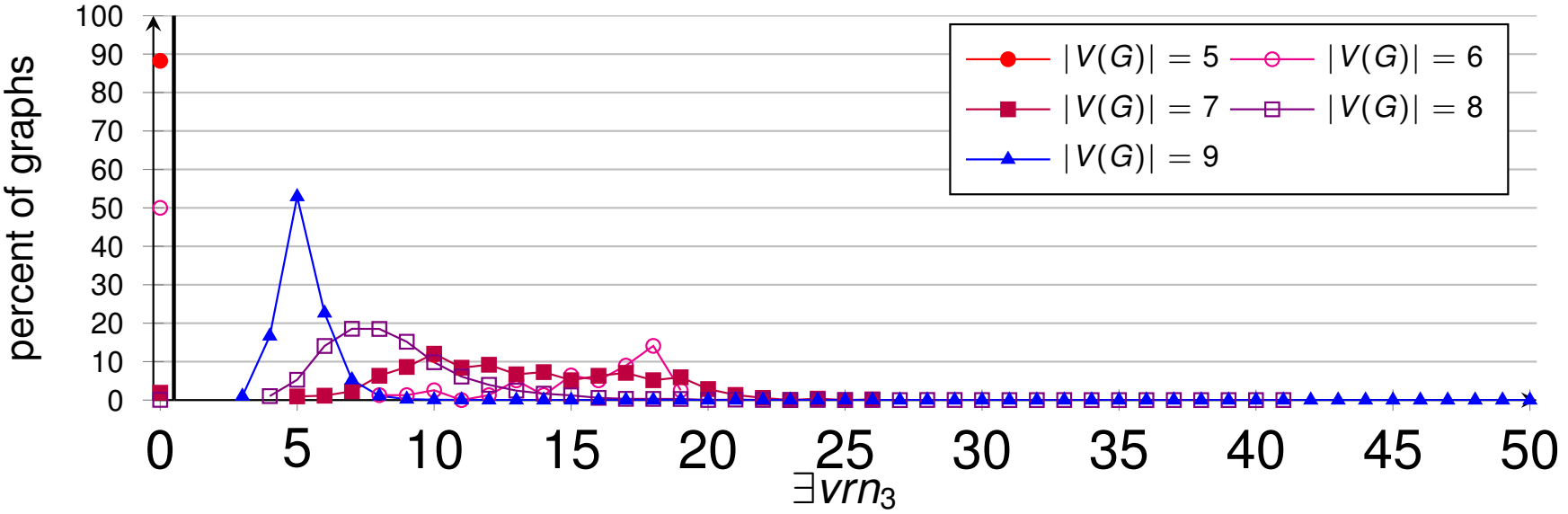
$\forall vrn_2$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

		graph order				
		5	6	7	8	9
unique graphs		34	156	1044	12346	274668
not reconstructible		30	78	20	8	0
$\exists vrn_3$	min	10	8	5	4	3
	max	10	19	26	41	50
	mean	10.00	15.62	13.12	8.56	5.20
	std dev	0.00	2.82	3.96	2.73	0.95
$\forall vrn_3$	min	10	17	26	37	50
	max	10	19	34	54	79
	mean	10.00	18.82	32.39	48.74	66.86
	std dev	0.00	0.50	1.13	1.64	2.59
% $\exists vrn_3 = 3$		0	0.0	0.00	0.000	1.0048


percent of graphs
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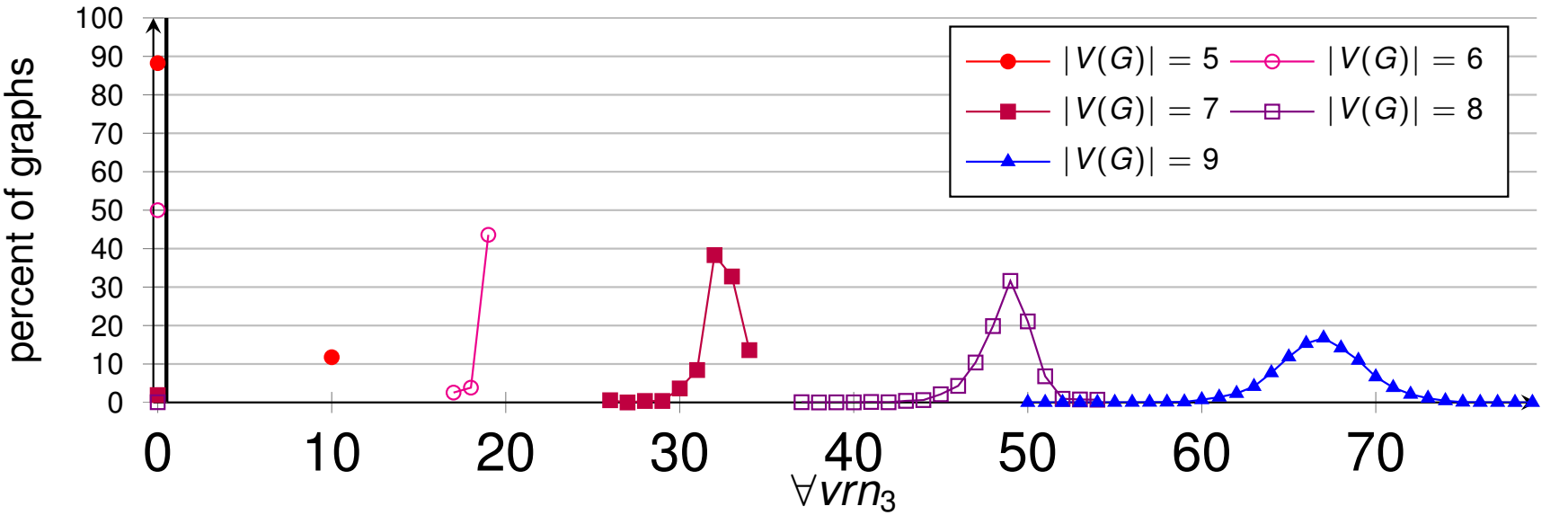
$\exists vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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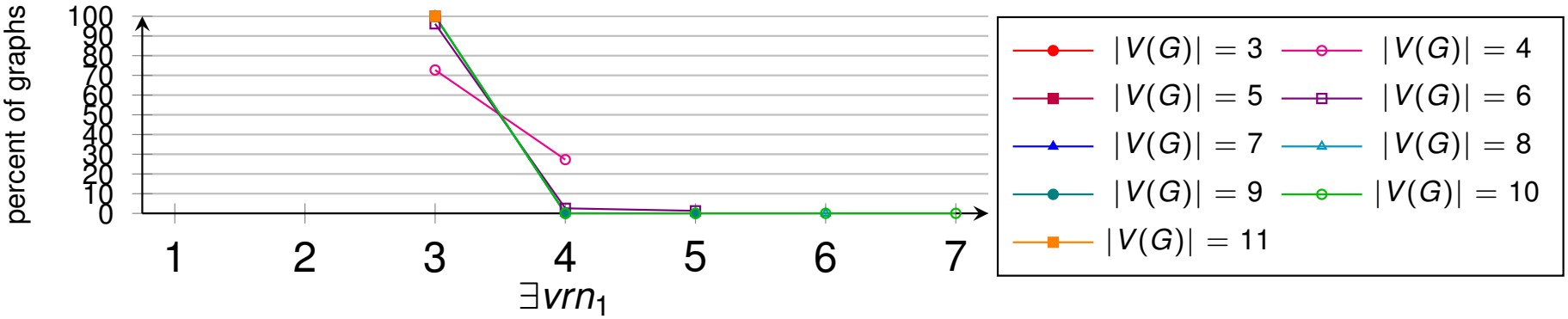
$\forall vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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$\exists vrn_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
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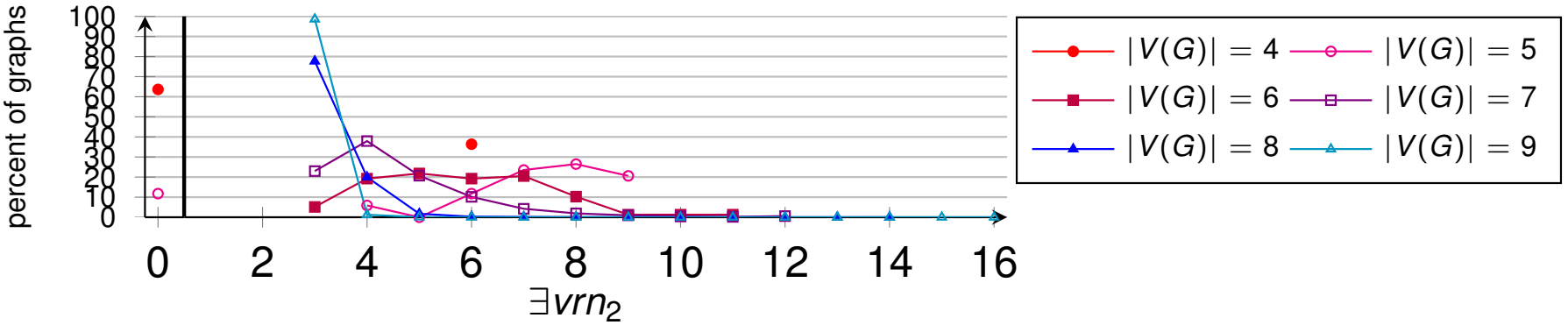
$|V(G)| = 10$
$|V(G)| = 11$


percent of graphs
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$\exists vrn_2$
$|V(G)| = 4$
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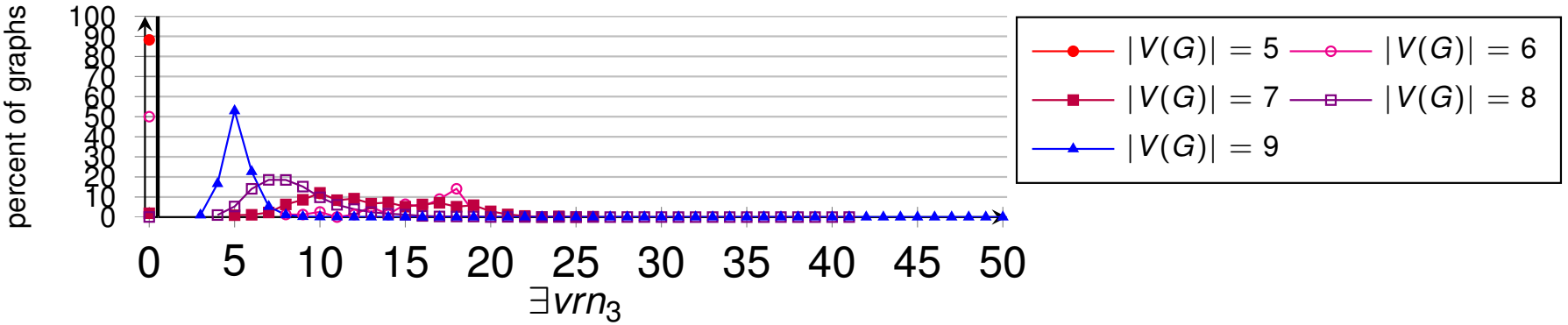
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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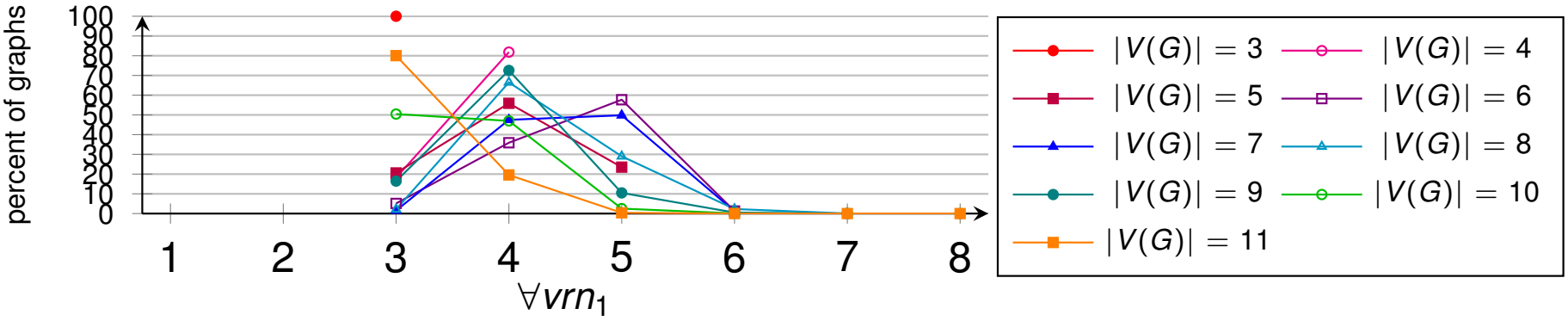
$\exists vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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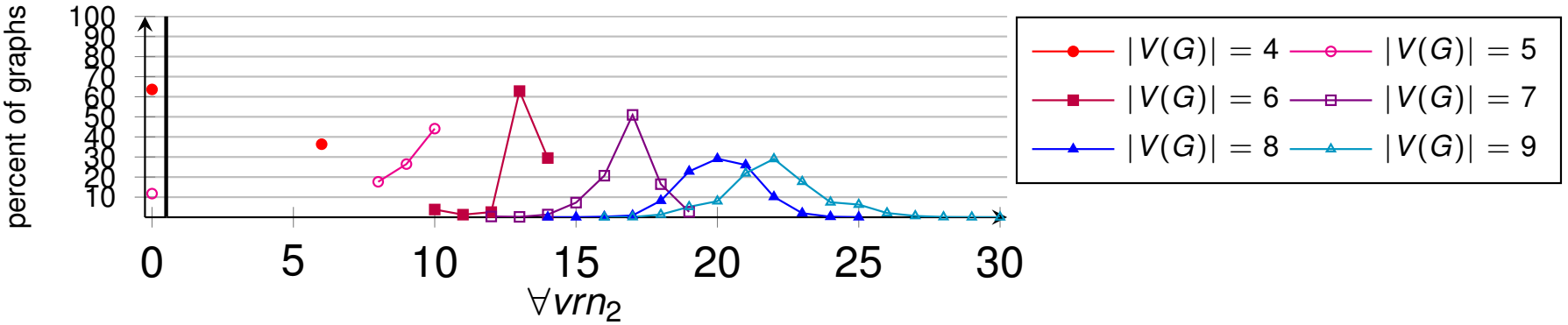
$\forall vrn_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$


