Some Edge Folkman Numbers

Ramsey arrowing of triangles

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Erdős and Hajnal

Research Problem 2-5, JCT 2, p. 105, 1967

Construct a graph G which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color.

done by R.L. Graham in 1968

The proposers expect that for every cardinal m there is a graph G which contains no complete quadrilateral such that for every coloring of the edges by m colors there is a triangle all of whose edges have the same color.

proved for $m = 2$ by Folkman in 1970
proved in general by Nešetřil and Rödl in 1976

History of the Most Wanted Folkman Number

What is the smallest order n of a K_4-free graph
which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	– 10^{10}?	Erdős offers \$100 for proof
1986	– 8×10^{11}	Frankl-Rödl (almost won)
1988	– 3×10^{9}	Spencer
1998	– 10^{6}?	Chung-Graham offer \$100 for the answer
1999	16 –	Piwakowski-R-Urbański (implicit)
2007	19 –	R-Xu
2008	– 9697	Lu
2008	– 941	Dudek-Rödl
2012	– 786	Lange-R-Xu
2012	– 100?	Graham offers \$100 for proof
2014	– 127?	working hard ...

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \to (s, t)$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \to (G, H)$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2

Edge Folkman graphs
$\mathcal{F}_e(s, t; k) = \{F \mid F \to (s, t),\ K_k \not\subseteq F\}$

Edge Folkman numbers
$F_e(s, t; k) =$ the smallest order of graphs in $\mathcal{F}_e(s, t; k)$

on slide 2 we discussed $F_e(3, 3; 4)$

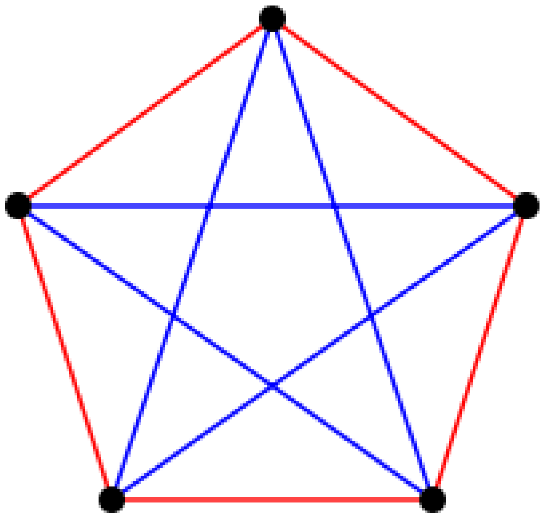
Theorem (Folkman 1970)
If $k > \max(s, t)$, then $F_e(s, t; k)$ and $F_v(s, t; k)$ exist.

from now, all arrowing is edge-arrowing unless specified as vertex-arrowing

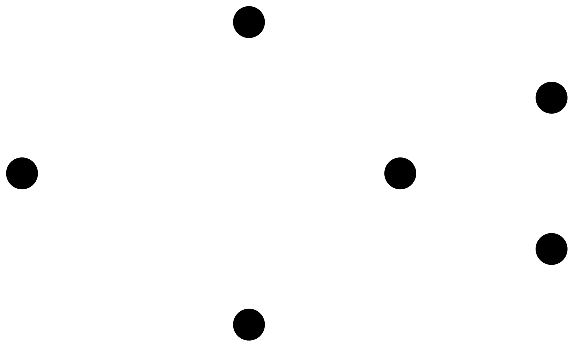
Classics

$R(3,3) = 6$

$K_5 \not\rightarrow (3,3)$



$K_6 \rightarrow (3,3)$



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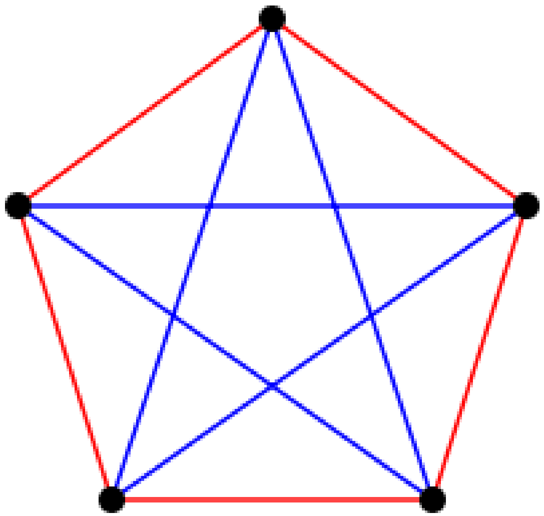

