The Resolution of Keller's Conjecture

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John Mackey (CMU)   **David Narváez (RIT)**

Topics in Advanced Algorithms, Spring 2021

Overview

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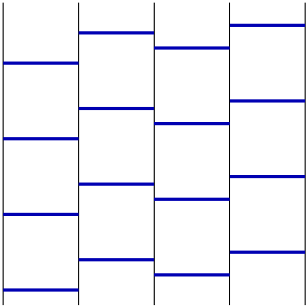
Introduction

Consider tiling a floor with square tiles, all of the same size. Is it the case that any gap-free tiling results in at least two fully connected tiles, i.e., tiles that have an entire edge in common?



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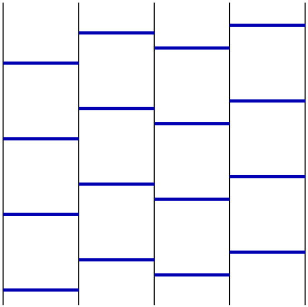

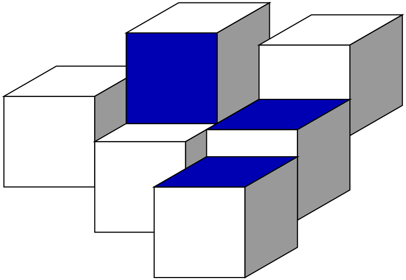

Keller's Conjecture

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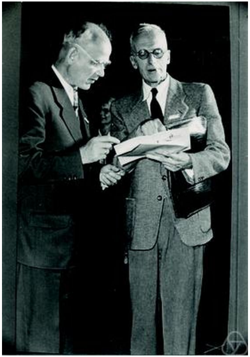

[Wikipedia, CC BY-SA]

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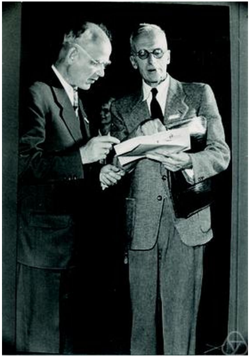
Keller's Conjecture. For all $n \geq 1$, every tiling of n-dimensional space with unit cubes has two which fully share a face.



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What about dimension 7?

Main Result

Theorem (Brakensiek, Heule, Mackey, and Narváez, 2020).
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- The SAT formula is very difficult to solve, required extensive symmetry breaking.
- Total proof size is over 200 gigabytes! Verified by a proof checker.

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Formal Description

- We define the Keller graph $G_{n,s}$ to has $(2s)^n$ vertices/cubes. Each has n dimensions/dots have one of $2s$ colors which come in complementary pairs: e.g. black/white and red/green.

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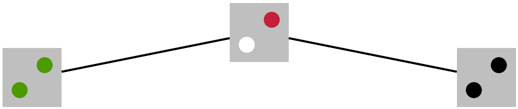

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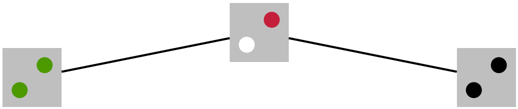
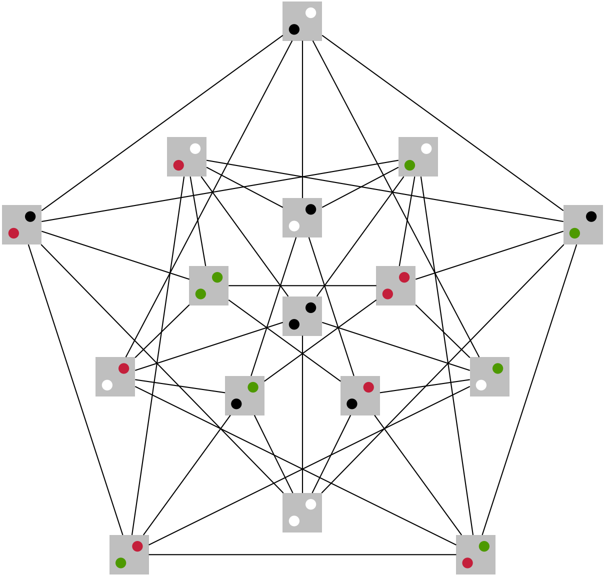


- Corrádi and Szabó's work (1990) showed that there is a counterexample to Keller's conjecture in some dimension n if one can show $G_{n,s}$ has a clique of size 2^n.