percent of graphs
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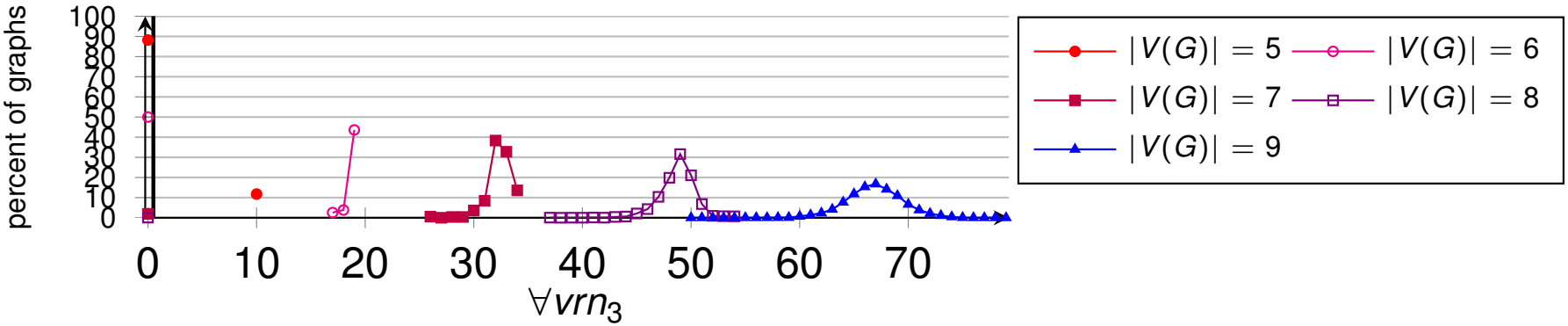
$\forall vrn_2$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$


percent of graphs
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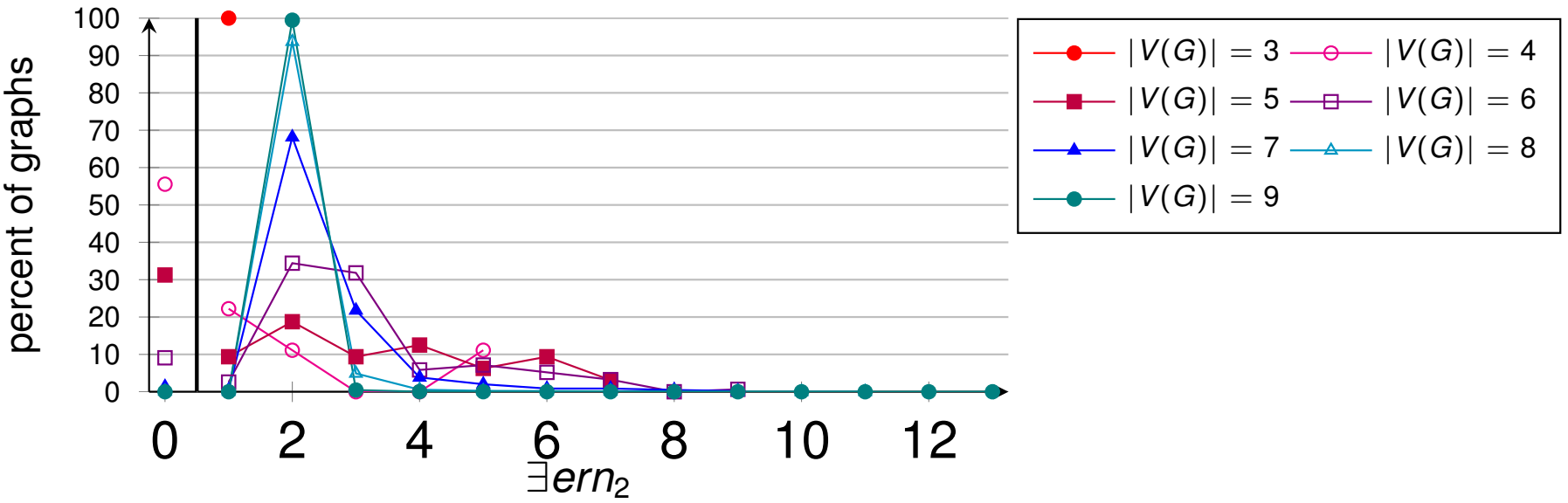
$\forall vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

		graph order						
		3	4	5	6	7	8	9
unique graphs		2	9	32	154	1042	12344	274666
not reconstructible		0	5	10	14	14	14	14
$\exists ern_2$	min	1	1	1	1	1	1	1
	max	1	5	7	9	9	11	13
	mean	1.00	2.25	3.41	3.14	2.48	2.09	2.01
	std dev	0.00	1.64	1.78	1.48	0.98	0.44	0.12
$\forall ern_2$	min	1	1	1	1	1	1	1
	max	1	6	22	56	124	224	382
	mean	1.00	3.50	11.50	19.71	31.55	43.23	51.34
	std dev	0.00	2.50	5.56	10.30	15.26	18.08	18.50
% $\exists ern_2 = 2$		0	11	19	34.4	68.14	93.705	99.4535

- The graphs with $\exists ern_2 = \forall ern_2 = 1$ are: K_n, $K_n - e$


percent of graphs
100
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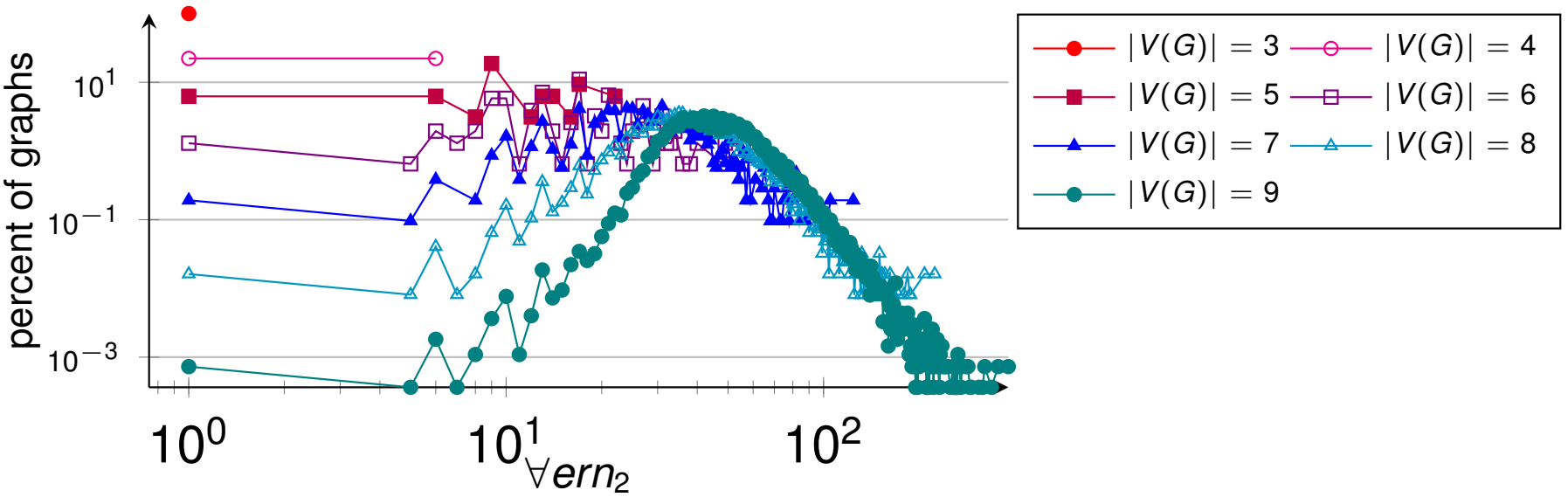
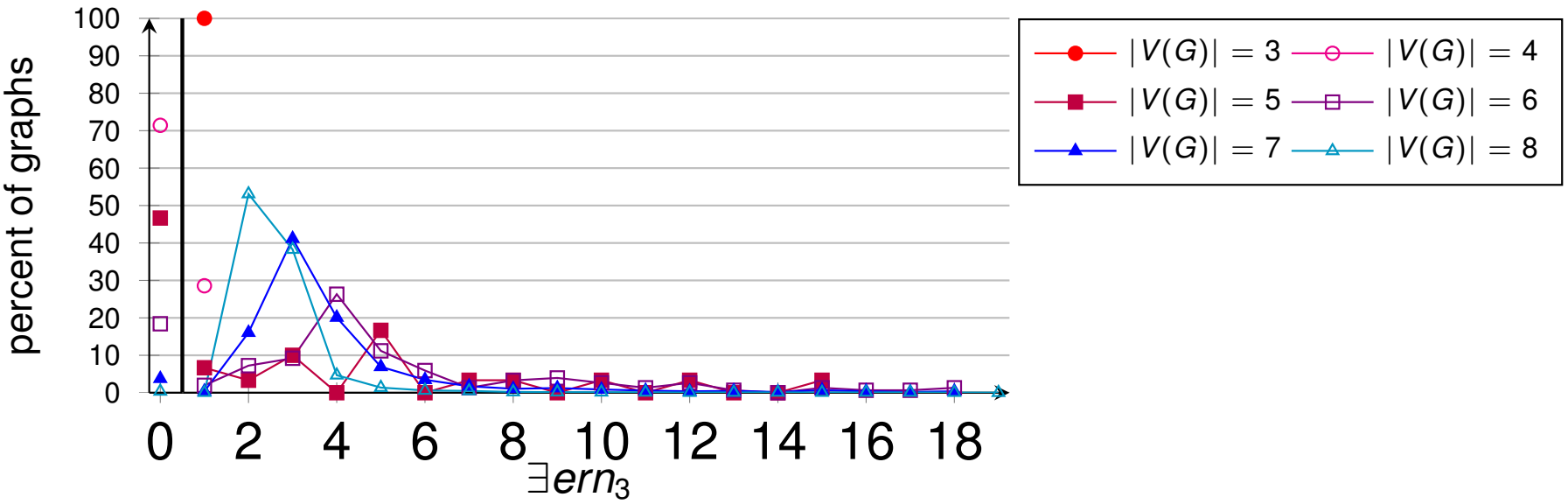
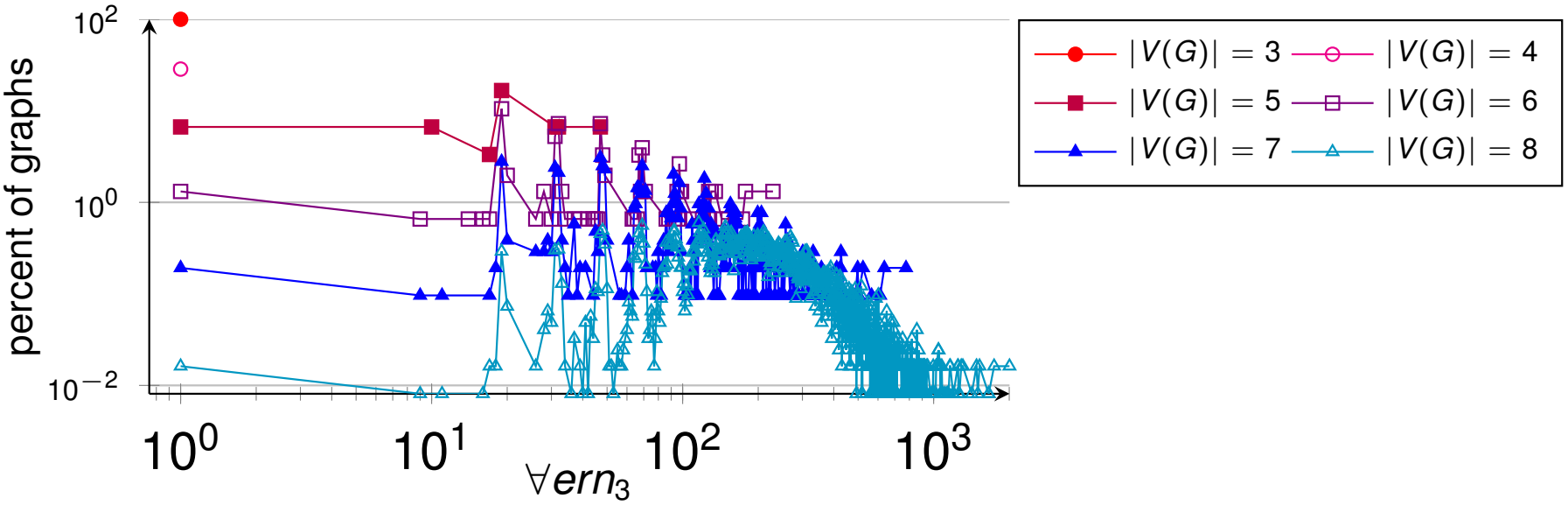
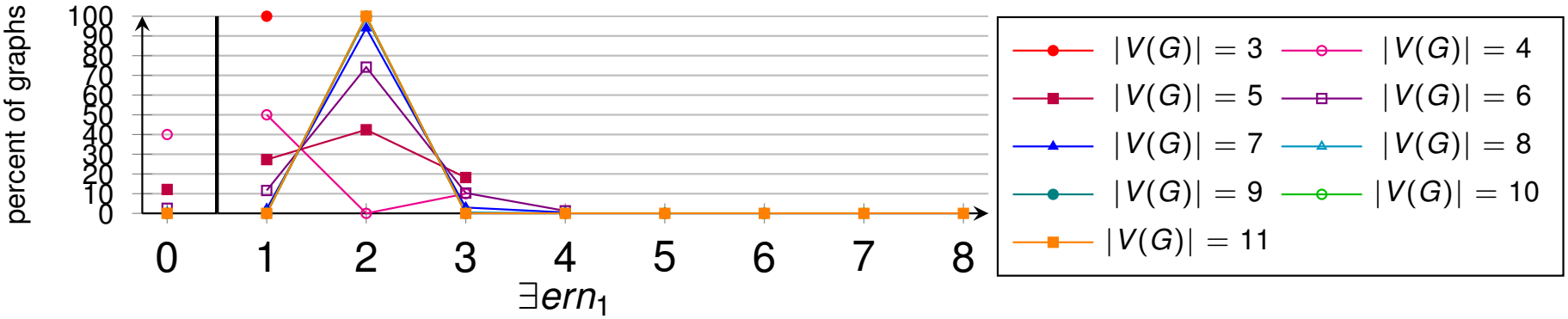
$\exists ern_2$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

