CSCI-761 Combinatorial Computing

Assignment 6

due April 8, 2021

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1. **Pruning Backtracking**

   CNF-SAT is a problem of finding satisfying assignments to a Boolean formula α in conjunctive normal form, i.e. where α is given by a set of clauses, each of which is a disjunction of literals, each of which is a variable or a negated variable. Suppose you are solving instances of CNF-SAT by backtracking. Describe informally a bounding function (in the spirit of bounding functions used in the maximum clique search algorithms considered in class) which could be used to prune the search space for satisfying assignments to instances of CNF-SAT.

   To follow the struggles of many researchers trying to do it better and better already for decades, explore `http://www.satcompetition.org`, which is a website of worldwide competitions for designing state-of-art SAT-solvers.

   Since each clause is a disjunction of literals, only one literal needs to be true in order for a clause to be true. In any satisfying assignment, all clauses must be true, so all clausse must have at least one literal that is true.

   Thus, we can use the number of clauses containing a true literal as a bounding function. In any complete solution this will be equal to the number of clauses, and for any partial assignment X, assigning additional variables will never falsify a clause that is currently true, so this is a true bounding function.

2. **Ramsey/Folkman numbers**

(a) Which of the following are true?
Give yes/no answer and a brief justification for each.

- $C_5 \to (3,3)^e$

  no

  there is no K_3 subgraph of C_5, so no coloring could ever produce a monochromatic K_3 in any color

- $C_5 \to (3,3)^v$

  no

  same as above

- $C_5 \to (2,2)^v$

  yes

  since C_5 is an odd cycle any 2-coloring must have two adjacent vertices of the same color, which constitutes a monochromatic K_2

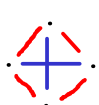
- $C_5 \to (2,2,2)^v$

  no

  a 5-cycle can be colored as e.g. (1, 2, 3, 1, 2) with no monochromatic K_2

- $K_4 \to (3,3)^e$

  no



  see left for a 2-coloring with no monochromatic triangles (also R(3,3) = 6)

- $K_5 \to (3,3)^e$

  no

  if true, R(3,3) would be <=5, but R(3,3) = 6

- $K_6 \to (3,3)^e$

  yes

  R(3,3) = 6

- $K_5 \to (3,3)^v$

  yes

  any coloring must have at least three vertices
  of the same color, which will all be connected by definition

- $K_5 \to (2,2,2)^v$

  yes

  as above, there must be at least two vertices
  of the same color

- $K_5 \to (2,2,2,2)^v$

  yes

  same as previous two

- $K_5 \to (2,2,2,2,2)^v$

  no

  all vertices can be given unique colors,
  hence no monochromatic K_2

(b) Is it easier to prove a lower or upper bound for a Ramsey number? Is it easier to prove a lower or upper bound for a Folkman number? Explain your answers.

It is easier to prove a lower bound for a Ramsey number, since a single graph on n vertices with no s-clique and no t-independent set is a witness to the fact that R(s, t) > n.

It is easier to prove an upper bound for a Folkman number, since the situation is reversed: a single graph on n vertices with no K_k-subgraph and a monochromatic K_s or a monochromatic K_t in all possible 2-colorings is a witness to the fact that F_e(s, t; k) <= n.

(c) Prove that $k > R(s,t)$ implies $F_e(s,t;k) = R(s,t)$.

Hint: It is not hard, just combine the definitions.

Let n = R(s, t). Consequentially, all 2-colorings of K_n must have a monochromatic K_s or a monochromatic K_t, and if k > n then K_n is a witness to the fact that F_e(s, t; k) <= n.

Suppose for a contradiction that F_e(s, t; k) = m < n. Then there must exist some graph on m vertices such that all 2-colorings produce a monochromatic K_s or K_t. But any such 2-coloring can be converted to a graph on m vertices that satisfies the Ramsey properties for R(s, t), which is impossible since R(s, t) = n. So F_e(s, t; k) >= n.

Thus, F_e(s, t; k) = n.

(d) Prove that $K_3 + C_5 \to (3,3)^e$.