From Keller's Conjecture to Graph Theory: $G_{2,2}$



Towards Resolving Dimension 7

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- Between 2013 and 2017, Łysakowska and Kisielewicz showed that if one of $G_{7,3}$, $G_{7,4}$ or $G_{7,6}$ has no clique of size 2^7, then Keller's conjecture is true in dimension 7.

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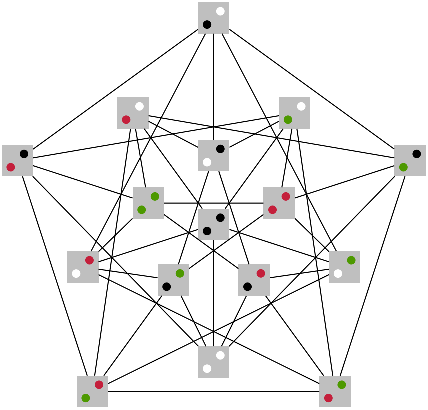
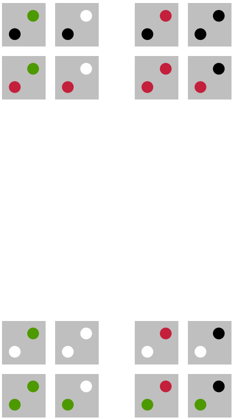
Succinct Encoding: Groups

$G_{n,s}$ can be partitioned into 2^n independent sets (groups)

Key Observation: If there is a clique of size 2^n, each group has exactly one vertex in the clique.





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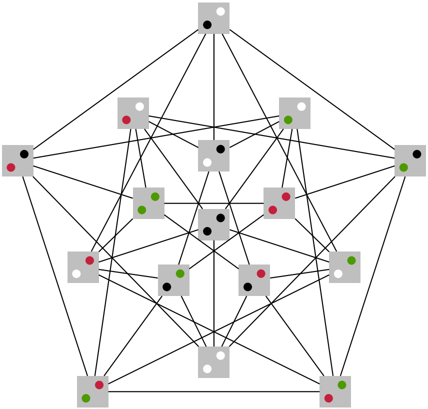

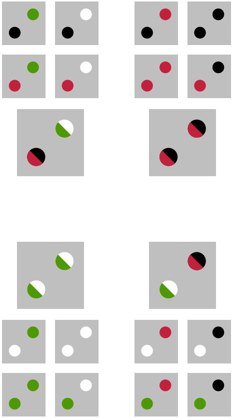

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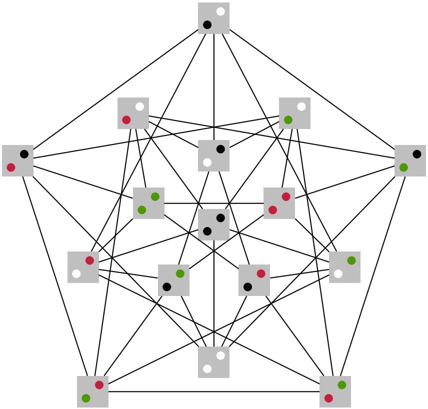

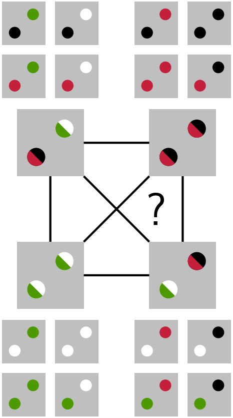

Succint Encoding: Under the Hood

The colors represent numbers, k and $s + k$ are complementary colors.