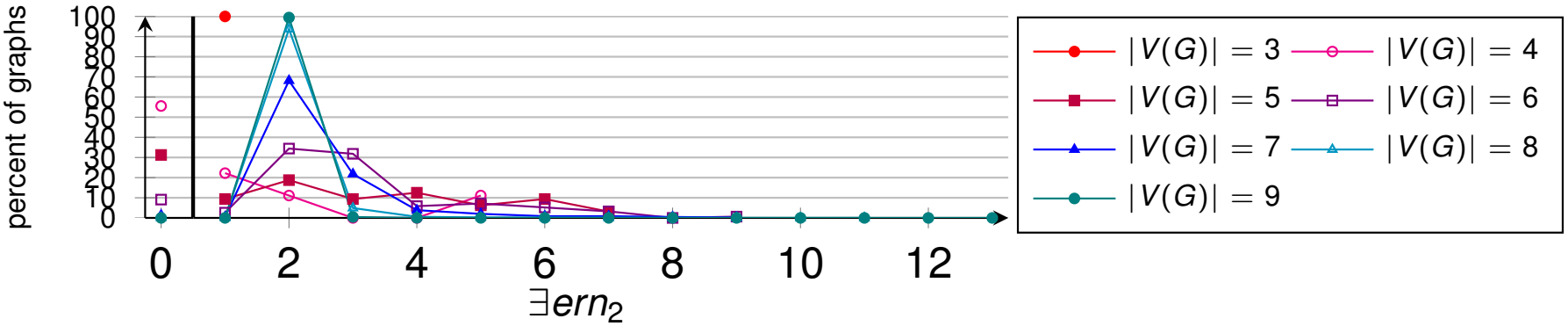

percent of graphs
10^1
10^{-1}
10^{-3}
10^0
10^1
10^2
$\forall ern_2$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

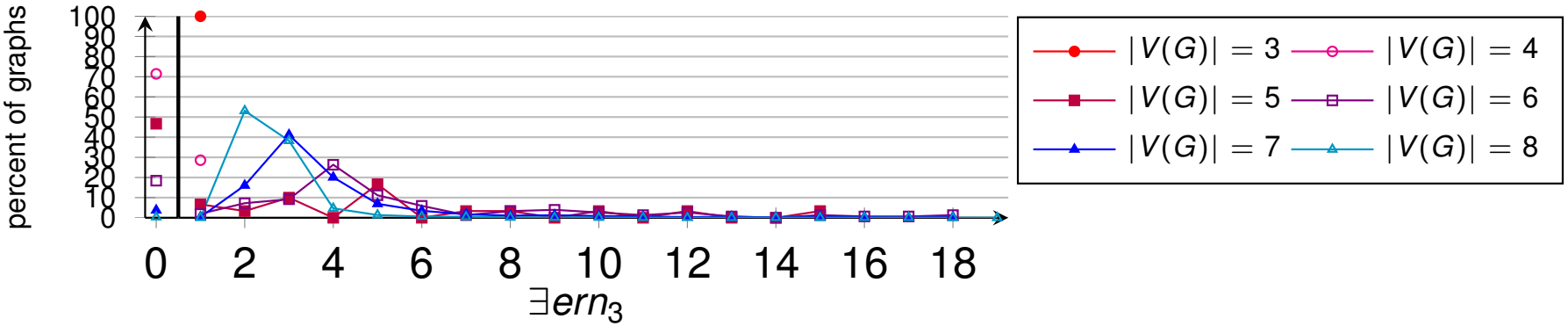
		graph order					
		3	4	5	6	7	8
unique graphs		1	7	30	152	1040	12342
not reconstructible		0	5	14	28	39	45
$\exists ern_3$	min	1	1	1	1	1	1
	max	1	1	15	18	18	19
	mean	1.00	1.00	5.62	5.61	3.93	2.68
	std dev	0.00	0.00	3.82	3.58	2.45	1.29
$\forall ern_3$	min	1	1	1	1	1	1
	max	1	1	47	229	776	2005
	mean	1.00	1.00	22.12	59.44	130.56	250.90
	std dev	0.00	0.00	13.35	44.57	98.54	163.56
% $\exists ern_3 = 2$		0	0	3	7.2	16.06	53.095

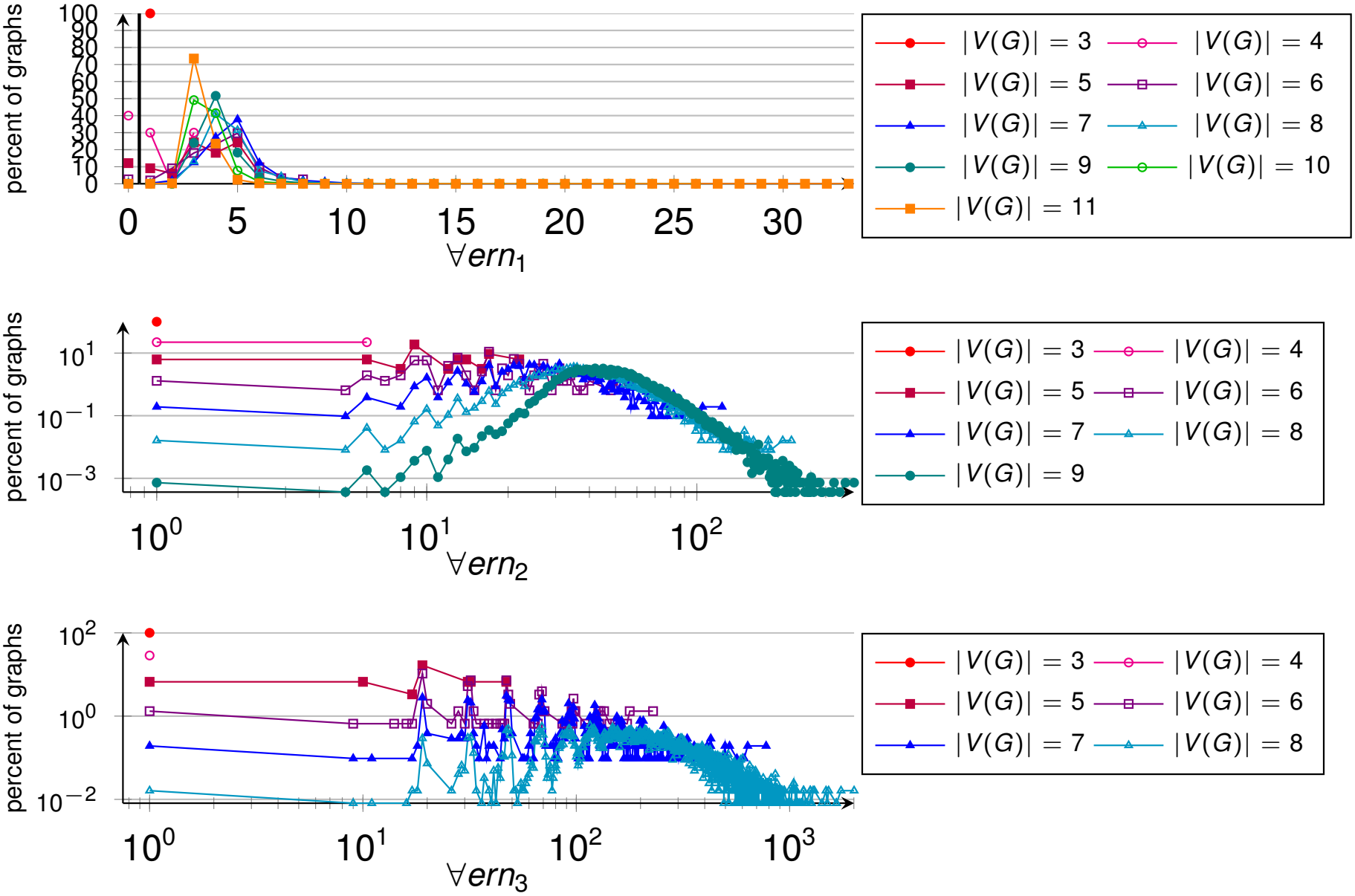

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0 2 4 6 8 10 12 14 16 18
$\exists ern_3$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$


percent of graphs
10^2
10^0
10^{-2}
10^0 10^1 10^2 10^3
$\forall ern_3$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

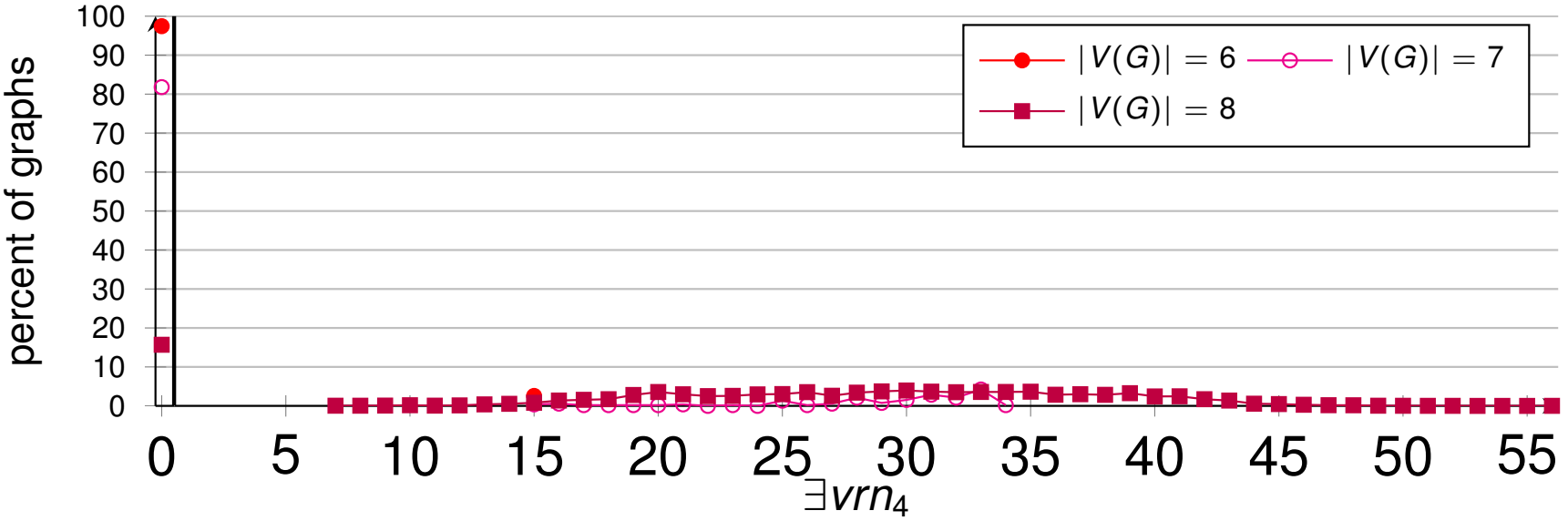

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0 1 2 3 4 5 6 7 8
∃ern1
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8
|V(G)| = 9
|V(G)| = 10
|V(G)| = 11