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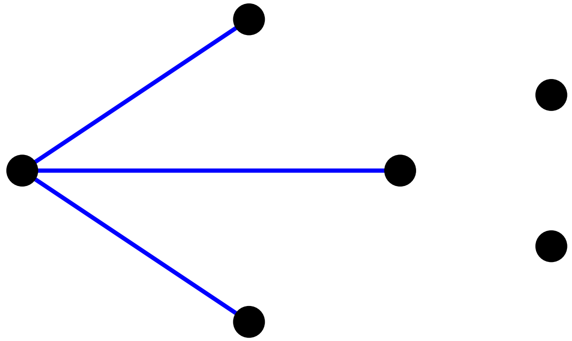

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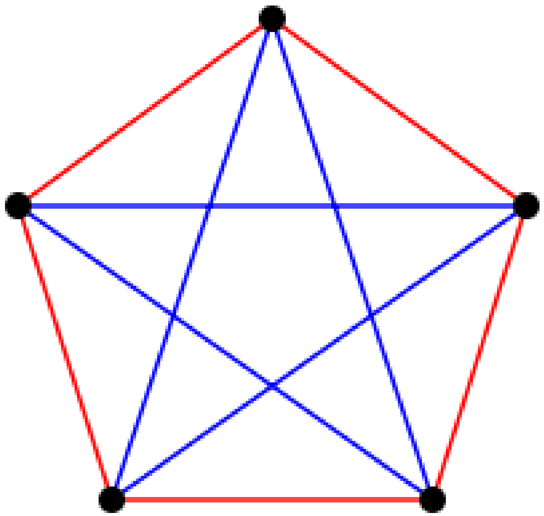

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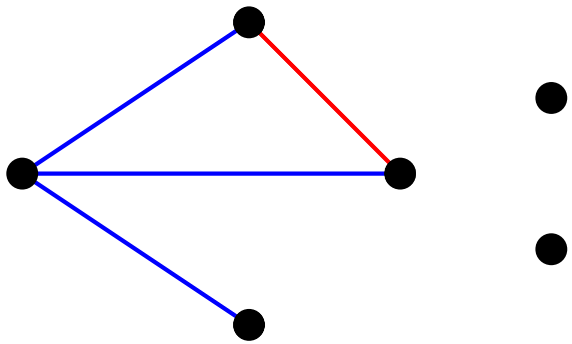

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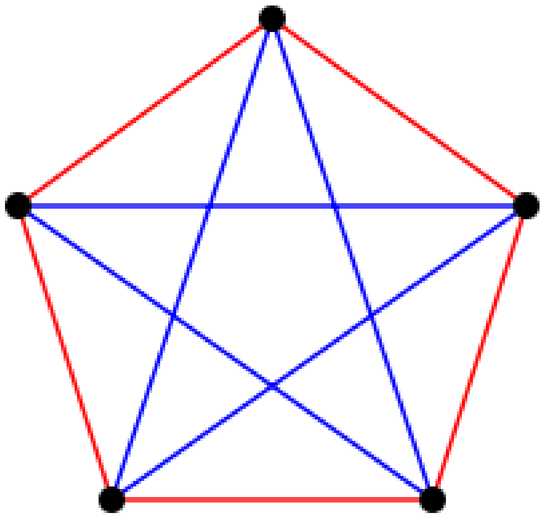

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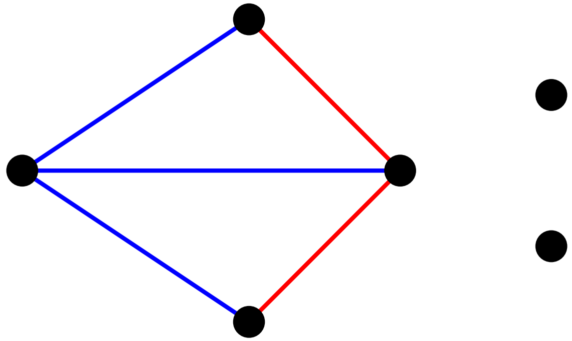

