Assignment 5

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While there is probably a way for me to draw on the provided `.pdf` file, I will be creating a LaTeX file from scratch[1] instead.

[1] I actually just copy the file from the last homework every time and delete the content to keep the header with all the packages I like

1 Pruning Backtracking

CNF-SAT is a problem of finding satisfying assignments to a Boolean formula α in conjunctive normal form, i.e. where α is given by a set of clauses, each of which is a disjunction of literals, each of which is a variable or negated variable. Suppose you are solving instances of CNF-SAT by backtracking. Describe informally a bounding function (in the spirit of bounding functions used in the maximum clique search algorithms considered in class) which could be used to prune the search space for satisfying assignment to instances of CNF-SAT.

Almost everybody is using the so called DIMACS-CNF format for encoding propositional formulas in the CNF form, see for example: `https://people.sc.fsu.edu/~jburkardt/data/cnf/cnf.html`, or `https://users.aalto.fi/~tjunttil/2021-DP-AUT/notes-sat/solving.html`.

To follow the struggles of many researchers trying to do it better and better already for decades, explore `https://www.satcompetition.org`, which is a website of worldwide competitions for designing state-of-art SAT-solvers.

Suppose we are looking at a short term. Because we're dealing with conjunctive normal form, this term must be satisfied for the entire expression to be satisfied. Short terms will only depend on a few of the variables, so we can check the term with some values and, if false, conclude that these variables cannot be these values for a satisfying solution, regardless of any of the other variables' values assigned. Thus, we can skip checking any values for the variables where we know this smaller term will be false.

2 Ramsey/Folkman numbers

(a) Which of the following are true?
Give yes/no answer and a brief justification for each.

- $C_5 \to (3,3)^e$

  No. Pentagons famously contain no triangles, so we can't have a monochromatic triangle because there aren't any triangles.

- $C_5 \to (3,3)^v$

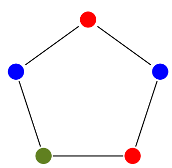
  No. Pentagons still don't contain triangles.

- $C_5 \to (2,2)^v$

  Yes! Without loss of generality, let's color a vertex the first color. The two vertices adjacent to that vertex would then be the second color. Of the two remaining vertices, they are both adjacent to vertices of the second color, so one of them is forced to be the first color. The remaining vertex is adjacent to vertices of both colors, so we are forced to have a pair of adjacent vertices of the same color.

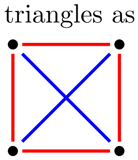
- $C_5 \to (2,2,2)^v$

  No. See the example below. No two adjacent vertices are of the same color in the coloring.

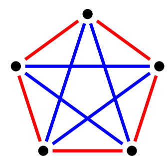
- $K_4 \to (3,3)^e$

  No. See how we can color the edges in such a way that no triangle is of one color below. There is a red square and a blue pair of disjoint edges. Squares contain no triangles, and disjoint edges contain no triangles as well.

- $K_5 \to (3,3)^e$

  No. See the edge coloring below; there are no monochromatic triangles.

  The famous counterexample.

- $K_6 \to (3,3)^e$

  Yes! Let's begin with one of the vertices of the graph. It is adjacent to five vertices, so at least three edges incident to this vertex will be of the same color. Let's suppose this color is the first color without loss of generality. These three or more edges must be incident to at least three other vertices. If the edge between any two of these vertices is the first color, that would create a triangle of those vertices and the original vertex. Thus all of the edges between the incident vertices must be of the second color. This would create a triangle in the second color, so we are forced to have a monochromatic triangle.

- $K_5 \to (3,3)^v$

  Yes! For any three vertices of K_5, a triangle is formed by the three vertices. Thus, we cannot have three vertices of the same color in the entire coloring. There are five vertices to be colored, so we must color at least three vertices the same color. This is a contradiction, so we are forced to have a monochromatic triangle in K_5.

- $K_5 \to (2,2,2)^v$

  Yes! For any two vertices of K_5, there exists an edge between the vertices. Thus, we cannot have two vertices of the same color in the entire coloring. Because there are five vertices to be colored, we must color at least two vertices the same color by the pigeon hole principle. This is a contradiction, so we must have a monochromatic pair of adjacent vertices.

- $K_5 \to (2,2,2,2)^v$

  Yes! For any two vertices of K_5, there exists an edge between the vertices. Thus, we cannot have two vertices of the same color in the entire coloring. Because there are five vertices to be colored, we must color at least two vertices the same color by the pigeon hole principle. This is a contradiction, so we must have a monochromatic pair of adjacent vertices.