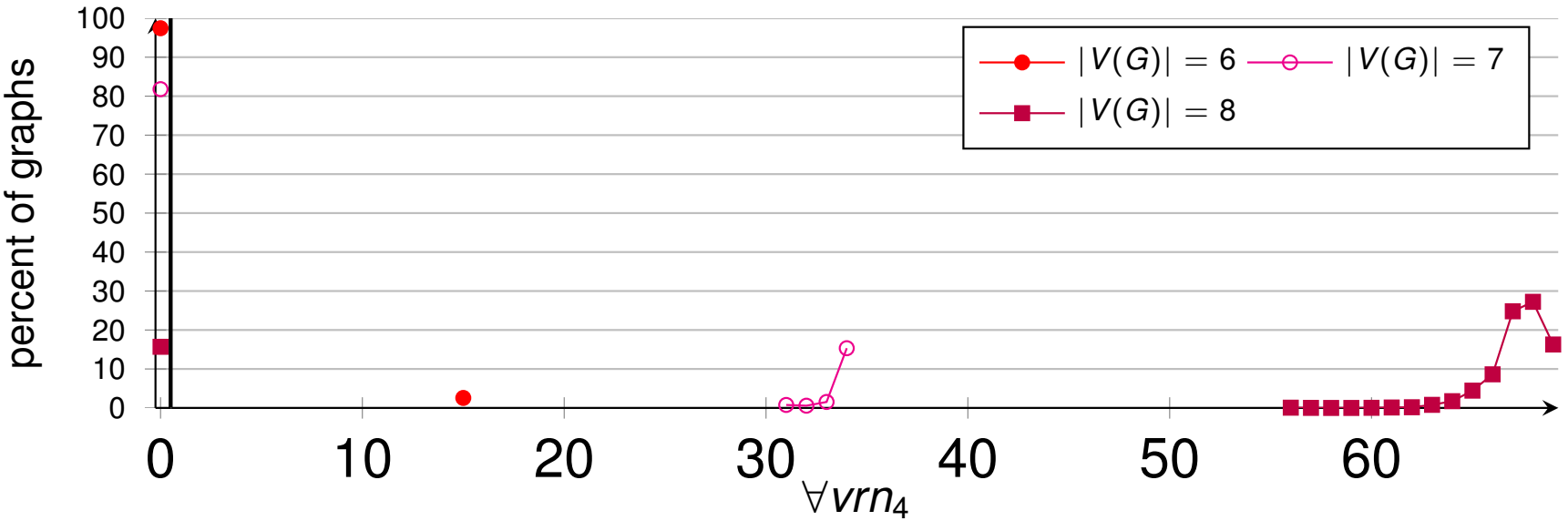

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0 2 4 6 8 10 12
∃ern2
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8
|V(G)| = 9


percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0 2 4 6 8 10 12 14 16 18
∃ern3
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8

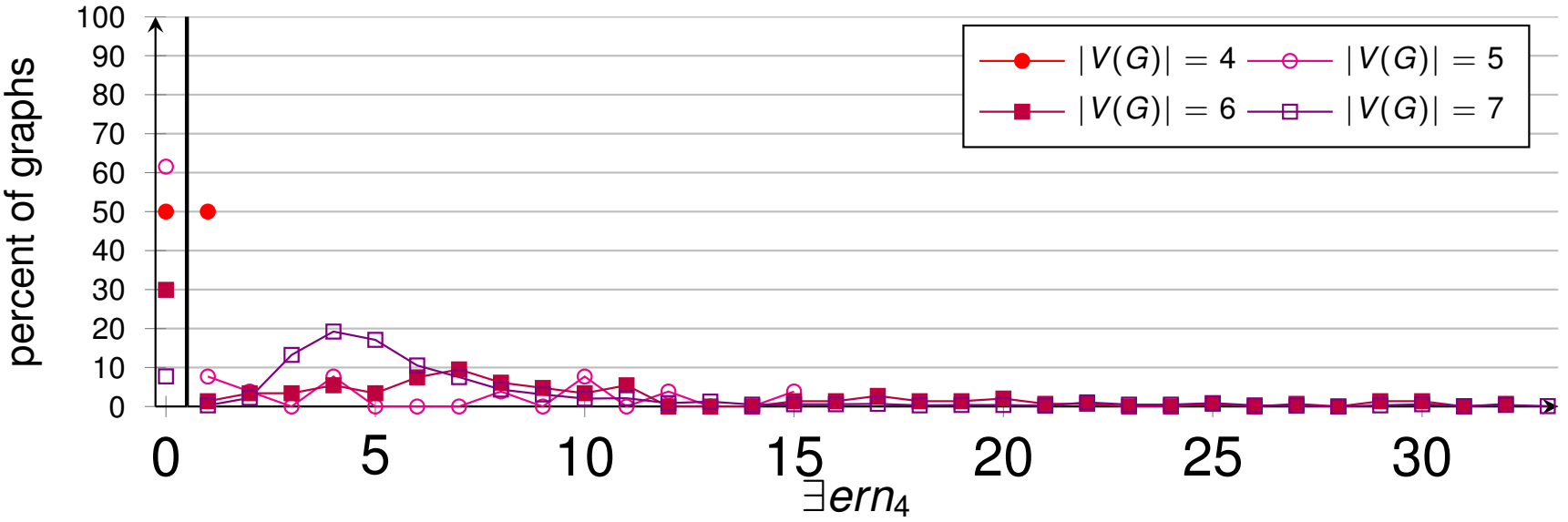

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
$\forall ern_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$
percent of graphs
10^1
10^{-1}
10^{-3}
10^0
10^1
10^2
$\forall ern_2$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
percent of graphs
10^2
10^0
10^{-2}
10^0
10^1
10^2
10^3
$\forall ern_3$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

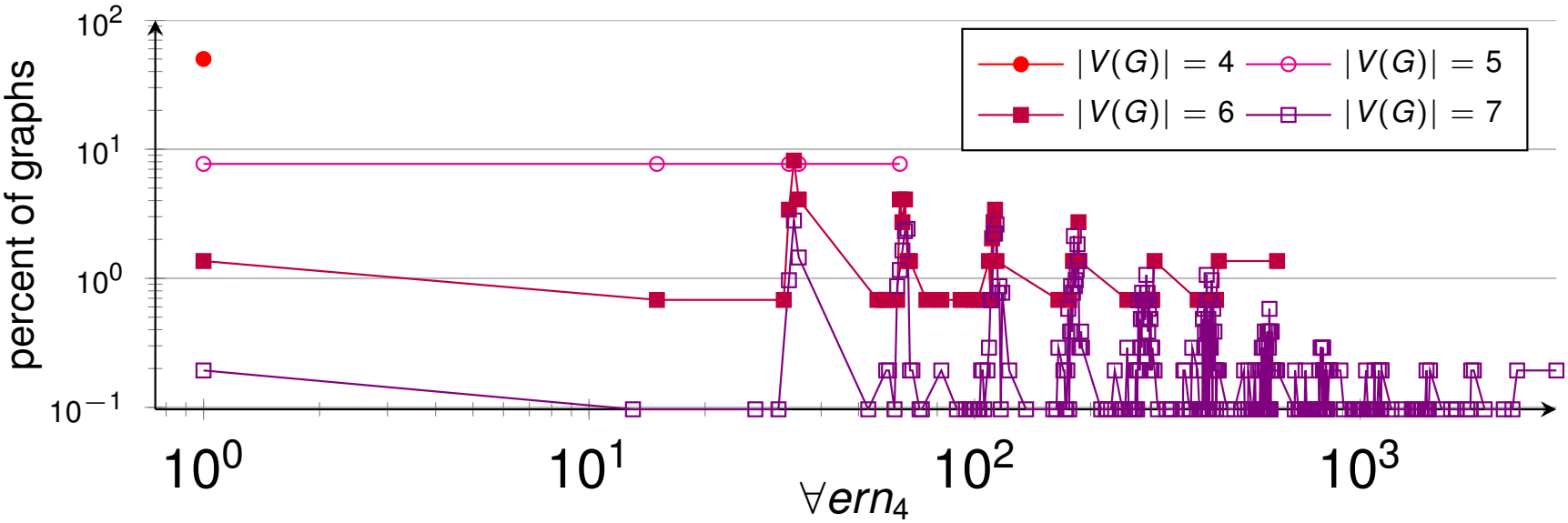
		graph order		
		6	7	8
unique graphs		156	1044	12346
not reconstructible		152	854	1937
$\exists vrn_4$	min	15	15	7
	max	15	34	56
	mean	15.00	28.84	29.61
	std dev	0.00	4.81	7.92
$\forall vrn_4$	min	15	31	56
	max	15	34	69
	mean	15.00	33.73	67.37
	std dev	0.00	0.72	1.36
% $\exists vrn_4 = 3$		0.0	0.00	0.000

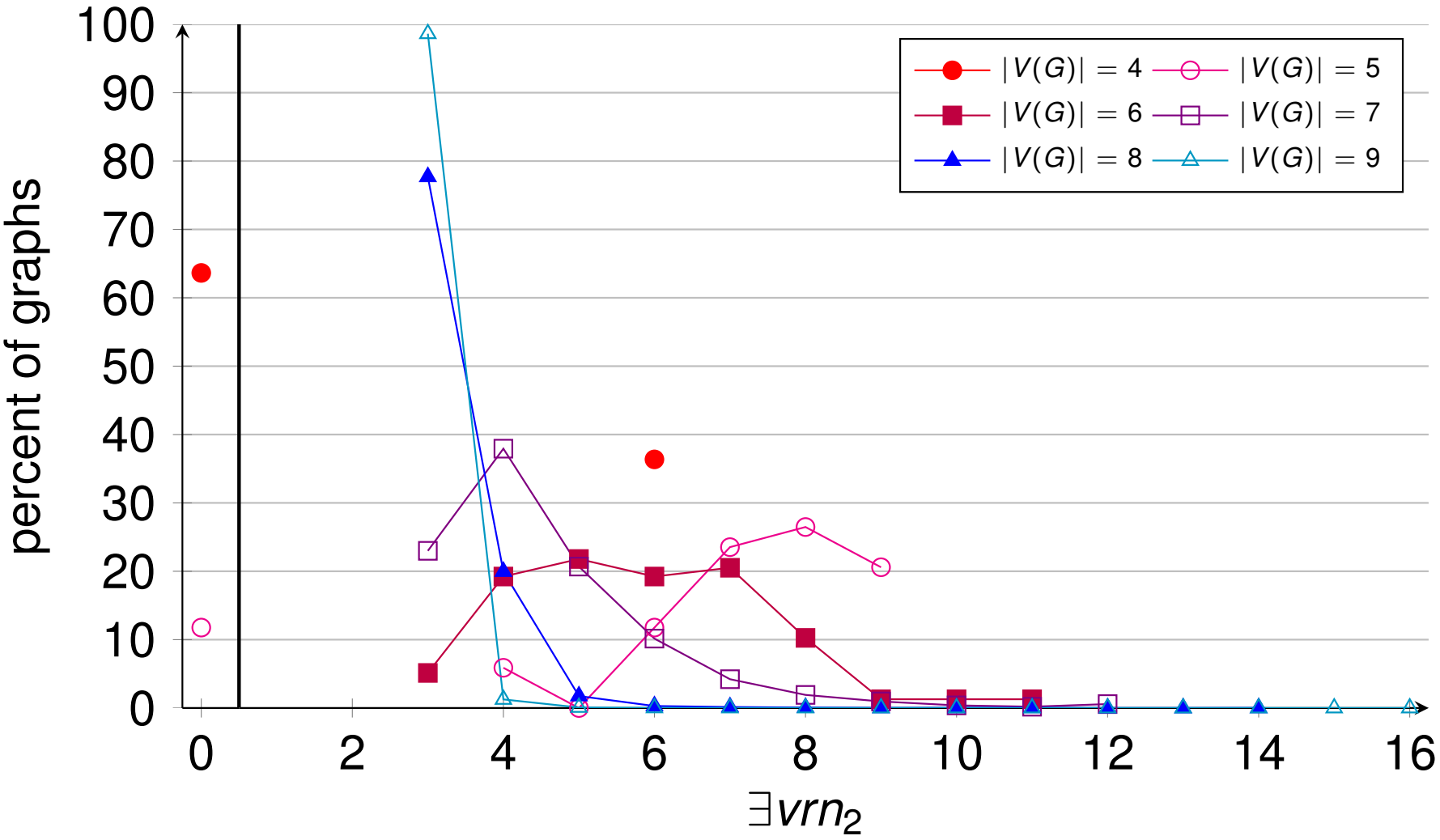

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
40
45
50
55
$\exists vrn_4$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