Suppose for a contradiction that it were possible to 2-color the edges of K_3 + C_5 in such a way that there are no monochromatic triangles in either color.

Clearly, both K_3 and C_5 can be 2-colored in such a manner, but in both cases there must be at least one monochromatic K_2. Consider the neighborhoods of each vertex in K_3; without loss of generality, orient the vertices in the neighborhood such that the monochromatic K_2 comes first.

I think this is the right approach but not sure how to proceed from here...

3. **Progress on Folkman problems**

In 2014, Christopher Wood wrote a comprehensive survey of the area, at `https://www.cs.rit.edu/~spr/COURSES/CCOMP/cawfolk.pdf`. In 2018, Aleksandar Bikov defended his PhD dissertation titled "Computation and Bounding of Folkman Numbers", available at `https://arxiv.org/abs/1806.09601` (both are also linked from the item 14 on the course page). List four results which are in the latter but not in the former.

(a) F_v(3,6;7) = 18.

(b) F_v(4,4,;5) >= 19.

(c) 20 <= F_v(2,2,2,3;4) <= 22.

(d) For m >= 9, F_v(2_{m-7}, 7; m-1) = m + 11.

4. **Making Graphs**

Make nauty canonical labelings of the following five graphs. In each case describe the steps how you made them. In each case state the main combinatorial properties they have.

(a) The Petersen Graph

```
IsP@OkWHG
$ geng -l -d3 -D3 10 15:15 | pickg -z2 -Z2 # (using properties from Wikipedia)
```

It is the smallest bridgeless graph with constant degree 3 and no three-edge-coloring. It has a number of "weird" properties, such as not having a Hamiltonian cycle (despite having a Hamiltonian path) and being a "snark", that is, a bridgeless cubic graph with chromatic index 4. There is a conjecture that every snark contains the Petersen graph as a minor; according to Wikipedia it seems there is a proof but it is mostly unpublished.

(b) The Grötzsch Graph

```
J?AKagjXfo?
$ geng -lt 11 20:20 | pickg -z2 -Z2 -a10 # (using properties from Wikipedia)
```

It is the Mycielskian of C_5, and so is triangle-free, and seems to be mainly used as a counterexample to theories about graph coloring (being a member of the Mycielskian sequence used to prove that triangle-free graphs can have arbitrary chromatic number). It shares many properties with the Clebsch graph.

(c) The Clebsch Graph

```
OsaBA`GP@`dIHWEcas_]O
$ geng -lt 16 -d5D5 40:40 | pickg -z2 -Z2 -a1920 # (using properties from Wikipedia)
```

Either this graph or its complement seem to be referred to as the Clebsch graph. The complement was used as a witness to the fact that R(3,3,3) >= 17, since K_16 can be 3-colored using 3 copies of the Clebsch graph, as part of a proof that R(3,3,3) = 17. The subgraph induced by the 10 non-neighbors of any vertex is isomorphic to the Petersen graph (cool! :D)

(d) The Paley Graph P_{17}

P}qTKukXaUja[IBjanPeMI\K

For this one I tried to use geng but it was taking a long time so I wrote a short program to generate the graph using its construction and output it in g6 format.

Per Wikipedia, this graph is constructed from elements of a finite field of prime order q, such that q is a prime power and congruent to 1 (mod 4). Elements are connected by an edge if and only if they differ by a quadratic residue mod q, i.e., if their difference is congruent to some x^2 (mod q).

(e) The Schläfli Graph

Z~~vnZjvUtw~nSmis{{k~a^||QBtQJNHLU[VQ^BxkFnDK\zEEvn@Tn^_Tn^w

As above, I tried using geng but ultimately wrote a short program to construct it "by hand". For this construction I used the one given on Wikipedia where the vertices correspond to 8-dimensional vectors which are connected by edges iff their dot product is 1.

The most salient property seems to be that it is the complement of a "generalized quadrangle", which is listed on both Wolfram Mathworld and Wikipedia, but I am not very familiar with this notion or with generalized polygons in general. However, Wikipedia says that the neighborhood of every vertex is isomorphic to the complement of the Clebsch graph, which is neat.