Let $n = 5$ and $s = 4$. Consider a vertex:

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Say $(1,0,1,0,0)$ is the "characteristic vector" of v: two vertices with the same characteristic vector cannot be connected.

Succint Encoding: Under the Hood

- There are 2^n possible characteristic vectors, identified by a corresponding binary number (LSB first):
  - $v = (6, 1, 5, 0, 2)$ is in group 5.

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- We build a clique by picking a vertex from each group.
- Variables: $x_{v,d,c}$ encodes vertex picked from group v at dimension/dot d has color/value c.
  - If v in base 2 has digit 1 in position d, the value is $s + c$.

Succinct Encoding: Constraints

- First, every dimension/dot must have exactly one color.
  $(x_{v,d,0} \lor x_{v,d,1} \lor \cdots \lor x_{v,d,s-1}) \land \bigwedge_{c \neq c'} (\bar{x}_{v,d,c} \lor \bar{x}_{v,d,c'})$

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- Second, each pair of vertices must have different colors/values in some other dimension/dot.
  - Note that this only matters for pairs of vertices v, v' such that $v \oplus v'$ has only one bit.
- Third, each pair of vertices must have complementary colors/values in some dimension/dot.

Encoding Size

Keller Graph	Cube Count	Variable Count	Clause Count
$G_{7,3}$	279 936	39 424	200 320
$G_{7,4}$	2 097 152	43 008	265 728
$G_{7,6}$	35 831 808	50 176	399 232

the number of clauses is smaller than the number of cubes

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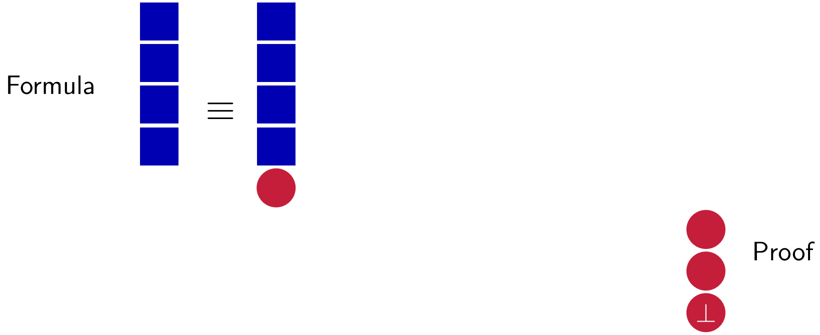

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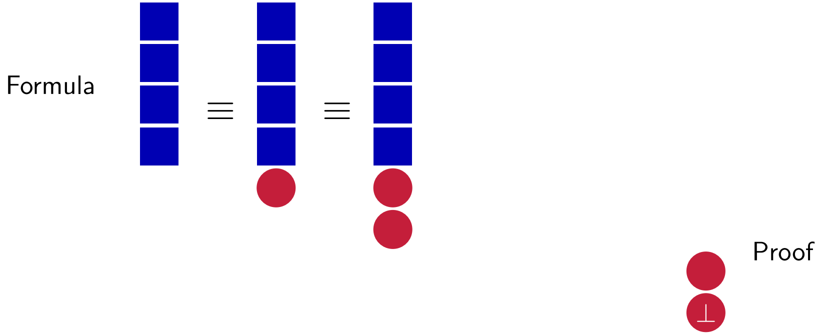

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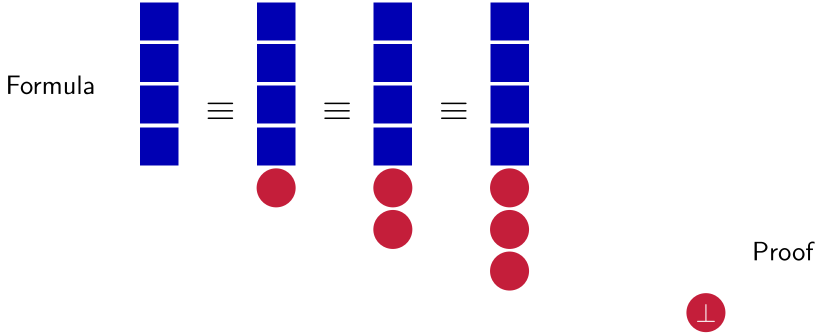