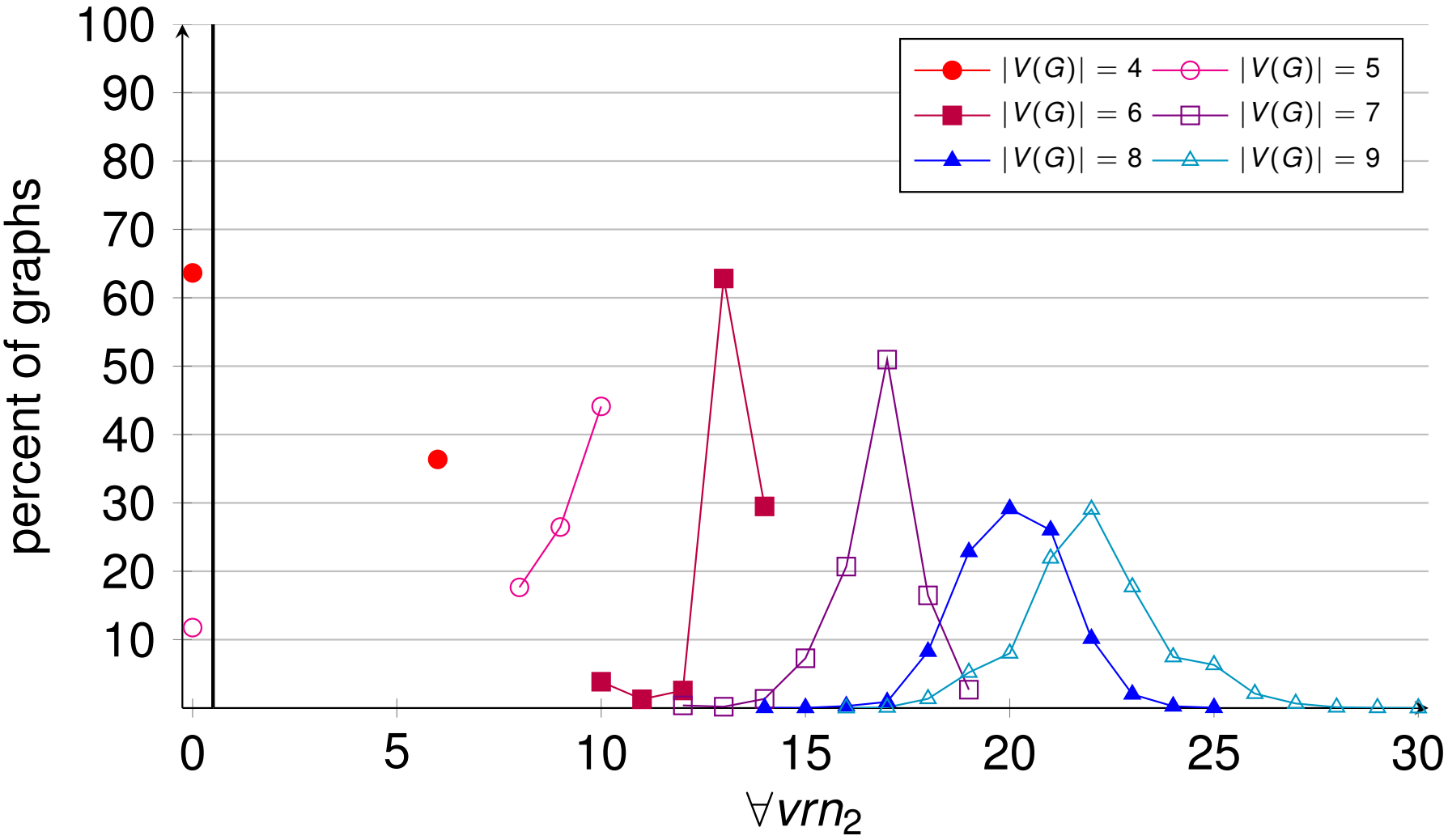

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
$\forall vrn_4$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

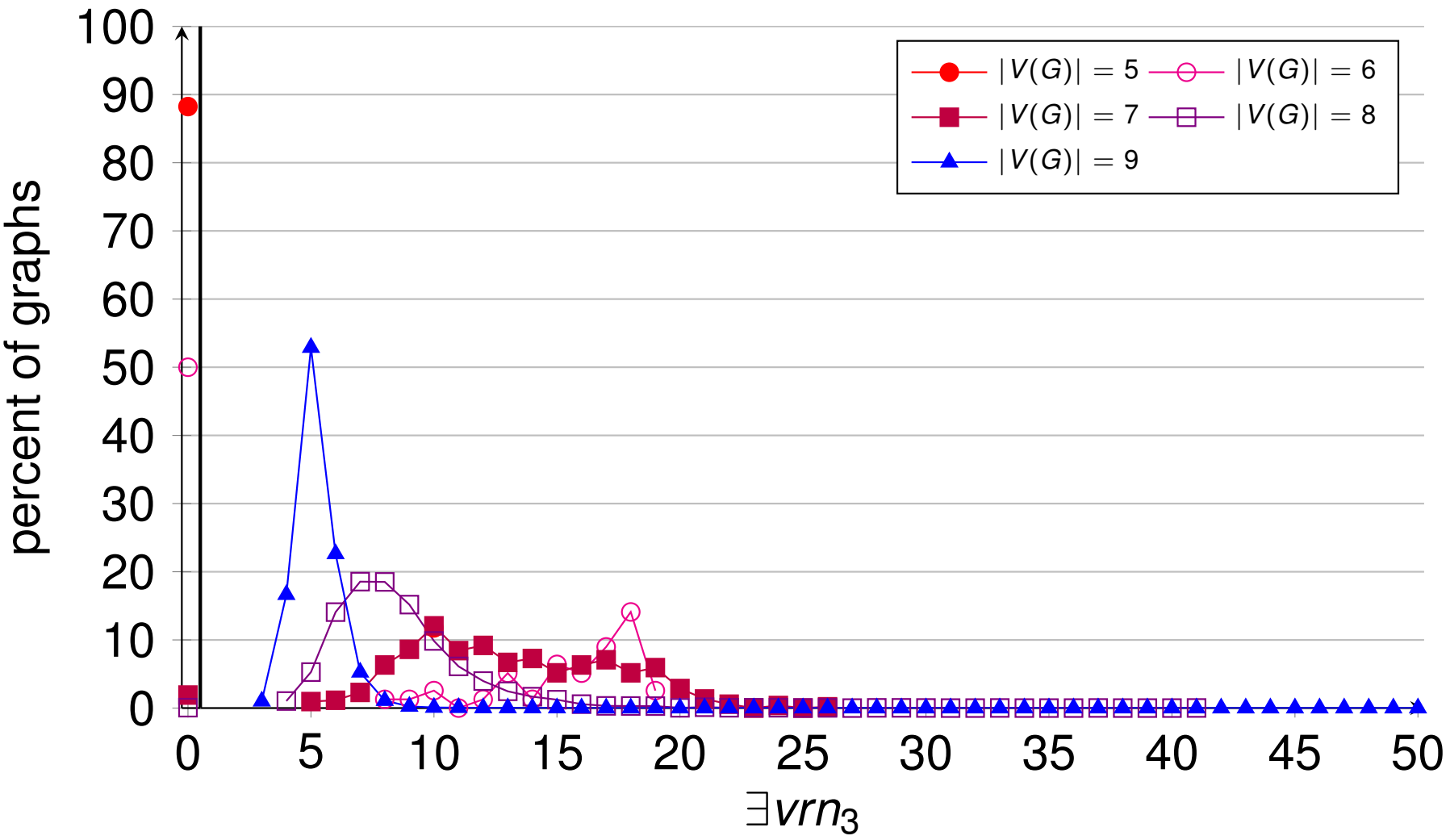
		graph order			
		4	5	6	7
unique graphs		4	26	147	1035
not reconstructible		2	16	44	80
$\exists ern_4$	min	1	1	1	1
	max	1	15	32	33
	mean	1.00	6.70	10.00	6.95
	std dev	0.00	4.71	7.10	5.52
$\forall ern_4$	min	1	1	1	1
	max	1	64	609	3230
	mean	1.00	29.60	127.28	371.36
	std dev	0.00	21.24	120.57	391.73
% $\exists ern_4 = 2$		0	4	3.4	2.22

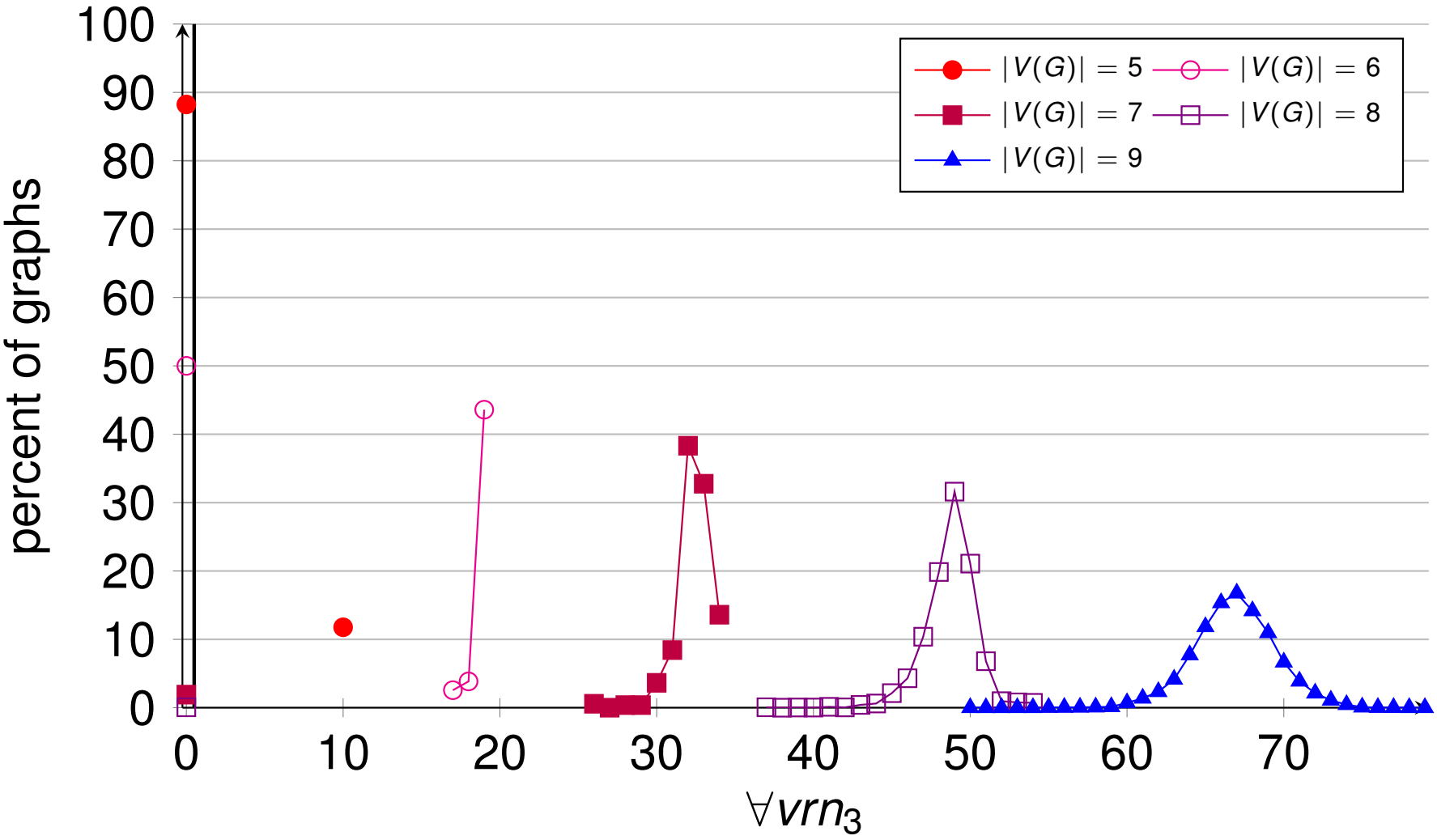

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
$\exists ern_4$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$

