CSCI-761 – Assignment 6

Zohair Hassan (zh5337)

April 2021

1. Pruning Backtracking

Every instance of CNF-SAT can be reduced to an instance of CLIQUE; we can construct a graph that represents the CNF-SAT instance, such that there must exist a clique of size k if the formula is satisfiable, where k is the number of clauses in the formula.

Keeping this in mind, we can use the graph representations of CNF-SAT formulas to construct a “bounding function”. Whenever we are checking if an assignment is valid, we can construct the graph that represents the “remaining” formula, which includes the clauses that have not yet been satisfied with variables that have not yet been assigned. We can run the greedy coloring algorithm on this graph and reject the assignment if the chromatic number achieved is smaller than the number of clauses (i.e. the clique size required).

2. Ramsey/Folkman Numbers

(a) Which of the following are true?

- $C_5 \to (3,3)^e$, $C_5 \to (3,3)^v$

  No. There are no triangles in C_5, so there can't be any monochromatic triangles in any coloring of vertices/edges of C_5.

- $C_5 \to (2,2)^v$

  Yes. $\chi(C_5) \geq 3$. Hence, every 2-coloring of C_5's vertices will include a monochromatic edge.

- $C_5 \to (2,2,2)^v$

  No. Using Nenov and Lin's Theorem: $m = 2+2+2-1-1-1+1 = 4$, so if $C_5 \to (2,2,2)^v$, $\chi(C_5)$ must be ≥ 4. However, it is known that $\chi(C_5) = 3$.

- $K_4 \to (3,3)^e$

  No. Using Nenov and Lin's Theorem: $M = R(3,3) = 6$, so if $K_4 \to (3,3)^e$, $\chi(K_4)$ must be ≥ 6. However, it is clear that $\chi(K_4) = 4$.

- $K_5 \to (3,3)^e$

  No. Using Nenov and Lin's Theorem: $M = R(3,3) = 6$, so if $K_5 \to (3,3)^e$, $\chi(K_5)$ must be ≥ 6. However, it is clear that $\chi(K_5) = 5$.

- $K_6 \to (3,3)^e$

  Yes. It is known that $R(3,3) = 6$, so by definition $K_6 \to (3,3)^e$ is true.

- $K_5 \to (3,3)^v$

  Yes. We prove by contradiction; assume that the statement is false and a coloring exists. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Consider the triangle (v_1, v_2, v_3). There must be two vertices with the same color. Assume w.l.o.g. that v_1 and v_2 have the same color. W.l.o.g. let that color be color A. Then it must be the case that v_3, v_4, and v_5 have color B, otherwise they form a monochromatic triangle with v_1 and v_2. However, in this case, (v_3, v_4, v_5) form a monochromatic triangle. Hence, no monochromatic triangle-free coloring exists.

- $K_5 \to (2,2,2)^v$

  Yes. $\chi(K_5) = 5$, so any 3-coloring will have an edge that shares the same colors.

- $K_5 \to (2,2,2,2)^v$

  Yes. $\chi(K_5) = 5$, so any 4-coloring will have an edge that shares the same colors.

- $K_5 \rightarrow (2,2,2,2,2)^v$

  No. Using Nenov and Lin's Theorem: $m = 2+2+2+2+2-1-1-1-1-1+1 = 6$, so if $K_5 \rightarrow (2,2,2,2,2)^v$, $\chi(K_5)$ must be ≥ 6. However, it is clear that $\chi(K_5) = 5$.

(b) Is is easier to prove a lower of upper bound for a Ramsey number? Is is easier to prove a lower of upper bound for a Folkman number?

It is easier to prove a lower bound for a Ramsey number; to show a lower bound, we only have to provide a valid witness coloring. To show an upper bound, we have to prove that no such coloring exists.

It is easier to prove an upper bound for Folkman numbers because to show a valid lower bound we have to prove that no coloring of the edges of all graphs below the bound can have the desired property.

(c) Prove that $k > R(s,t)$ implies $F_e(s,t;k) = R(s,t)$

$k > R(s,t)$ means that $\mathcal{F}_e(s,t;k)$ may include graphs which include cliques of size $R(s,t)$. By the definition of Ramsey numbers, $K_{R(s,t)}$ is the smallest clique such that each of its 2-edge-colorings must include K_s or K_t. Also by definition of Ramsey numbers, no smaller graph has this property. Hence, $F_e(s,t;k) = R(s,t)$.