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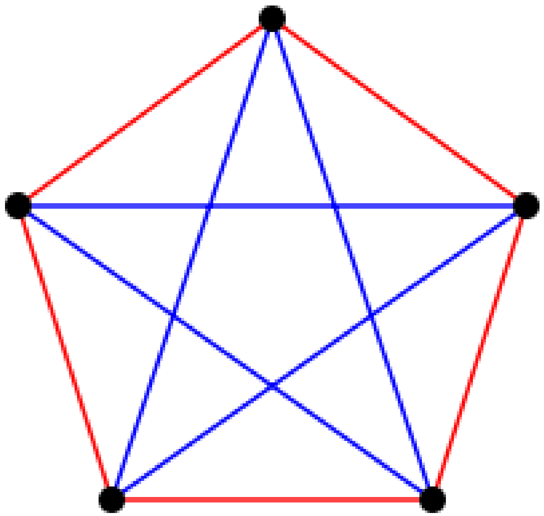
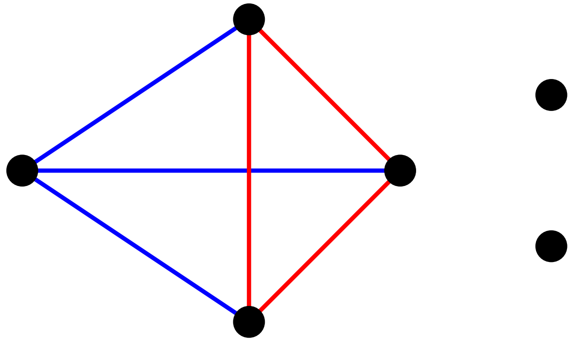




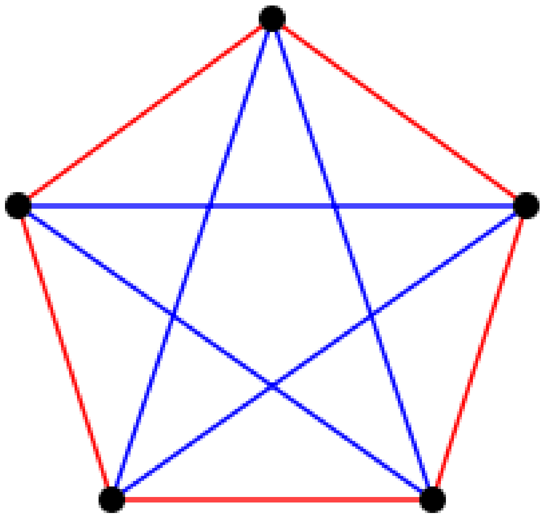
What if we want F to be K_6-free? (i.e. in $\mathcal{F}_e(3,3;6)$)

- Graham 1968: $K_6 \not\subseteq K_8 - C_5 = C_5 + K_3 \rightarrow (3,3)$

Classics

$R(3,3) = 6$

$K_5 \not\to (3,3)$ $\qquad$ $K_6 \to (3,3)$



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$F_e(3,3;k)$

k	$F_e(3,3;k)$	graphs	who
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham 1968
5	15	659 graphs	Piwakowski-R-Urbański 1999
4	$19 - 786$	L_{786}	R-Xu 2007, Lange-R-Xu 2013

$k > R(s,t) \implies F_e(s,t;k) = R(s,t)$

$k \leq R(s,t)$, very little known in general

Asymptotics for Edges

Rödl, Ruciński and Schacht, 2014

Theorem 1. For all $r \geq 2$ and large k

$$f(k;r) = F_e(k_r; k+1) \leq 2^{O(k^4 \log k + k^3 r \log r)}.$$

Theorem 3. For all $0 < \alpha < 1/4$ and large $k \leq \alpha l$

$$f(k,l) = F_e(k,k;l) \leq 2^{4k/(1-4\alpha)}.$$

Challenge. Obtain any reasonable lower bound.

Asymptotics for Vertices

Dudek and Rödl, 2010

Theorem 1. For all $r \geq 2$ there exists c_r, such that for all k

$$F_v(k_r; k+1) \leq c_r n^2 \log^4 k.$$

Theorem 2. For all $r \geq 2$ and arbitrarily small $\epsilon > 0$, there exists $c = c(r, \epsilon)$, such that for all k

$$F_v(k_r; \lceil (2+\epsilon)k \rceil) \leq ck.$$

Some facts on $F_e(s,t;k)$

- $G \in \mathcal{F}_e(s,t;k) \quad \Rightarrow \quad \chi(G) \geq R(s,t)$
  easy, no k in the bound!

