Note

On Edgewise 2-Colored Graphs with Monochromatic Triangles and Containing No Complete Hexagon

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The following question was raised by Erdős and Hajnal [1] recently: Construct a graph $G$ which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color. It is easily checked that $G$ must have more than 7 vertices. In this note we present such a graph $G$ with 8 vertices.

Let $G$ denote the graph formed from the complete graph on the vertices \{1, 2, ..., 8\} by removing the 5 edges \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}. Assume that the edges of $G$ can be partitioned into two sets $A$ and $B$ such that neither set contains a triangle. We can further assume that \{6, 7\} $\in A$, \{7, 8\} $\in A$, and \{6, 8\} $\in B$. Thus, for $x \in \{1, 2, 3, 4, 5\}$ we must have \{7, x\} $\in B$ since otherwise \{7, x\} $\in A$ implies either at least one of \{6, x\}, \{8, x\} $\in A$ (forming a triangle in $A$) or both \{6, x\} $\in B$, \{8, x\} $\in B$ (forming a triangle in $B$). This forces all the edges \{1, 3\}, \{3, 5\}, \{5, 2\}, \{2, 4\}, \{4, 1\} $\in A$. Now for any three distinct points $x, y, z \in \{1, 2, 3, 4, 5\}$ we cannot have \{6, x\} $\in A$, \{6, y\} $\in A$, and \{6, z\} $\in A$ since some pair \{x, y\}, \{x, z\}, \{y, z\} is an edge of $G$ in $A$. Hence there must exist at least three distinct points $a, b, c \in \{1, 2, 3, 4, 5\}$ such that \{6, a\} $\in B$, \{6, b\} $\in B$, \{6, c\} $\in B$. A similar argument applied to vertex 8 forces the existence of distinct points $a', b', c' \in \{1, 2, 3, 4, 5\}$ such that \{8, a'\} $\in B$, \{8, b'\} $\in B$, \{8, c'\} $\in B$. But there must exist $w \in \{a, b, c\} \cap \{a', b', c'\}$ and the triangle with vertices \{6, 8, w\} is in $B$ which is a contradiction. $G$ clearly does not contain a complete hexagon and the proof is complete.

To the best of the author’s knowledge, the first example of a graph satisfying the conditions of Erdős and Hajnal was given by J. H. van Lint; subsequently L. Pósa showed the existence of such a graph containing no complete pentagon and Jon Folkman constructed such a graph containing no complete quadrilateral (all unpublished).

Reference