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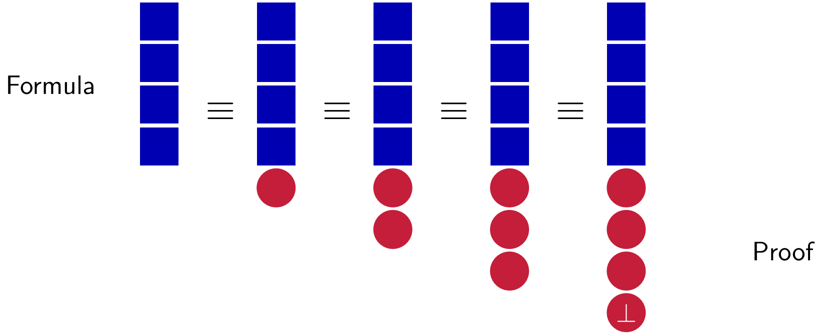

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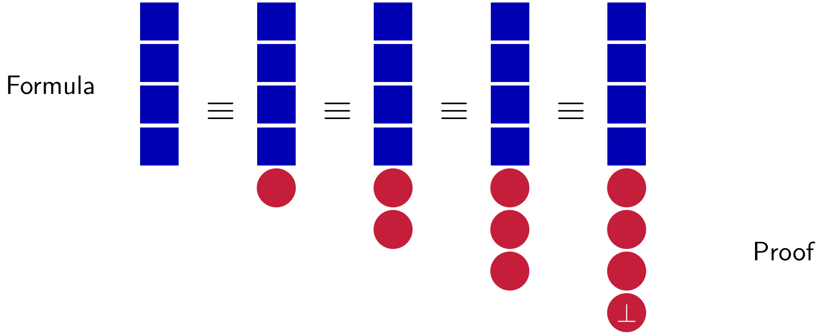
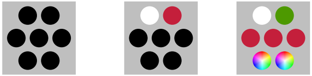


- Checking the redundancy of a clause in polynomial time
- Clausal proofs are easy to emit from modern SAT solvers
- Symmetry breaking can be expressed using clausal proofs

Symmetry Breaking Overview

The symmetry breaking consists of three parts:

1. Manual proof that we can assume two 5-facesharing cubes and a third, mostly colored cube:



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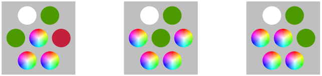

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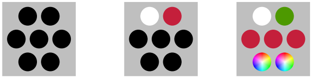

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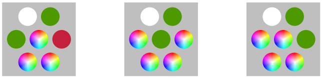
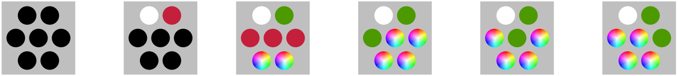


3. Enumerate and filter all options for the rainbow dimensions/dots

Symmetry Breaking: Under the Hood



The 6 vertices involved in the previous slide are:

$$
\begin{aligned}
c_0 &= (0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,) \\
c_1 &= (s\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,) \\
c_3 &= (s\,,\ s+1\,,\ \star_s\,,\ \star_s\,,\ 1\,,\ 1\,,\ 1\,) \\
c_{19} &= (s\,,\ s+1\,,\ \star_s\,,\ \star_s\,,\ s+1\,,\ \star_s\,,\ \star_s\,) \\
c_{35} &= (s\,,\ s+1\,,\ \star_s\,,\ \star_s\,,\ \star_s\,,\ s+1\,,\ \star_s\,) \\
c_{67} &= (s\,,\ s+1\,,\ \star_s\,,\ \star_s\,,\ \star_s\,,\ \star_s\,,\ s+1)
\end{aligned}
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(Note that the 4th cube corresponding to c_{19} has one extra rainbow dot.)