- $\mathcal{F}_e(s,t;k) = R(s,t)$ for $k > R(s,t)$ easy
- $\mathcal{F}_e(s,t;R(s,t)) = R(s,t) + c$ so, so
  in most cases c is small (2, 4, 5)
- $\mathcal{F}_e(s,t;k) \geq R(s,t) + 4$ for $k < R(s,t)$ hard

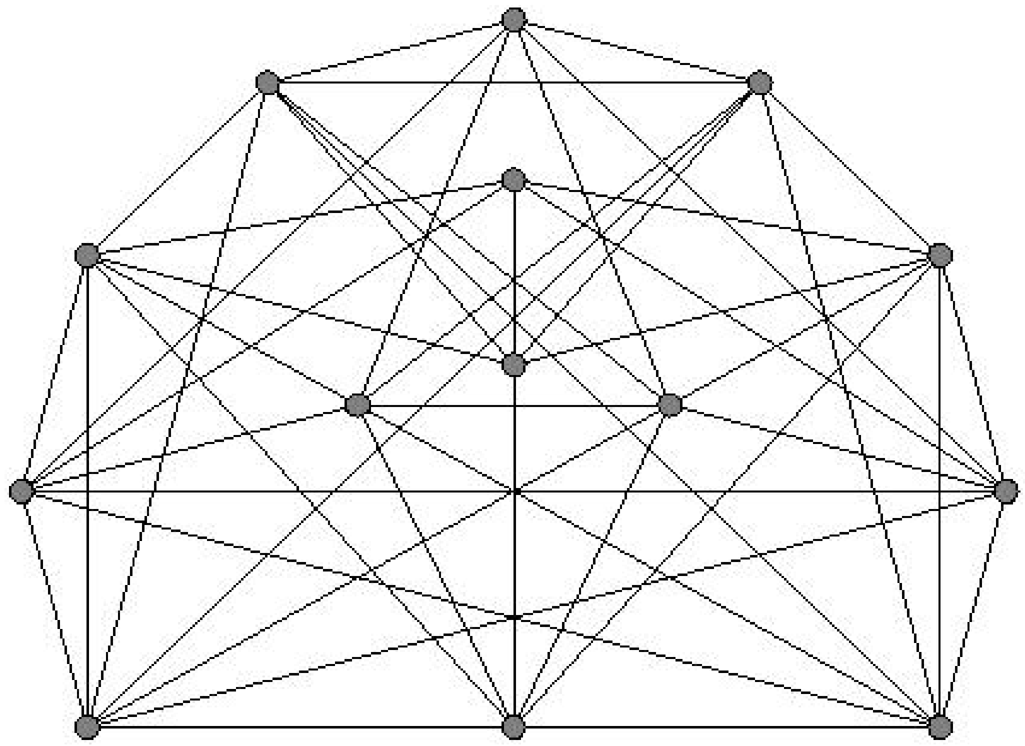
- $G \in \mathcal{F}_v(R(s-1,t), R(s,t-1); k-1) \Rightarrow G + x \in \mathcal{F}_e(s,t;k)$

  or equivalently

  $G + x \not\rightarrow (s,t)^e \Rightarrow G \not\rightarrow (R(s-1,t), R(s,t-1))^v$,
  and clearly $cl(G+x) = cl(G) + 1$

$F_e(3,3;5) = 15$, and $F_v(3,3;4) = 14$

$G + x \rightarrow (3,3)^e$, and $G \rightarrow (3,3)^v$



unique 14-vertex bicritical $F_v(3,3;4)$-graph G [PRU 1999]

Arrowing is Hard

more difficult for edges than vertices

Complexity

- Determining if $F \rightarrow (3,3)^e$ is **coNP**-complete, Burr 1976
- Testing $F \rightarrow (G,H)^e$ is Π_2^{P}-complete, Schaefer 2001

Leads to Ramsey numbers

$R(s,t) = \min\{n \mid K_n \rightarrow (s,t)\}$

$$43 \leq R(5,5) \leq 49: \quad K_{42} \not\rightarrow (5,5)^e, K_{49} \rightarrow (5,5)^e$$

Arrowing triangles reduces to 3-SAT (actually, 3-NAE-SAT)
For all (edge) triangles xyz, we add the clauses
$(x \vee y \vee z)$ and $(\bar{x} \vee \bar{y} \vee \bar{z})$ to ϕ. Then

$$G \not\rightarrow (3,3)^e \quad \text{iff} \quad \phi(G) \text{ is satisfiable}$$

Counting Triangles

in order to deduce $F \to (3,3)^e$

For any blue-red edge coloring of graph G, let

- $T_{\text{BB}}(v)$, $T_{\text{RR}}(v)$, and $T_{\text{BR}}(v)$ count triangles vuw where $\{v,u\}$ and $\{v,w\}$ are blue-blue, red-red, and blue-red
- T_{blue}, T_{red}, and $T_{\text{blue-red}}$ count the number of all blue, red and blue-red triangles

Then

- $\sum_{v \in V(G)} T_{\text{BR}}(v) = 2T_{\text{blue-red}}$
- $\sum_{v \in V(G)} \left(T_{\text{BB}}(v) + T_{\text{RR}}(v)\right) = 3(T_{\text{blue}} + T_{\text{red}}) + T_{\text{blue-red}}$

