Small Folkman Numbers

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January 4, 2014

Abstract

This survey contains a comprehensive overview of the results related to Folkman numbers, a topic in general Ramsey Theory. Folkman numbers are founded upon the notion of Ramsey arrowing. For a graph G, we say that $G \to (a_1, \ldots, a_r; q)^v$ or $G \to (a_1, \ldots, a_r; q)^e$ iff G is K_q-free and for every vertex- or edge-coloring of G with r colors, respectively, there exists a monochromatic copy of K_{a_i} in color i for some $i \in \{1, \ldots, r\}$. Vertex and edge Folkman numbers are defined as $F_v(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q)^v\}$ and $F_e(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q)^e\}$, respectively. In a more general case one may use the constraint that the graph G is H-free instead of K_q-free, where H is any arbitrary graph. The diversity of problems related to Folkman numbers have made them a subject of challenging research for more than five decades. In this survey we try to report and comment on all known results related to Folkman numbers, including ties with complexity theory, with as complete references as we could collect. While we do discuss asymptotic results, our focus is on bounds and exact values.

Contents

1 Introduction

Ramsey theory studies conditions under which a combinatorial object, often a graph, necessarily contains some smaller objects of interest. It is often regarded as the study of how order emerges from randomness. The breadth and depth of Ramsey-related literature is massive, encompassing about a century of mathematical research and development starting with the original paper of Ramsey [92] and being heavily influenced by later mathemticians such as Paul Erdős.

The central concept of Ramsey theory is that of *arrowing*. Given graphs G, H, and I, we say that G arrows the vertices of (H, I), denoted $G \to (H, I)^v$, if for every red and blue coloring of the *vertices* of G, G contains a red H or blue I. Analogously, we say that G arrows the edges of (H, I), denoted $G \to (H, I)^e$, if for every red and blue coloring of the *edges* of G, G contains a red H or blue I. The role of *Ramsey numbers* are to quantify the general existential properties of Ramsey theory. In particular, the *generalized Ramsey number* $R(H, I)$ of graphs H and I is the smallest n such that $K_n \to (H, I)^e$, where K_n denotes the complete graph on n vertices. If $H = K_p$ and $I = K_q$, then we denote this Ramsey number as $R(p, q)$. The existential properties of arrowing problems, as well as the corresponding Ramsey numbers, can be extended to multiple colors (i.e., arrowing multiple graphs) and hypergraphs. Interested readers are referred to [91] for a regularly updated survey on known results for various types of Ramsey numbers.

In this survey we only consider finite, simple graphs, i.e., those without loops and multiple edges. We denote the vertex and edge set of a graph G as $V(G)$ and $E(G)$, respectively. $N(v)$ is the open neighborhood of vertex $v \in V(G)$. $G[V]$ for $V \subset V(G)$ denotes the subgraph induced by the vertices V. Also, $\alpha(G)$ and $\omega(G)$ denote the cardinality of a maximum independent set of G and the cardinality of the largest clique in G. The chromatic number of G, which is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices share the same color [32], is denoted by $\chi(G)$. The girth of a graph G, denoted as $g(G)$, is the length of the shortest cycle in G. Finally, we denote the complement of a graph G by $\overline{G}$.

For $v \in V(G)$ and $e \in E(G)$, we let $G - v = G[V(G) \setminus v]$. We also denote $G - e$ as the subgraph of G such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Similarly, if $e \notin E(G)$, we denote $G + e$ as the supergraph of G such that $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$. Finally, for graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, we denote the join of G_1 and G_2 as $G_1 + G_2$, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$. From this definition, it is clear that $\omega(G_1 + G_2) = \omega(G_1) + \omega(G_2)$. Unless explicitly stated, the $+$ operator will always denote a graph join. We denote $K_n - C_m$, $m \leq n$, as the induced subgraph of K_n obtained by removing all edges on some cycle C_m. The k-th power of a graph G, denoted as G^k, is the graph $(V(G), E(G) \bigcup \{(u, v) : d(u, v) < k\})$.

A *hypergraph* $\mathcal{G}$ is the pair $(V(\mathcal{G}), E(\mathcal{G}))$ where $V(\mathcal{G})$ is the set of vertices and $E(\mathcal{G}) \subseteq 2^{V(\mathcal{G})}$ is the set of hyperedges. Traditional definitions for graph order and clique sizes are similar to hypergraphs. A hypergraph is *k-uniform* if for every edge $e \in E(\mathcal{G})$ it holds that $|e| = k$.