(d) Prove that $K_3 + C_5 \rightarrow (3,3)^e$

Let $G = K_3 + C_5$. Let $V(G) = \{x,y,z,a,b,c,d,e\}$, where x,y,z are the vertices of K_3 and a,b,c,d,e are the vertices of C_5. We show by contradiction that no coloring exists for G without monochromatic triangles. Hence, we assume that a coloring exists. First, we make the following observation:

Observation 1. *Any sub-graph of C_5 with more than three vertices has at least one edge.*

This is easy to see because for any two independent vertices in C_5, a third vertex must be incident to at least one of them. Now, assume w.l.o.g. that the triangle (x,y,z) is colored such that (x,y) and (y,z) is colored A, and (x,z) is colored B. We can now consider the following cases:

Case 1. Vertex z has ≥ 3 edges colored A going to C_5. W.l.o.g. let $N = \{a,b,c\}$ be the neighbors of z such that (z,v) is colored A for each $v \in N$. Clearly, if (z,v) is colored A for any $v \in N$, then (y,v) can not be colored A, otherwise a monochromatic triangle is formed on (y,z,v). Hence, (y,v) is colored B for each $v \in N$. However, due to Observation 1, we know that at least one edge exists in N. Assume w.l.o.g. that the edge is (a,b). If (a,b) is colored A, then (z,a,b) forms a monochromatic triangle. If (a,b) is colored B, then (y,a,b) forms a monochromatic triangle.

Case 2. Vertex z has ≤ 2 edges colored A going to C_5. Clearly, z now has ≥ 3 edges colored B going to C_5. We can now make the same argument as in the previous case, but with x instead of y. For the sake of completeness, the argument is repeated below.

W.l.o.g. let $N = \{a, b, c\}$ be the neighbors of z such that (z, v) is colored B for each $v \in N$. Clearly, if (z, v) is colored B for any $v \in N$, then (x, v) can not be colored B, otherwise a monochromatic triangle is formed on (x, z, v). Hence, (y, v) is colored A for each $v \in N$. However, due to Observation 1, we know that at least one edge exists in N. Assume w.l.o.g. that the edge is (a, b). If (a, b) is colored A, then (x, a, b) forms a monochromatic triangle. If (a, b) is colored B, then (z, a, b) forms a monochromatic triangle.

In both cases we reach a contradiction. Hence, no coloring of G's edges can evade a monochromatic triangle, and the arrowing holds.

3. Progress of Folkman problems

The following four things were found in [Bik18] but not in [Woo14].

(a) In [Woo14], it is mentioned that $F_e(3, 3; 4) \geq 19$ is the best lower bound known so far. It is shown in [Bik18] that $F_e(3, 3; 4) \geq 20$.

(b) The result $32 \leq F_v(2, 2, 2, 2, 2; 3) \leq 40$ is mentioned in [Bik18], but no results relevant to this Folkman number are discussed in [Woo14].

(c) In [Woo14], the bounds for $F_v(2, 2, 2, 3; 4)$ are: $18 \leq F_v(2, 2, 2, 3; 4) \leq 30$. However, in [Bik18], it is show than $20 \leq F_v(2, 2, 2, 3; 4) \leq 22$.

(d) In [Woo14], it is only mentioned that $F_v(2, 3, 3; 4) \geq 19$, while in [Bik18], it is shown that $20 \leq F_v(2, 3, 3; 4) \leq 24$.

4. Making Graphs

(a) The Petersen Graph

The Petersen Graph has 10 vertices and 15 edges. It has 120 automorphism groups. It is triangle-free, and has a chromatic number of 3.

The Petersen Graph is the complement of the line graph[1] of K_5 [Pet]. Hence, we can use the following command to generate the Petersen Graph:

```
./genspecialg -k5 | ./linegraphg | ./complg
```

The canonical label is:

```
:I`ACWqHKhhccTF
```

(b) The Grötzsch Graph

The Grötzsch Graph is a graph on 11 vertices and 20 edges. It is triangle-free and Hamiltonian. It has a chromatic number of 4 and only has 10 automorphisms [Grt].

The graph was generated using the following `nauty` command:

```
./geng 11 20 | ./pickg - b -T0 -a10 | ./labelg
```

This specifies that the graph must not be bipartite (i.e. chromatic number is greater than 2), has no triangles, and has 10 automorphisms. The canonical label is:

```
J?AKagjXfo?
```

(c) The Clebsch Graph

The Clebsch graph is a graph on 16 vertices and 40 edges. The graph regular, Hamiltonian, and triangle-free. The 3-edge-coloring of K_{16} which contains no monochromatic triangles splits the graph into 3 copies of the Clebsch graph [Cle].

The Clebsch graph is 5-regular, and for each vertex, the non-neighbors form an isomorphic copy of the Petersen graph [Cle]. Keeping this in mind, the following graph G was constructed: $G =$ Petersen Graph $\cup$ 5-Star. Clearly, G is contained in the Clebsch graph. G has 5 edges from the star, and 15 from the Petersen Graph. So, adding 20 edges to G such that no new triangle is formed and each vertex has degree ≤ 5 will give us graphs that match the Clebsch graph description. The `dreadnaut` version of G was constructed and converted to `g6` format. Then, the following command was used to generate graphs with one edge added that do not break the desired properties:

```
./addedgeg -t -D5 | ./shortg
```

[1] A line graph $L(G) = (L_V, L_E)$ of a graph $G = (V, E)$ is a graph where $L_V = E$ and $(u, v) \in L_E$ iff edges u and v are incident on the same vertex in G

This was repeated 20 times, and the only graph obtained was the Clebsch graph, which has the following canonical label:

```
OsaBA`GP@`dIHWEcas_]O
```

(d) The Paley Graph P_{17}

P_{17} has 17 vertices and 68 edges. The Payley Graph P_{17}, like all Payley graphs, is Hamiltonian and regular. It is 8-regular.