- $K_5 \to (2,2,2,2,2)^v$

  No. We can color each vertex its own color, so there cannot exist any pair of vertices of the same color that are adjacent because there is no pair of vertices of the same color.

(b) Is it easier to prove a lower or upper bound for a Ramsey number?
Is it easier to prove a lower or upper bound for a Folkman number? Explain your answers.

For Ramsey numbers, a lower bound is easier because you only need to show there exists a graph with the desired properties, whereas an upper bound requires showing that every single coloring you could try will not work, a "for all" statement.

For Folkman numbers, an upper bound is easier because you only need to show there exists a graph with the desired properties, whereas a lower bound requires showing there are no smaller graphs that have the desired properties, another "for all" statement.

In general, it is easier to show that something exists than proving something for all cases because the former only involves one case where the latter involves every single case.

(c) Prove that $k > R(s,t)$ implies $F_e(s,t;k) = R(s,t)$.
Hint: It is not hard, just combine the definitions.

Let's recall those definitions. For Ramsey numbers, we're looking for the fewest vertices such that there must be a K_s subgraph or an independent set of t vertices. This independent set can also be seen as a K_t subgraph of the complement of the graph. For Folkman numbers, we've got an edge coloring with two colors where we want to find the minimum order of a graph that is K_k-free, but must have a K_s of the first color or a K_t of the second color.

Given that $k > R(s,t)$, consider $G = K_{R(s,t)}$. G cannot contain any K_k because a K_k has k vertices and G only has $R(s,t)$ vertices, which is a smaller number. If we take the pair of colors "exists" and "does not exist," we can say that there must either be a K_s subgraph of G or an independent set on t vertices, which is a K_t subgraph in the "does not exist" edge color. Thus, $F_e(s,t;k) = R(s,t)$.

(d) Prove that $K_3 + C_5 \to (3,3)^e$

Let's begin with the K_3. We cannot color all three of its edges one color because that would make a monochromatic triangle. Thus, there must be two edges of one color and one edge of another color. Let's call the first color red and the second color blue without loss of generality. Now, for the edges between the K_3 and the C_5. Let's call the vertex incident to both the red edges v_r for "red." Let's call the other two triangle vertices v_0 and v_1. For all of the edges between v_r and the vertices of the C_5, we must color the edges blue. Why? Suppose one of these edges are red. Let's call the vertex of the C_5 incident to this edge $v_\heartsuit$. What colors should the edges $\{v_\heartsuit, v_0\}$ and $\{v_\heartsuit, v_1\}$ be? They can't both be blue because that would form a blue triangle between v_0, v_1, and $v_\heartsuit$. One of them must then be red in that case, but that would form a red triangle between the vertex incident to that edge on the triangle side, $v_\heartsuit$, and v_r. Thus, all of the edges between v_r and the vertices of the C_5 must be blue. Now, for the edges of the C_5, suppose one of them is blue. This would create a triangle of the vertices incident to the blue edge and v_r. Thus, all of the edges of the C_5 must be red.

At this point, we will need to label another vertex from the C_5. Let us choose a point from the C_5 without loss of generality and call it v_α. Consider the edges $\{v_\alpha, v_0\}$ and $\{v_\alpha, v_1\}$. If both of these edges were blue, then a blue triangle would be formed between v_α, v_0, and v_1. Thus, one of these edges must be red. Without loss of generality, let's take $\{v_\alpha, v_0\}$ to be red. Because $\{v_0, v_\alpha\}$ and the edges incident to v_α of the pentagon are red, it must be the case that the edges between v_0 and the vertices adjacent to v_α of the pentagon are blue, else a red triangle would be formed. As for the edges between v_1 and the vertices of the pentagon adjacent to v_α, we must color them red because they would otherwise form a blue triangle. Of the edges between v_0 and the vertices of the pentagon not adjacent to v_α, one edge must be blue, for a red triangle between the nonadjacent pentagon vertices and v_0 would be formed. Consider the pentagonal vertex incident to one of these blue edges. What color will the edge between that vertex and v_1 be? If it were red, a red triangle would be formed between v_1, the vertex, and one of the vertices adjacent to v_α. If it were blue, a blue triangle would be formed between v_0, v_1, and the vertex. Thus, a monochromatic triangle must occur, and $K_3 + C_5 \to (3,3)^e$.