Counting Triangles

in order to deduce $F \rightarrow (3,3)^e$

This leads to

$$\sum_{v \in V(G)} T_{\text{BR}}(v) = \left(2 \sum_{v \in V(G)} T_{\text{BB}}(v) + T_{\text{RR}}(v) \right) - 6(T_{\text{blue}} + T_{\text{red}}).$$

$G \rightarrow (3,3)$ iff $T_{\text{blue}} + T_{\text{red}} > 0$ for every coloring

so,

$G \rightarrow (3,3)$ iff for every coloring

$$\sum_{v \in V(G)} T_{\text{BR}}(v) < 2 \sum_{v \in V(G)} \left(T_{\text{BB}}(v) + T_{\text{RR}}(v) \right) \tag{1}$$

From Arrowing to MAX-CUT

only edge-arrowing for a while

A *cut* of G is a bipartition of its vertices, $S \subseteq V(G)$, $\overline{S} = V(G) \setminus S$,

the *size* of a cut is $|\{ \{u, v\} \in E(G) \mid u \in S,\ v \in \overline{S} \}|$,

let $MC(G)$ be the **maximum cut** size of G.

Theorem: (Frankl-Rödl 1986, Spencer 1988)
If

$$\sum_{v \in V(G)} MC\left(G\left[N(v)\right]\right) < \frac{2}{3} \sum_{v \in V(G)} |\left(G\left[N(v)\right]\right)|,$$

then $G \rightarrow (3, 3)$.

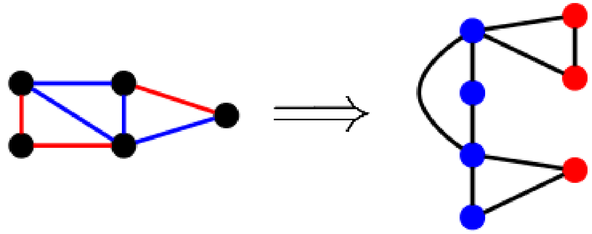
From Arrowing to MAX-CUT

another way to count

Define graph H

$$V(H) = E(G),$$
$$E(H) = \{\{e,f\} \mid e,f \in E(G),\ efg \text{ is a } \triangle \text{ in } G \text{ for some edge } g\}.$$

Then $|E(H)| = 3t_{\triangle}(G)$



$4 \leq MC(H) \leq 6,\ t_{\triangle}(G) = 3$

Theorem: (Dudek-Rödl 2008)

$$G \to (3,3) \quad \text{iff} \quad MC(H) < 2t_{\triangle}(G) \tag{2}$$

MAX-CUT Problem

MAX-CUT(H, k)
For graph H and integer k, is there a cut of H whose size is at least k?

- One of Karp's original **NP**-complete problems (Karp 1972)

Dudek-Rödl theorem gives:

$$G \to (3,3) \quad \text{iff} \quad \textbf{MAX-CUT}\,(H, 2t_{\triangle}(G)) = \text{NO}$$

We will approximate the upper bound to show arrowing.

Sahni-Gonzalez 1976

$\frac{1}{2}$-approximation algorithm for MAX-CUT

- Main idea:
  1. Pick two vertices and place one in S and one in $\overline{S}$
  2. Iterate through all remaining vertices, placing them in whichever set maximizes the current cut
- Can be extended to a $\frac{1}{k}$-approximation for k sets
- P-time, greedy

Minimum Eigenvalue Method

MAX-CUT via eigenvalues

Proposition (Alon 1996)

$$MC(H) \leq \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}|V(H)|}{4}$$

Proof. For graph H, let

- λ_{min} = smallest eigenvalue of A, the adjacency matrix of H,
- $V(H) = \{1, 2, \ldots, n\}$,
- $\mathbf{x} = (x_1, \ldots, x_n)$, $x_i \in \{-1, 1\}$.

Then

$$\sum_{\{i,j\} \in E(H)} (x_i - x_j)^2 = \sum_{i=1}^{n} d_i x_i^2 - \sum_{i \neq j} a_{ij} x_i x_j$$
$$= 2|E(H)| - \mathbf{x}^T A \mathbf{x} \ \ldots$$

Dudek-Rödl Technique, 2008

arrowing via maxcut via eigenvalues

1. For graph G, construct graph H where $E(G) = V(H)$,
   $E(H) = \{\{e,f\} \mid e,f \in E(G),\ efg$ is a $\triangle$ in G for some edge $g\}$
2. Let

$$\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}(H)|V(H)|}{4},$$
$$\beta = 2t_{\triangle}(G)$$

3. If $\alpha < \beta$, then $G \to (3,3)$

Example

Proving that $K_6 \to (3,3)$

Construct graph H from K_6

- $|E(K_6)| = |V(H)| = 15$
- $t_\triangle(K_6) = 20 \implies |E(H)| = 60$

Compute $\lambda_{\text{min}}(H) = -2$. Then

- $\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}(H)|V(H)|}{4} = 37.5$,
- $\beta = 2t_\triangle(G) = 40$.