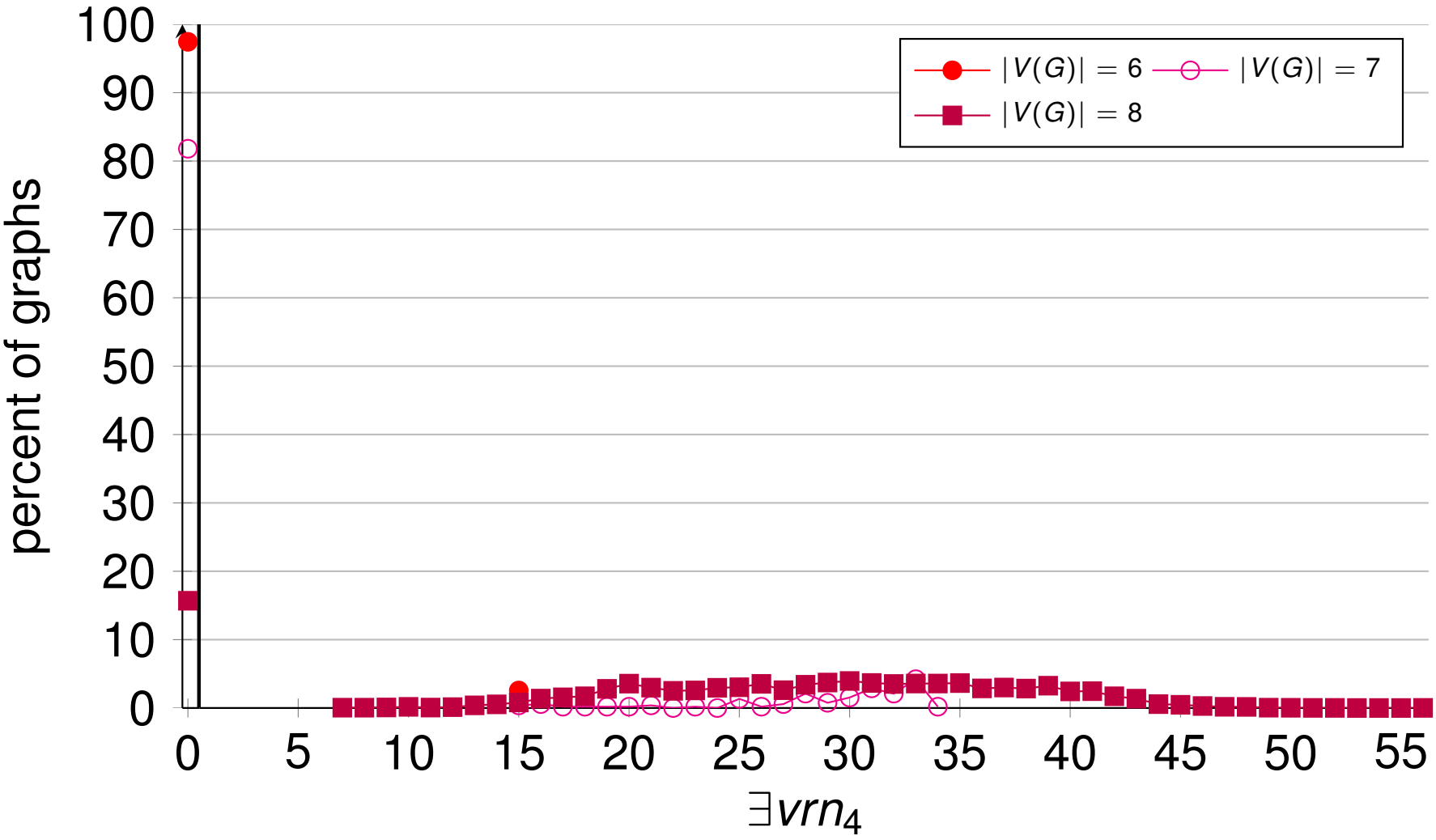

percent of graphs
10^2
10^1
10^0
10^{-1}
10^0
10^1
10^2
10^3
$\forall ern_4$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$

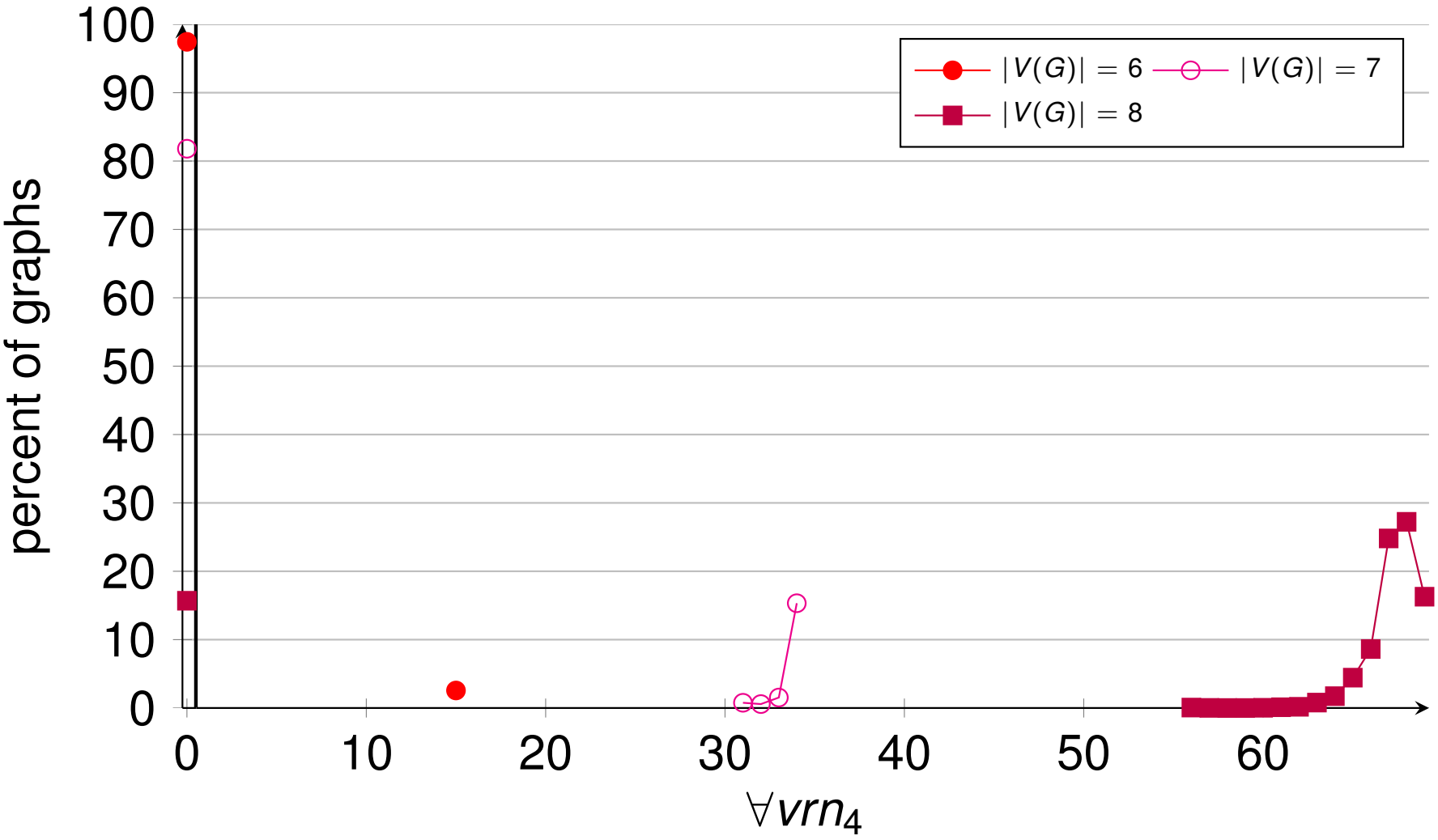

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
2
4
6
8
10
12
14
16
$\exists vrn_2$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

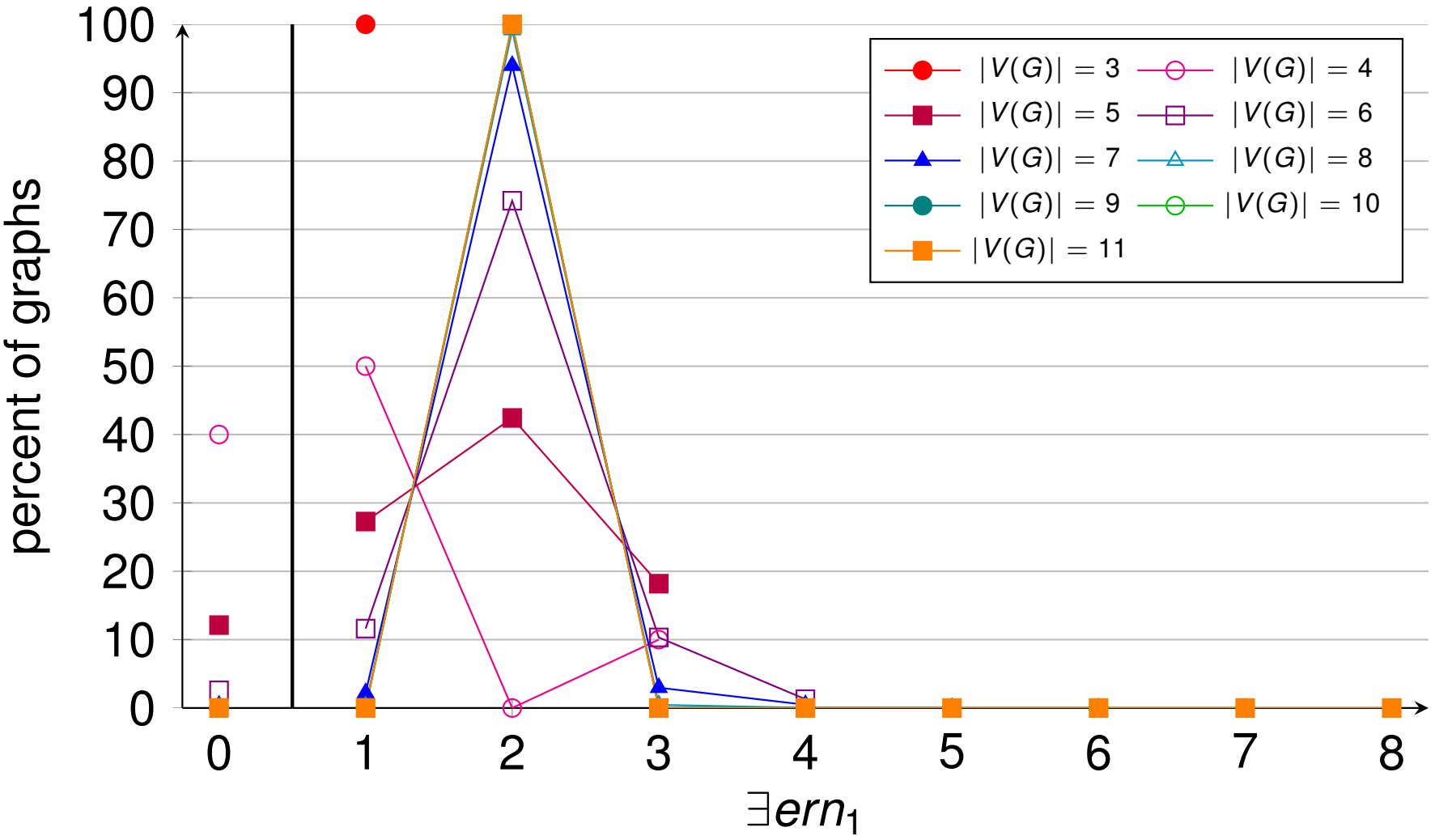

100
90
80
70
60
50
40
30
20
10
percent of graphs
0
5
10
15
20
25
30
$\forall vrn_2$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

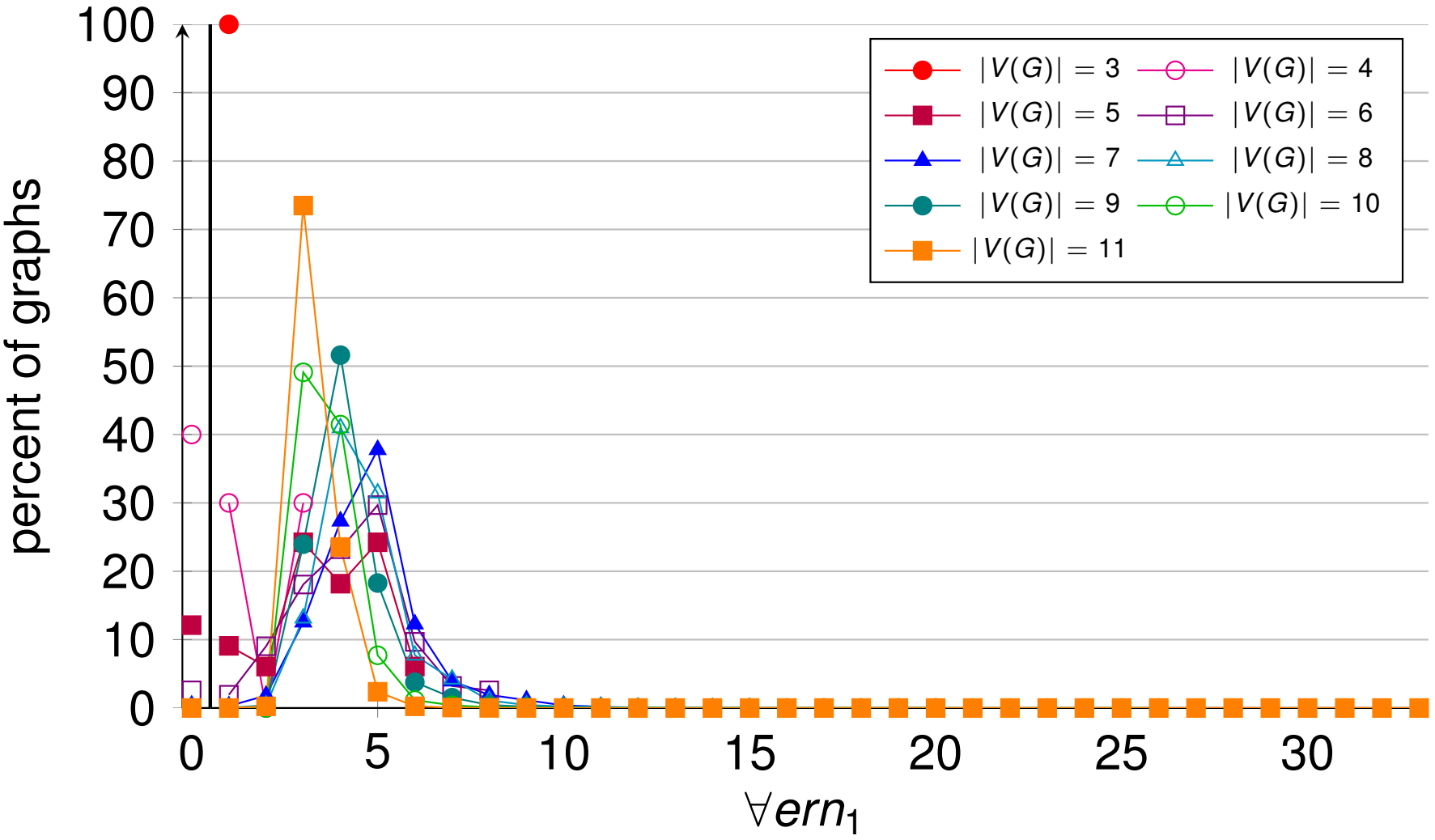

percent of graphs
100
90
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
40
45
50
$\exists vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

