2. Ramsey/Folkman Numbers

(a) Which of the following are true?

- $C_5 \to (3,3)^e$, $C_5 \to (3,3)^v$

  No. There are no triangles in C_5, so there can't be any monochromatic triangles in any coloring of vertices/edges of C_5.

- $C_5 \to (2,2)^v$

  Yes. $\chi(C_5) \geq 3$. Hence, every 2-coloring of C_5's vertices will include a monochromatic edge.

- $C_5 \to (2,2,2)^v$

  No. Using Nenov and Lin's Theorem: $m = 2+2+2-1-1-1+1 = 4$, so if $C_5 \to (2,2,2)^v$, $\chi(C_5)$ must be ≥ 4. However, it is known that $\chi(C_5) = 3$.

- $K_4 \to (3,3)^e$

  No. Using Nenov and Lin's Theorem: $M = R(3,3) = 6$, so if $K_4 \to (3,3)^e$, $\chi(K_4)$ must be ≥ 6. However, it is clear that $\chi(K_4) = 4$.

- $K_5 \to (3,3)^e$

  No. Using Nenov and Lin's Theorem: $M = R(3,3) = 6$, so if $K_5 \to (3,3)^e$, $\chi(K_5)$ must be ≥ 6. However, it is clear that $\chi(K_5) = 5$.

- $K_6 \to (3,3)^e$

  Yes. It is known that $R(3,3) = 6$, so by definition $K_6 \to (3,3)^e$ is true.

- $K_5 \to (3,3)^v$

  Yes. We prove by contradiction; assume that the statement is false and a coloring exists. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Consider the triangle (v_1, v_2, v_3). There must be two vertices with the same color. Assume w.l.o.g. that v_1 and v_2 have the same color. W.l.o.g. let that color be color A. Then it must be the case that v_3, v_4, and v_5 have color B, otherwise they form a monochromatic triangle with v_1 and v_2. However, in this case, (v_3, v_4, v_5) form a monochromatic triangle. Hence, no monochromatic triangle-free coloring exists.

- $K_5 \to (2,2,2)^v$

  Yes. $\chi(K_5) = 5$, so any 3-coloring will have an edge that shares the same colors.

- $K_5 \to (2,2,2,2)^v$

  Yes. $\chi(K_5) = 5$, so any 4-coloring will have an edge that shares the same colors.

- $K_5 \rightarrow (2,2,2,2,2)^v$

  No. Using Nenov and Lin's Theorem: $m = 2+2+2+2+2-1-1-1-1-1+1 = 6$, so if $K_5 \rightarrow (2,2,2,2,2)^v$, $\chi(K_5)$ must be ≥ 6. However, it is clear that $\chi(K_5) = 5$.

(b) Is is easier to prove a lower of upper bound for a Ramsey number? Is is easier to prove a lower of upper bound for a Folkman number?

It is easier to prove a lower bound for a Ramsey number; to show a lower bound, we only have to provide a valid witness coloring. To show an upper bound, we have to prove that no such coloring exists.

It is easier to prove an upper bound for Folkman numbers because to show a valid lower bound we have to prove that no coloring of the edges of all graphs below the bound can have the desired property.

(c) Prove that $k > R(s,t)$ implies $F_e(s,t;k) = R(s,t)$

$k > R(s,t)$ means that $\mathcal{F}_e(s,t;k)$ may include graphs which include cliques of size $R(s,t)$. By the definition of Ramsey numbers, $K_{R(s,t)}$ is the smallest clique such that each of its 2-edge-colorings must include K_s or K_t. Also by definition of Ramsey numbers, no smaller graph has this property. Hence, $F_e(s,t;k) = R(s,t)$.

(d) Prove that $K_3 + C_5 \rightarrow (3,3)^e$

Let $G = K_3 + C_5$. Let $V(G) = \{x,y,z,a,b,c,d,e\}$, where x,y,z are the vertices of K_3 and a,b,c,d,e are the vertices of C_5. We show by contradiction that no coloring exists for G without monochromatic triangles. Hence, we assume that a coloring exists. First, we make the following observation:

Observation 1. *Any sub-graph of C_5 with more than three vertices has at least one edge.*

This is easy to see because for any two independent vertices in C_5, a third vertex must be incident to at least one of them. Now, assume w.l.o.g. that the triangle (x,y,z) is colored such that (x,y) and (y,z) is colored A, and (x,z) is colored B. We can now consider the following cases:

Case 1. Vertex z has ≥ 3 edges colored A going to C_5. W.l.o.g. let $N = \{a,b,c\}$ be the neighbors of z such that (z,v) is colored A for each $v \in N$. Clearly, if (z,v) is colored A for any $v \in N$, then (y,v) can not be colored A, otherwise a monochromatic triangle is formed on (y,z,v). Hence, (y,v) is colored B for each $v \in N$. However, due to Observation 1, we know that at least one edge exists in N. Assume w.l.o.g. that the edge is (a,b). If (a,b) is colored A, then (z,a,b) forms a monochromatic triangle. If (a,b) is colored B, then (y,a,b) forms a monochromatic triangle.

Case 2. Vertex z has ≤ 2 edges colored A going to C_5. Clearly, z now has ≥ 3 edges colored B going to C_5. We can now make the same argument as in the previous case, but with x instead of y. For the sake of completeness, the argument is repeated below.

W.l.o.g. let $N = \{a, b, c\}$ be the neighbors of z such that (z, v) is colored B for each $v \in N$. Clearly, if (z, v) is colored B for any $v \in N$, then (x, v) can not be colored B, otherwise a monochromatic triangle is formed on (x, z, v). Hence, (y, v) is colored A for each $v \in N$. However, due to Observation 1, we know that at least one edge exists in N. Assume w.l.o.g. that the edge is (a, b). If (a, b) is colored A, then (x, a, b) forms a monochromatic triangle. If (a, b) is colored B, then (z, a, b) forms a monochromatic triangle.

In both cases we reach a contradiction. Hence, no coloring of G's edges can evade a monochromatic triangle, and the arrowing holds.