Since $\alpha < \beta$, then $K_6 \to (3,3)$

$F_e(3,3;4) \leq 941$

Dudek-Rödl 2008

Define circulant graph $G(n,r)$ by

- $V(G) = \mathbb{Z}_n$
- $E(G) = \{\{x,y\} \mid x - y = s^r \bmod n,\ \ 0 \neq s \in \mathbb{Z}_n\}$

Closeness $\rho = \frac{\alpha-\beta}{\alpha}$

n	r	ρ
127	3	0.0309
281	4	0.0423
457	4	0.0304
571	5	0.0441
701	5	0.0295
937	6	0.0485
941	**5**	**-0.0127**

Lange-R-Xu 2013:

A subgraph with 860 vertices yields $\rho = -0.000056$

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Example

trying to prove that $C_5 + K_4 \to (3,3)$

Let $G = C_5 + K_4$

- Obtain $V(H)$:

$$|E(G)| = 31 = |V(H)|$$

- Obtain $E(H)$:

$$t_\triangle(G) = 54 \implies |E(H)| = 162$$

Compute:

- $\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}(H)|V(H)|}{4} > 108$
- $\beta = 2t_\triangle(G) = 108$

$\alpha \not< \beta$, so this does not imply $C_5 + K_4 \to (3,3)$

Goemans-Williamson Method, 1995

approximation algorithm for MAX-CUT via SDP

- Randomized approximation algorithm
- Expected value is at least $\alpha_{\text{GW}} \approx .87856$ times the optimal value
  - First improvement on the $1/2$ constant from Sahni-Gonzales
- Relaxes the problem to a semidefinite program
  - Novel use of semidefinite programming in approximation algorithms
- Khot, Kindler and Mossel (2005): Assuming the Unique Games Conjecture and $P \neq NP$, Goemans-Williamson approximation algorithm is optimal

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Goemans-Williamson Method

main idea

$V = \{1, \ldots, n\}$, weights $w_{ij} \geq 0$ (no edge $w_{ij} = 0$),
write $MC(G)$ as the integer quadratic program

$$\text{Maximize} \quad \frac{1}{2} \sum_{i<j} w_{ij}(1 - y_i y_j) \tag{3}$$

$$\text{subject to:} \quad y_i \in \{-1, 1\} \quad \forall\, i \in V$$

Cut $S = \{i \mid y_i = 1\}$.

Goemans-Williamson Method

main idea

Relax (3), extend it to a larger space:

- think of y_i as a restriction to a single dimension
- extend y_i to $\mathbf{v}_i \in \mathbb{R}^n$ such that $\|\mathbf{v}_i\| = 1$,
- replace $y_i y_j$ with $y_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$,
- for matrix $Y = X^T X$, let $y_{ii} = 1$ and the i-th column of $X = \mathbf{v}_i$.

New semidefinite program for symmetric matrix Y:

$$\text{Maximize} \quad \frac{1}{2}\sum_{i<j} w_{ij}(1 - y_{ij}) \tag{4}$$

$$\text{subject to:} \quad y_{ii} = \|\mathbf{v}_i\| = 1 \quad \forall\, i \in V$$

$$Y \succeq 0$$

Goemans-Williamson Method

the algorithm

1. Solve (4) using an SDP solver (this is all we need)
2. Decompose solution Y into $X^T X$ where $X = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ using Cholesky decomposition
3. Choose random, uniformly distributed vector $\mathbf{r}$
4. $S = \{i \mid \mathbf{v}_i \cdot \mathbf{r} \geq 0\}$

Back to our example

$K_4 + C_5 \to (3,3)$?

Recall:

- $\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}(H)|V(H)|}{4} > 108$
- $\beta = 2t_{\triangle}(G) = 108$

The SDP solution gives an upper bound of **104**
Therefore, $K_4 + C_5 \to (3,3)$

Note: the actual MAX-CUT is 102

$F_e(3,3;4) \leq 786$

construction and arrowing

Define graph $L(n,s)$:

- $V\big(\ L(n,s)\ \big) = \mathbb{Z}_n$
- $E\big(\ L(n,s)\ \big) = \big\{(u,v) \mid u \neq v$ and $u - v \equiv s^i \mod n$ for some $i \in \{0,1,2,\ldots,m-1\}\big\}$, where m is the smallest positive integer such that $s^m \equiv 1 \mod n$.

