

to be a folklore result: people either know it, or can prove it overnight; however, we could not find it written up anywhere. Thus, we decide to give a proof for the sake of future references.

Theorem 4. For every integer $r \geq 5$,

$$F^v(\underbrace{2, 2, \dots, 2}_r; r) = r + 5,$$

i.e., a smallest $(r + 1)$ -chromatic, K_r -free graph has $r + 5$ vertices.

Proof. Let G be a K_r -free graph on $r + 4$ vertices. We shall show that $\chi(G) \leq r$. Let M be a maximal matching in G^c . Since $G \not\supset K_r$, the matching M consists of at least three edges. If $|M| \geq 4$ then it is easy to properly color G with at most r colors: just assign the same color to both endpoints of each edge in M .

This takes $|M|$ colors, and the remaining $r + 4 - 2|M|$ vertices are colored each with a different color. Assume thus that M consists of exactly three edges $\{u_i, v_i\}$, $i = 1, 2, 3$. The remaining $r - 2$ vertices form an independent set I in G^c . If for some i , both u_i and v_i send an edge into I , then either there is another matching in G^c of size 4, or there is a triangle consisting of u_i, v_i and a vertex in I , and again $\chi(G) \leq r$. Let S be the subset of $V(M)$ consisting of the vertices which do not send an edge into I . Since $G \not\supset K_r$, the induced subgraph $G^c[S]$ must be a clique. Hence, for $|S| \geq 4$ we have $\chi(G) \leq (6 - |S|) + (r - 2) \leq r$. In view of the above remark, the only case left is when $S = \{v_1, v_2, v_3\}$. Then all u_1, u_2 and u_3 have the same unique neighbor w in I . (Otherwise, again, there would be a matching of size 4.) This, in turn, implies, that there is at least one edge in the induced subgraph $G^c[u_1, u_2, u_3]$. Otherwise, the set $I \setminus \{w\} \cup \{u_1, u_2, u_3\}$ would form a clique K_r in G . Assume that $\{u_1, u_2\}$ is an edge of G^c . Then a proper r -coloring of G can be obtained by coloring v_1, v_2, v_3 by one color, u_1, u_2, w by a second color, and the remaining vertices by $r - 2$ different colors. \square

Finally, in this section, we briefly mention the classical Ramsey case, when one requires a monochromatic triangle. It turns out that the Folkman number $F^v(\underbrace{3, 3, \dots, 3}_r; 2r)$, for $r \geq 4$ can be estimated quite precisely. The upper bound for this number is given in Proposition 5. The proof of the lower bound is similar but more tedious than that of Theorem 4, so we decided to omit it here.

Proposition 6. For $r \geq 4$

$$2r + 7 \leq F^v(\underbrace{3, 3, \dots, 3}_r; 2r) \leq 2r + 10.$$

Recently, Piwakowski et al. [8] showed that $F^v(3, 3; 4) = 14$ which settles the case $r = 2$. For the case $r = 3$, we know only that $11 \leq F^v(3, 3, 3; 6) \leq 20$.