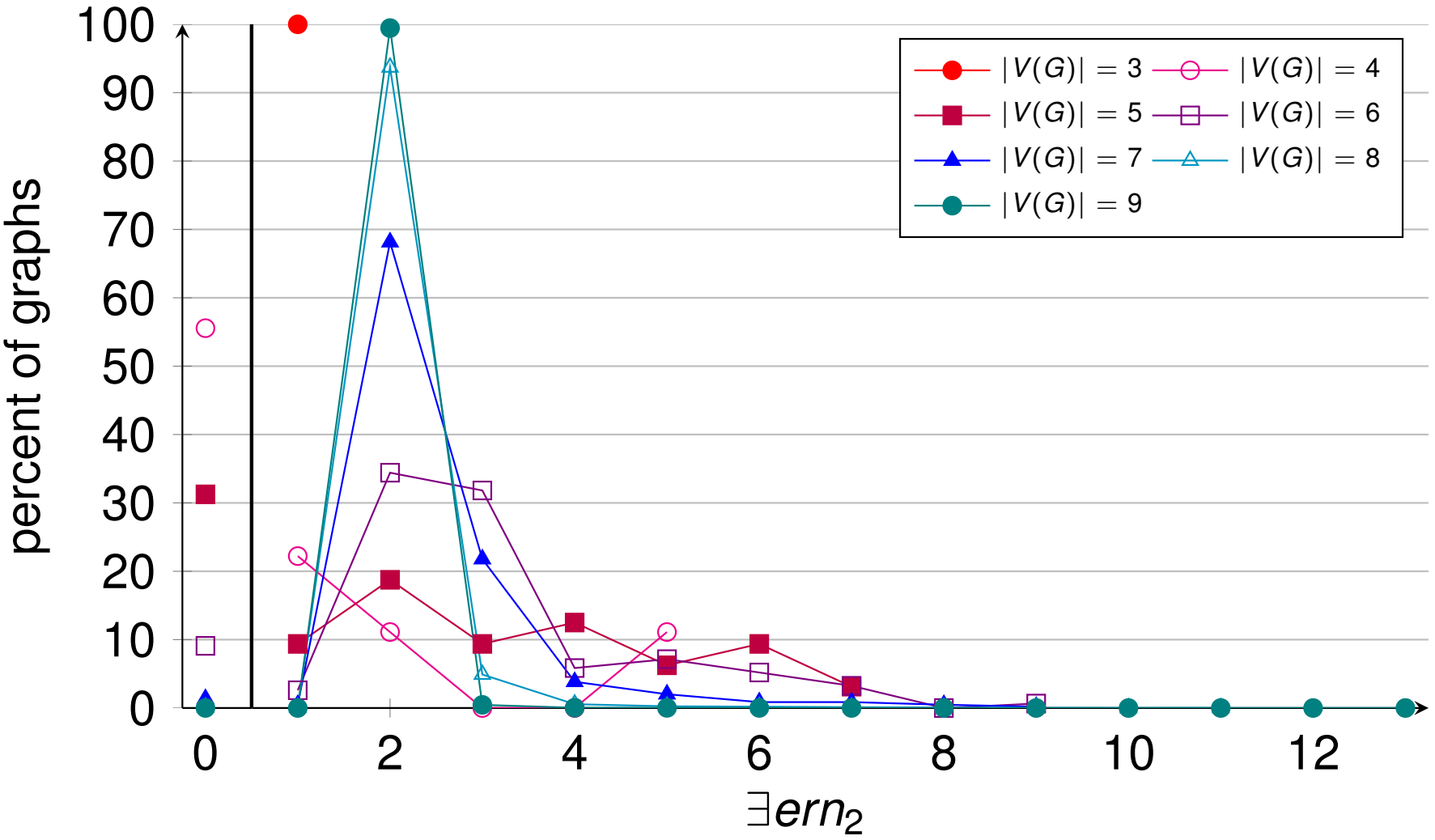

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
10
20
30
40
50
60
70
$\forall vrn_3$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$

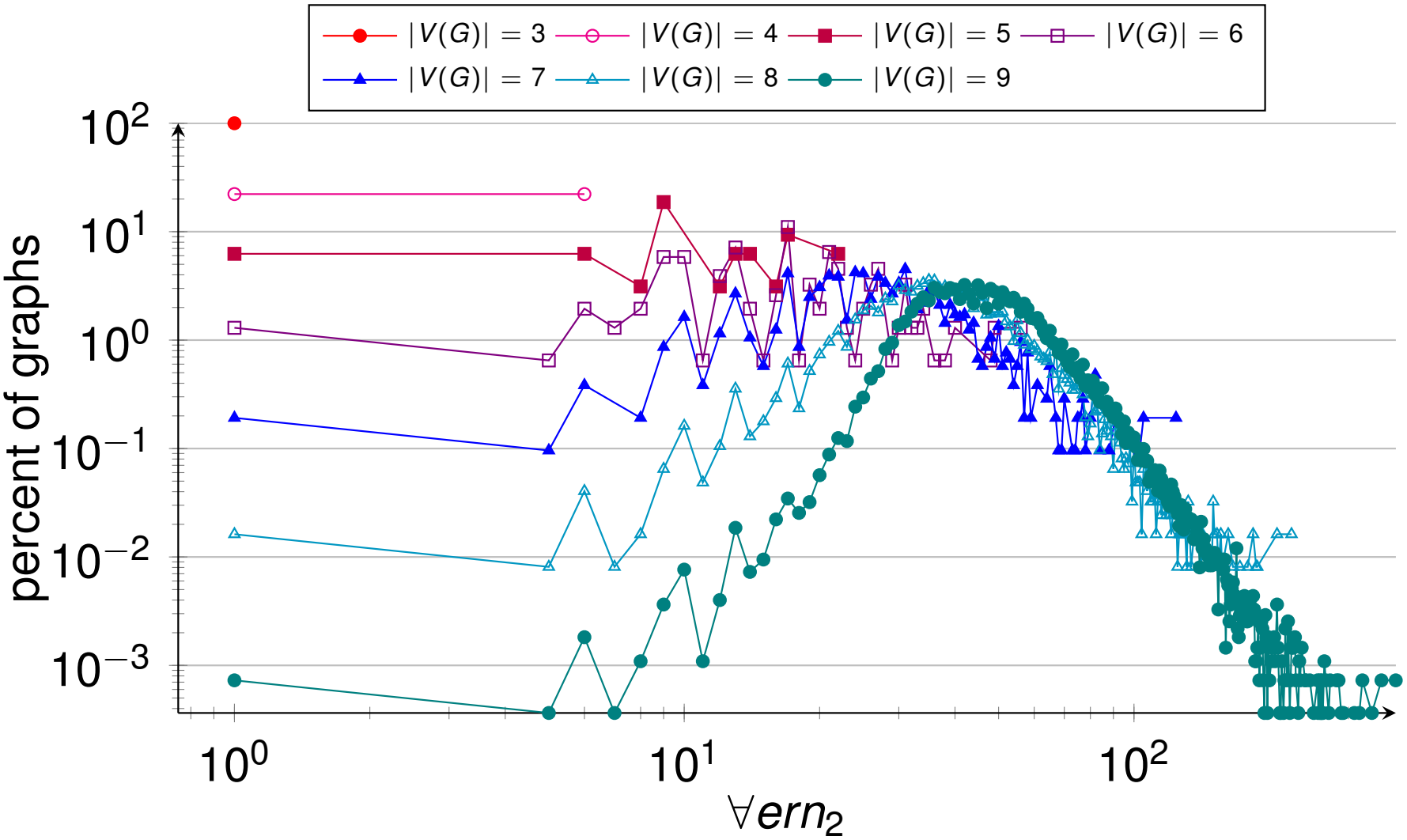

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
5
10
15
20
25
30
35
40
45
50
55
$\exists vrn_4$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

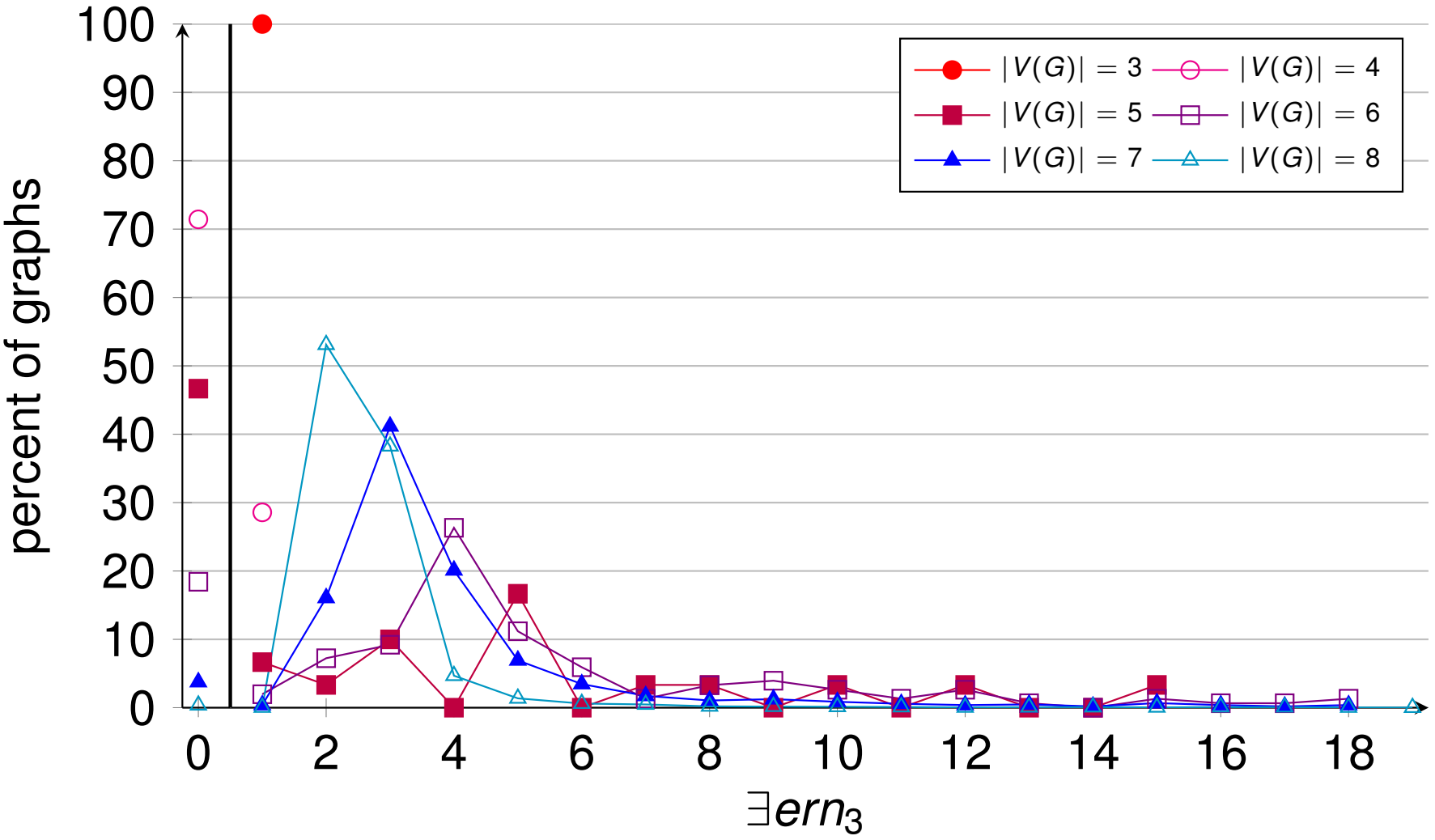

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
10
20
30
40
50
60
$\forall vrn_4$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$