Let L_{786} be $L(785, 53)$ with one additional vertex of degree 60,

`SDPLR-MC`, `SBmethod`, and `SpeeDP` all give bound at most 857753,

$$MC\left(H\left(L_{786}\right)\right) \leq 857753 < 2t_{\triangle}(L_{786}) = 857762.$$

Therefore, $L_{786} \rightarrow (3,3)$ and $F_e(3,3;4) \leq 786$.

Moving Forward

more techniques and problems

Minimum Eigenvalue vs. Goemans-Williamson

- Experiments show that SDP often provides better bounds
- However, MATLAB's `eigs` can handle larger instances easier
- Both can fail easy instances (like all $F_e(3,3;5)$ graphs)

	MinEigs	SDP
K_6	Pass	Pass
$K_3 + C_5$	Fail	Fail
$K_4 + C_5$	Fail	Pass

Other MAX-CUT Methods

- Directly solve integer program
- Rendl, Rinaldi, Wiegele: *Solving Max-Cut to Optimality by Intersecting Semidefinite and Polyhedral Relaxations*

G_{127}

Hill-Irving 1982

$G_{127} = (\mathcal{Z}_{127}, E)$
$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$

Ramsey $(4, 12)$-graph, a color in a $(4, 4, 4; 127)$-coloring
Exoo asked if $G_{127} \rightarrow (3, 3)^e$

- 127 vertices, 2667 edges, 9779 triangles
- no K_4's, independence number 11, regular of degree 42
- vertex- and edge-transitive
- 5334 $(= 127 * 42)$ automorphisms
- $(127, 42, 11, \{14, 16\})$ - regularity
- K_{127} can be partitioned into three G_{127}'s

Reducing $\{G \mid G \not\to (3,3)^e\}$ to 3-SAT

edges in $G \longmapsto$ variables of ϕ_G
each (edge)-triangle xyz in $G \longmapsto$ add to ϕ_G

$$(x + y + z) \wedge (\bar{x} + \bar{y} + \bar{z})$$

Clearly,

$$G \not\to (3,3)^e \iff \phi_G \quad \text{is} \quad \text{satisfiable}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 clauses,
2 for each of the 9779 triangles.

Note: By taking only positive clauses,
we get a reduction to NAE-3-SAT with 9779 clauses.

$G_{127} \to (3,3)^e$?

zChaff, picosat experiments on $\phi_{G_{127}}$

- Pick $H = G_{127}[S]$ on $m = |S|$ vertices.
  Use a SAT-solver to split H:
  - $m \leq 80$, H easily splittable
  - $m \approx 83$, phase transition ?
  - $m \geq 86$, splitting H is very difficult
- $\#(clauses)/\#(variables) = 7.483$ for G_{127},
  far above conjectured phase transition ratio $r \approx 4.2$ for 3-SAT.

Moving Forward

$F_e(3,3;4) \leq 100$?

Ronald Graham $\$100$ Challenge (2012):
Determine whether $F_e(3,3;4) \leq 100$

Conjecture (Exoo):

- $G_{127} = G(127,3) \to (3,3)$, moreover
- Removing 33 vertices from G_{127} (3 indsets of 11) gives a G_{94} which still looks good for arrowing, if so, worth $100

History of lower bounds on $F_e(3,3;4)$

- $10 \le F_e(3,3;4)$ Lin 1972
- $16 \le F_e(3,3;4)$ Piwakowski-Urbański-R 1999
  since $F_e(3,3;5) = 15$, all graphs in $\mathcal{F}_e(3,3;5)$ on 15 vertices are known, and all of them contain K_4's
- $19 \le F_e(3,3;4)$ R-Xu 2007
  $18 \le F_e(3,3;4)$ proof "by hand"
- ANY proof technique improving on 19 very likely will be of interest

Moving Forward

some cases to work on

Improve over $19 \leq F_e(3,3;4) \leq 786$

Improve over $19 \leq F_e(K_4 - e, K_4 - e; 4) \leq 30193$

Find $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$

Don't work on $F_e(3,3;K_4 - e)$

$F_e(3,3,3;k) = 17, 19, 21, 23, 25$ for $k = 18, ..., 14$
$F_e(3,3,3;13) \leq 30$ since (6-join of C_5) $\rightarrow (3,3)^e$, Kolev 2011
$F_e(3,3,3;4) \leq 3^{3^4}$? Dudek-Frankl-Rödl 2010