The squares in Z_{17} were identified as $S = \{1, -1, 2, -2, 4, -4, 8, -8\}$. An adjacency list was created for vertex set $\{0, 1, 2, \ldots, 17\}$ where vertex i is adjacent to $i + j \mod 17$ for each $j \in S$. The adjacency list was converted to a `dreadnaut` file and converted to `g6` format using `dretog`. The canonical label is:

```
P}qTKukXaUja[IBjanPeMI\K
```

(e) The Schläfli Graph

The Schläfli Graph is a graph on 27 vertices and 216 edges. It is Hamiltonian, 16-regular, and has 51,840 automorphisms. It is the single witness graph for $R(J_7, J_4) = 28$, i.e. it contains no J_7's or $\overline{J_4}$'s.

According to [Sch, BN92], the following can be used to construct the Schläfli Graph:

" The Schläfli graph may also be constructed from the system of eight-dimensional vectors

$(1, 0, 0, 0, 0, 0, 1, 0)$, $(1, 0, 0, 0, 0, 0, 0, 1)$, and $(-1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$, and the 24 other vectors obtained by permuting the first six coordinates of these three vectors. These 27 vectors correspond to the vertices of the Schläfli graph; two vertices are adjacent if and only if the corresponding two vectors have 1 as their inner product. "

The vectors were constructed, and the graph was generated as described above in adjacency list format. The corresponding `dreadnaut` file was made and converted to `g6` using `dretog`. The canonical label is:

```
Z~~vnZjvUtw~nSmis{{k~a^||QBtQJNHLU[VQ^BxkFnDK\zEEvn@Tn^_Tn^w
```

The following 27 vectors were used to generate the graph:

$$
\begin{array}{r}
(1,0,0,0,0,0,1,0)\\
(0,1,0,0,0,0,1,0)\\
(0,0,1,0,0,0,1,0)\\
(0,0,0,1,0,0,1,0)\\
(0,0,0,0,1,0,1,0)\\
(0,0,0,0,0,1,1,0)\\
(1,0,0,0,0,0,0,1)\\
(0,1,0,0,0,0,0,1)\\
(0,0,1,0,0,0,0,1)\\
(0,0,0,1,0,0,0,1)\\
(0,0,0,0,1,0,0,1)\\
(0,0,0,0,0,1,0,1)\\
(-1/2,-1/2,1/2,1/2,1/2,1/2,1/2,1/2)\\
(-1/2,1/2,-1/2,1/2,1/2,1/2,1/2,1/2)\\
(-1/2,1/2,1/2,-1/2,1/2,1/2,1/2,1/2)\\
(-1/2,1/2,1/2,1/2,-1/2,1/2,1/2,1/2)\\
(-1/2,1/2,1/2,1/2,1/2,-1/2,1/2,1/2)\\
(1/2,-1/2,-1/2,1/2,1/2,1/2,1/2,1/2)\\
(1/2,-1/2,1/2,-1/2,1/2,1/2,1/2,1/2)\\
(1/2,-1/2,1/2,1/2,-1/2,1/2,1/2,1/2)\\
(1/2,-1/2,1/2,1/2,1/2,-1/2,1/2,1/2)\\
(1/2,1/2,-1/2,-1/2,1/2,1/2,1/2,1/2)\\
(1/2,1/2,-1/2,1/2,-1/2,1/2,1/2,1/2)\\
(1/2,1/2,-1/2,1/2,1/2,-1/2,1/2,1/2)\\
(1/2,1/2,1/2,-1/2,-1/2,1/2,1/2,1/2)\\
(1/2,1/2,1/2,-1/2,1/2,-1/2,1/2,1/2)\\
(1/2,1/2,1/2,1/2,-1/2,-1/2,1/2,1/2)
\end{array}
$$

References

[Bik18] Aleksandar Bikov. Computation and bounding of folkman numbers. *arXiv preprint arXiv:1806.09601*, 2018.

[BN92] Frans C Bussemaker and Arnold Neumaier. Exceptional graphs with smallest eigenvalue-2 and related problems. *Mathematics of Computation*, 59(200):583–608, 1992.

[Cle] Clebsch graph - wikipedia. `https://en.wikipedia.org/wiki/Clebsch_graph`. (Accessed on 04/07/2021).

[Grt] Grötzsch graph - wikipedia. `https://en.wikipedia.org/wiki/Gr%C3%B6tzsch_graph`. (Accessed on 04/07/2021).

[Pet] Petersen graph - wikipedia. `https://en.wikipedia.org/wiki/Petersen_graph`. (Accessed on 04/07/2021).

[Sch] Schläfli graph - wikipedia. `https://en.wikipedia.org/wiki/Schl%C3%A4fli_graph`. (Accessed on 04/08/2021).

[Woo14] Christopher A Wood. Small folkman numbers. *press. Available: http://christopher-wood. com/papers/FolkmanSurvey. pdf, last accessed September*, 22, 2014.