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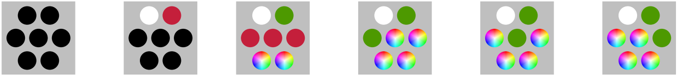

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- Naïvely: s^6 cases to try. We want to do much better.
- First observation: one of $c_{19,6}$ or $c_{35,5}$ have to be 1.
  - Add 3 clauses $(x_{19,6,1} \vee x_{35,5,1})$, $(x_{35,7,1} \vee x_{67,6,1})$, $(x_{67,5,1} \vee x_{19,7,1})$ to the proof.
  - No need to justify these clauses: their negation leads to a contradiction.
  - Takes care of $3(s-1)^2 s^4 - 3(s-1)^4 s^2 + (s-1)^6$.

Symmetry Breaking: Under the Hood

Next we look at symmetric configurations:

	5	6	7
c_{19}	$s+1$	1	2
c_{35}	2	$s+1$	2
c_{67}	1	1	$s+1$

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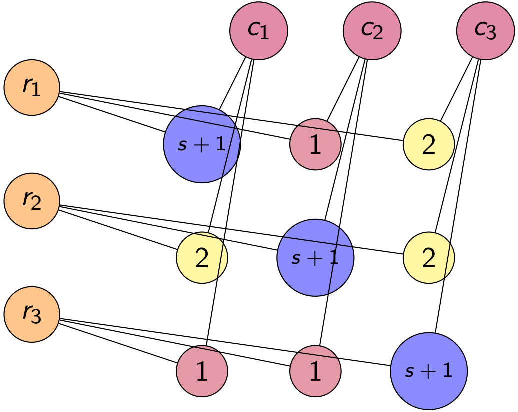
Game plan:

- Enumerate all $(2s-1)^3$ remaining matrices and group them by canonical representative.
- Block all remaining matrices except the canonical representative.

Symmetry Breaking: Under the Hood

Matrix symmetries are just (vertex-colored) graph symmetries:



Symmetry Breaking: Under the Hood

For a matrix M and a canonical matrix M' symmetrical to M:

- We block the configuration M using M' as a justification.
- We use a proof system that allows to add clauses to the proof, together with a justification.
  - Marijn J.H. Heule, Benjamin Kiesl, Martina Seidl, and Armin Biere. PRuning Through Satisfaction.

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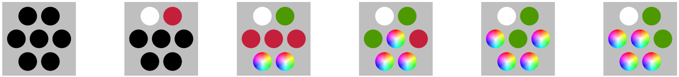
After all symmetries have been broken, we can assume $c_{19,6} = 1$.

Case Split

Given the cubes, in how many ways can we color rainbow dots?



Worst case for n rainbow dots without symmetry breaking is s^n

With symmetry breaking these can be reduced to:

- $s = 3$: 21 525 (instead of $3^{13} = 1\,594\,323$)
- $s = 4$: 37 128 (instead of $4^{13} = 67\,108\,864$)
- $s = 6$: 38 584 (instead of $6^{13} = 13\,060\,694\,016$)

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Conclusions

We resolved the remaining case of Keller's conjecture

- No clique of size 128 in $G_{7,3}$, $G_{7,4}$, $G_{7,6}$
- Designed a SAT compact encoding
- Combined parallel SAT solver and symmetry breaking
- Constructed a clausal proof of unsatisfiability
- Certified the proof with a formally-verified checker

Future Work

Towards a full formal proof of Keller's conjecture:

- Formalize Keller's conjecture
- Prove the relation between Keller graphs and the conjecture
- Prove the correctness of the encoding
- Solve the case $G_{7,64}$ directly

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Open questions:

- What is the largest clique in $G_{7,3}$, $G_{7,4}$, $G_{7,6}$?
- Is the clique of 256 in $G_{8,2}$ unique (modulo symmetries)?
- Why is there a transition between dimensions 7 and 8?

Fin: A Clique of Size 256 in $G_{8,2}$ [Mackey, 2002]