$F_v(2_r; k)$

includes many easier and moderate cases

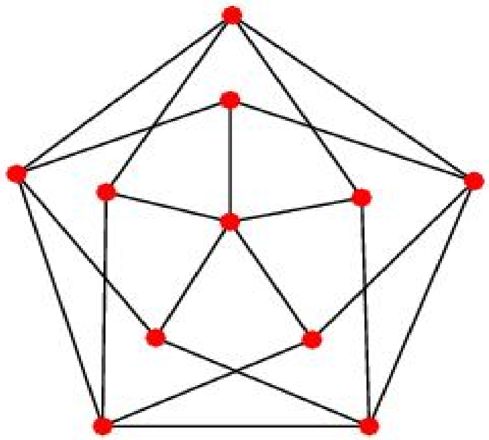
$F_v(2_r; k)$
is the order of the smallest K_k-free graph
with chromatic number larger than r.

This, and many other similar questions, summary of what we know, and what people are looking for, are collected in:

New dynamic survey *Small Folkman Numbers*
by Christopher Wood, 2014, to appear soon!

Vertex Folkman numbers pearls

$F_v(2, 2, 2; 3) = 11$
the smallest 4-chromatic triangle-free graph



Grőtzsch graph [mathworld.wolfram.com]

$F_v(2, 2, 2, 2; 3) = 22$, Jensen-Royle 1995
the smallest 5-chromatic triangle-free graph has 22 vertices

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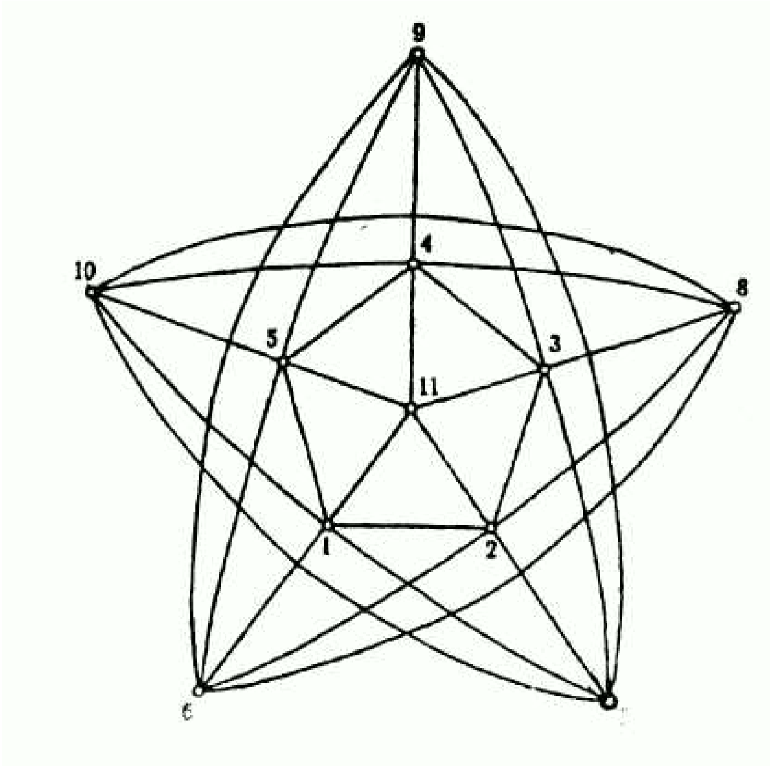
$F_v(2,2,2,2;4) = 11$, Nenov 1984, also 1993
the smallest 5-chromatic K_4-free graph has 11 vertices



$17 \leq F_v(4,4;5) \leq 23$, Xu-Luo-Shao 2010

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Theorem (ancient folklore + ŁRU 2001)
$F_v(\underbrace{2, \cdots, 2}_{r}; r) = r + 5$, for $r \geq 5$.

Proof. For the upper bound consider
as the critical graph $K_{r-5} + C_5 + C_5$
for the lower bound take any
K_r−free graph G on $r + 4$ vertices, then
assemble matchings in $\overline{G}$ to show $\chi(G) \leq r$ ∎

Theorem (Nenov 2003)
$F_v(\underbrace{3, \cdots, 3}_{r}; 2r) = 2r + 7$, for $r \geq 3$.

For $r = 2$, a small but hard case, $F_v(3, 3; 4) = 14$ (PRU 1999)

Some references

- Many by Dudek, Rödl, Ruciński, Soifer, and others ...

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  in preparation, Dynamic Survey, revision #0

Thanks for listening!