3 Progress on Folkman Problems

In 2014, Christopher Wood wrote a comprehensive survey on the area, at https://www.cs.rit.edu/~spr/COURSES/CCOMP/cawfolk.pdf. In 2018, Aleksandar Bikov defended his PhD dissertation titled "Computation and Bounding of Folkman Numbers," available at https://arxiv.org/abs/1806.09601 (both are also linked from the item 14 on the course page). List four results which are in the latter but not in the former.

(a) On page 118 of the latter, minimal graphs in $\mathcal{H}_e(3,3;13)$ are discussed.

(b) On page 123 of the latter, automorphism groups for $\mathcal{H}_e(3,3)$ minimal graphs with 10, 11, 12, and 13 vertices.

(c) On page 103 of the latter, maximal graphs in $\mathcal{H}_v(2,2,2,4;5;18)$ are discussed.

(d) On page 85 of the latter, maximal graphs in $\mathcal{H}_v(2,2,2,2,2,7;11;22)$ are discussed.

4 Making Graphs

Make nauty canonical labelings of the following graphs. In each case describe the steps how you made them.

(a) The Petersen Graph

```
:I‘ACWqHKhhccTF
```

```
genspecialg -P5,2 | labelg
```

There is just a way to make generalized Petersen graphs, so I used that with the parameters `genspecialg --help` said to use.

(b) The Grötzch Graph

```
:Ji?c@pEUPBFaGhg@CKf
```

```
genspecialg -X65
```

There is a way to make graphs with a name, too. They're even pre-labeled. "How did I find the magic number" I hear you ask? I used the following: `genspecialg --Xhelp | grep Gro`.

(c) The Clebsch Graph

```
:O`ACGPDC[QPJGYCqG\KafPK`ckeSqDsIWyn
```

```
genspecialg -X$(genspecialg --Xhelp | grep ClebschGraph | awk '{print $1}')
```

Putting the search into the command, I can now do the generation in a single line of `bash`. Well, technically I'm using `zsh`, but it's probably similar enough.

(d) The Paley Graph P_{17}

```
:P__@_@_@_B_AD_BCE_ACD`ADE`CEFH`BCDH`ABGIcDEGIJKbDFGHIKaBEFJKLaCFGHJMN
```

```
genspecialg -Y17 | labelg
```

This one is not in the handy `genspecialg` named graph thing. Why? Because `genspecialg -Y#` is for Paley graphs!

(e) The Scläfli Graph

```
:Z__@_@A_@AB_@AB_@ABD_@ACD_@ACEF_@BCDF_@BCEGH_@DEFGHI_ABCDFH
_ABCEGIK_ADEFGJKL_BDEHIJKLM_CFGHIJKLMN`ABCDFHK`ABCEGILP`ADEF
GJMPQ`BDEHIJNPQR`CFGHIJOPQRSaBDEKLMNPQRSaCFGKLMOPQRTUbCHIKLN
OPQSTUVdFHJKMNOPRSTUVWeGIJLMNOQRSTUVWX
```

```
genspecialg -X$(genspecialg --Xhelp | grep SchlaefliGraph | awk '{print $1}')
```

Back to graphs that are in the big graph list inside `genspecialg`. I just pulled the graph right out of there line the Clebsch Graph.

(f) The Nenov Graph for $F_v(2^4; K_4)$ on 11 vertices

```
JqKDmY^\bY_
```

```
genspecialg -c5 | addptg -n1 -j5 | addptg -n5 | listg -A | tail -n+3 > /tmp/nenov.g6
vim /tmp/nenov.g6
< /tmp/nenov.g6 amtog -n11 | labelg
```

What to write to `/tmp/nenov.g6`:

```
0 1 0 0 1 1 1 1 0 0 1
1 0 1 0 0 1 1 1 1 0 0
0 1 0 1 0 1 0 1 1 1 0
0 0 1 0 1 1 0 0 1 1 1
1 0 0 1 0 1 1 0 0 1 1
1 1 1 1 1 0 0 0 0 0 0
1 1 0 0 1 0 0 0 1 1 0
1 1 1 0 0 0 0 0 0 1 1
0 1 1 1 0 0 1 0 0 0 1
0 0 1 1 1 0 1 1 0 0 0
1 0 0 1 1 0 0 1 1 0 0
```

Alright, this one is a little more complicated. Let's get into it. To start off, I created the pentagon in the middle with `genspecialg -c5` and added the point in the center with `addptg -n1 -j5`. This also connects the center point to all the points of the pentagon. After that, I wasn't sure how I wanted to connect up the five remaining vertices on the outside, so I created the adjacency matrix and wrote it to a file. I edited the file manually and converted that into a graph.