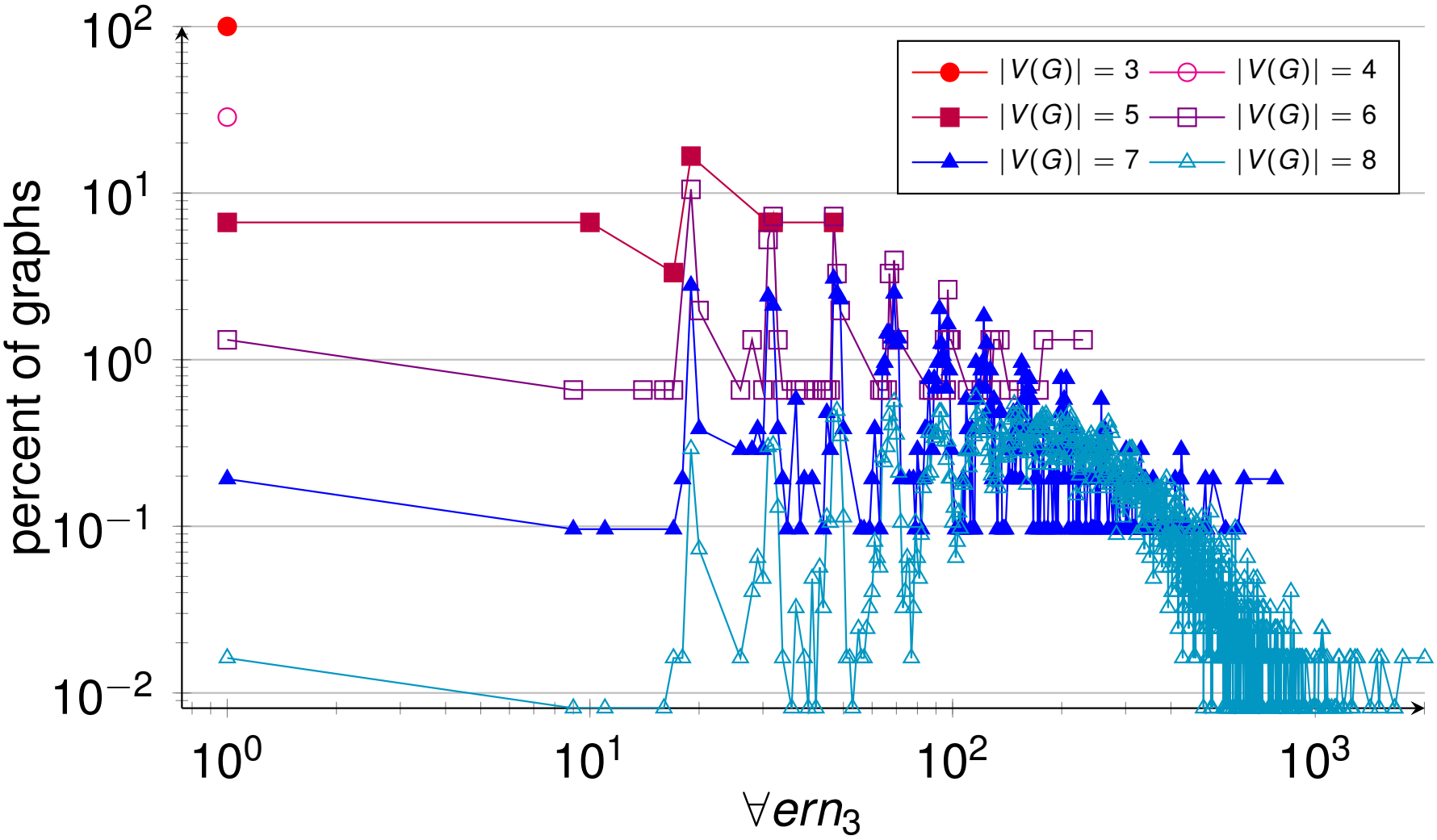

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
1
2
3
4
5
6
7
8
∃ern1
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8
|V(G)| = 9
|V(G)| = 10
|V(G)| = 11

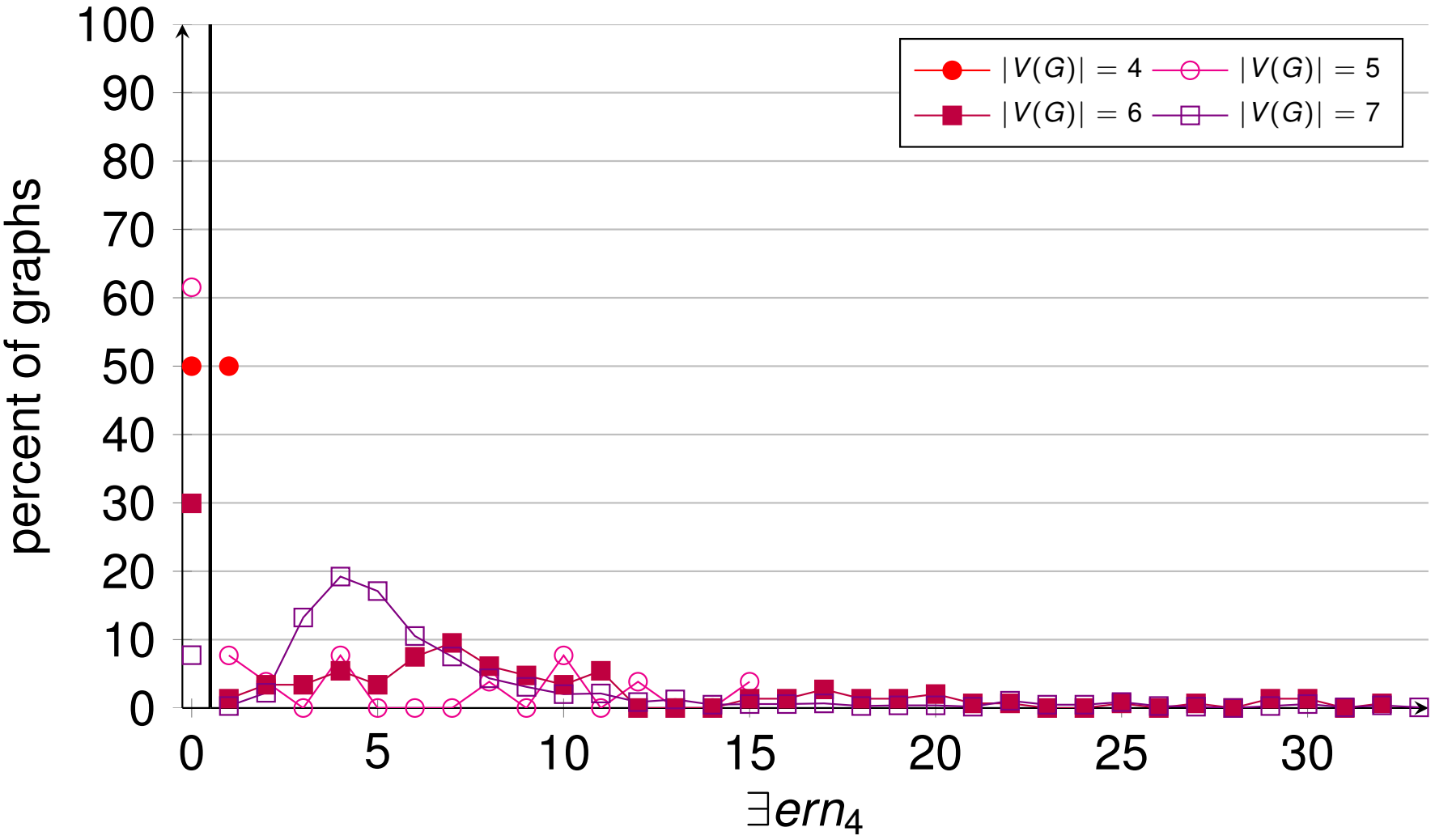

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
5
10
15
20
25
30
$\forall ern_1$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
$|V(G)| = 10$
$|V(G)| = 11$

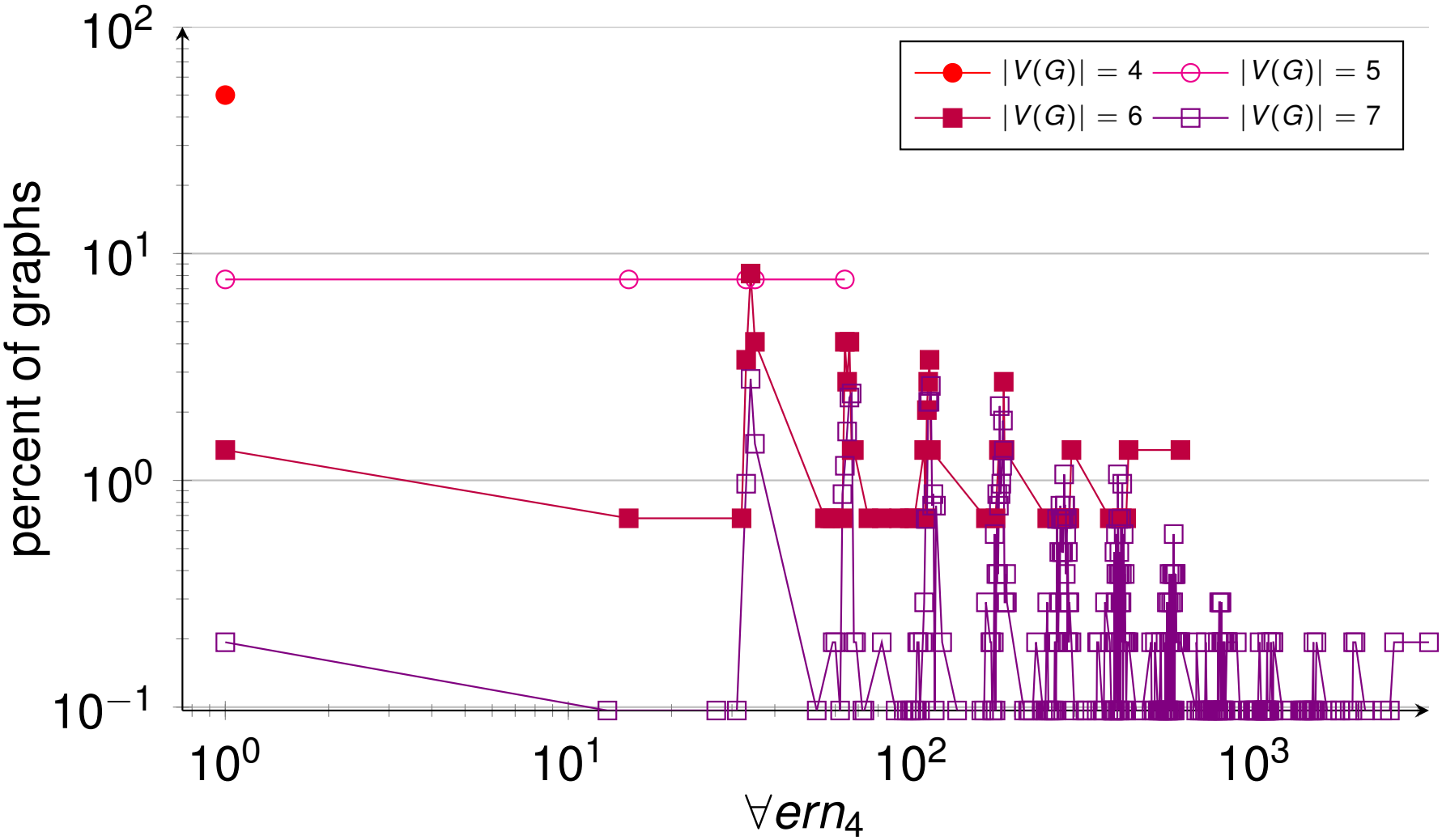

100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
2
4
6
8
10
12
∃ern2
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8
|V(G)| = 9


$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$
$|V(G)| = 9$
percent of graphs
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^0
10^1
10^2
$\forall ern_2$


100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
2
4
6
8
10
12
14
16
18
∃ern3
|V(G)| = 3
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7
|V(G)| = 8


10^2
10^1
10^0
10^{-1}
10^{-2}
percent of graphs
10^0
10^1
10^2
10^3
$\forall ern_3$
$|V(G)| = 3$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$
$|V(G)| = 8$


100
90
80
70
60
50
40
30
20
10
0
percent of graphs
0
5
10
15
20
25
30
∃ern4
|V(G)| = 4
|V(G)| = 5
|V(G)| = 6
|V(G)| = 7


10^2
10^1
10^0
10^{-1}
percent of graphs
10^0
10^1
10^2
10^3
$\forall ern_4$
$|V(G)| = 4$
$|V(G)| = 5$
$|V(G)| = 6$
$|V(G)| = 7$

Reconstruction Function Pairs

- Ordered pair of functions (f_D, f_E) which operate over two sets
  - $f_D : \mathcal{S}_1 \to \mathcal{M}(\mathcal{S}_2)$
    - takes an element of $\mathcal{S}_1$ and returns a multiset of elements from $\mathcal{S}_2$
  - $f_E : \mathcal{S}_2 \to \mathcal{P}(\mathcal{S}_1)$
    - takes an element of $\mathcal{S}_2$ and returns a set of elements from $\mathcal{S}_1$
- Must have this key property: $x_2 \in f_D(x_1) \iff x_1 \in f_E(x_2)$
  - If x_2 is in the deck of x_1, then x_1 is an extension of x_2, and vice-versa.
- Example with 2-vertex reconstruction:
  - $\mathcal{S}_1 =$ all graphs on 7 vertices
  - $\mathcal{S}_2 =$ all graphs on 5 vertices