Folkman numbers and *Folkman graphs* are concerned with arrowing questions that are less constrained. More specifically, given a graph G, we write $G \to (a_1, \ldots, a_r; q)^v$ iff for every vertex coloring of an undirected simple graph G that is K_q-free, there exists a monochromatic K_{a_i} in color i for some $i \in \{1, \ldots, r\}$. The *vertex Folkman number* is defined as

$$F_v(a_1, \ldots, a_k; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_k; q)^v\}.$$

Similarly, the *edge Folkman number* is defined as

$$F_e(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q)^e\}.$$

In 1970, Jon Folkman proved that for all $q > \max(s, t)$, vertex and edge Folkman numbers $F_v(s, t; q)$ and $F_e(s, t; q)$ exist. The sets $\mathcal{F}_v(s, t; q)$ and $\mathcal{F}_e(s, t; q)$, defined below, are called the *vertex* and *edge Folkman graphs*, respectively.

$$\mathcal{F}_v(s, t; q) = \{G : G \to (s, t)^v \wedge \omega(G) < q\}$$
$$\mathcal{F}_e(s, t; q) = \{G : G \to (s, t)^e \wedge \omega(G) < q\}$$

These definitions extend to multiple colors in the natural way. In the case of multicolor Folkman numbers, we also denote $F(a_1, \ldots, a_r; q)$ as $F(a_r; q)$ if $a_1 = a_2 = \cdots = a_r$, and use a similar notation for the multicolor Ramsey number. We also use $F(r, G)$ to denote

$$\min\{|V(H)| : H \to (G)_r \text{ and } \omega(H) = \omega(G)\},$$

where $H \to (G)_r$ is analogous for saying that for every r-coloring (of the vertices or edges) of H there exists a monochromatic copy of G. As usual, the superscript v or e denotes the vertex or edge version of this particular number. According to this definition we have that $F(r, G) = F(a_1, \ldots, a_r; \omega(G)+1)$. Furthermore, we will use the notation $F(r, k, q)$ to denote the (edge or vertex) Folkman number $F(a_1, \ldots, a_r; q)$ where $a_1 = \cdots = a_r = k$.

Induced Folkman numbers, marked with the ind superscript, are special types of Folkman numbers. Following the notation of Dudek, Ramadurai, and Rödl in [21], we write $H \underset{\mathsf{ind}}{\to} (G)^v_r$ if for every r-coloring of the vertices of H there exists a monochromatic and induced copy of G. As a result, we define the induced vertex Folkman number as

$$F_v^{\mathsf{ind}}(r, s; s+1) = \min\{|V(H)| : H \underset{\mathsf{ind}}{\to} (K_s)^v_r \text{ and } \omega(H) = \omega(K_s)\}.$$

Induced edge Folkman numbers have a similar definition.

Finally, the analogous induced vertex Folkman number variant for hypergraphs $\mathcal{G}$ and $\mathcal{H}$ is defined as

$$F_v^{\mathsf{ind}}(r, \mathcal{G}) = \min\{|V(\mathcal{H})| : \mathcal{H} \underset{\mathsf{ind}}{\to} (\mathcal{G})^v_r \text{ and } \omega(\mathcal{H}) = \omega(\mathcal{G})\}.$$

Again, a similar definition holds for induced edge Folkman numbers for hypergraphs.

We begin our exploration of Folkman numbers in Chapter 2 with a discussion of classical two-color Folkman numbers. We then discuss more general multicolor Folkman numbers in Chapter 3. Results on hypergraphs are then presented in Chapter 4, followed by a discussion of the complexity theoretical results regarding Folkman numbers and their computations in Chapter 5. Selected comments on asymptotic results appear throughout all sections where appropriate.

2 Two-Color Problems

In this section we present results for two-color vertex and edge Folkman numbers. Known bounds and values are discussed along with the relevant history leading to the current results.

2.1 $F_v(s, t; q)$

Vertex Folkman numbers are the most studied forms of Folkman numbers, most likely due to their natural relation to the problem of graph vertex coloring. In this section we present known results for classical vertex Folkman numbers of the form $F_v(s, t; q)$. The main results are shown in Table 1, with further comments and background information on relevant problems shown below.

$F_v(2, 2; 3)$

This can be easily proved using the graph C_5 by showing that $C_5 \to (2, 2)^v$. It is interesting to note that vertex Folkman numbers of the form $F_v(2, 2; q)$ (or more generally, $F_v(2, \ldots, 2; q)$), correspond to the chromatic number of a graph. In particular, $F_v(2, \ldots, 2; q)$ is the smallest order of a K_q-free graph G such that $\chi(G) = r$.

$F_v(3, 3; 4)$

The authors also showed that $F_e(3, 3; 5) \leq F_v(3, 3; 4) + 1$ and $F_v(3, 3; 4) = 14$. The inequality $F_v(3, 3; 4) \leq 14$ was proven by Nenov [62] in 1981 with a $R(3, 3)$ graph, and the inequality $F_v(3, 3; 4) \geq 14$ was proven by Piwakowski et al. [89] in 1999 using computer programs.

$F_v(3,4;5)$

The proof of this result hinged on a supporting theorem proven by Nenov, which states that for an n-vertex graph $G \in \mathcal{H}(3,4)$ it is true that $\alpha(G) \leq n-9$, and that equality in this bound implies that $n \leq 18$. The lower bound $F_v(3,4;5) \geq 13$ was proven by showing that no 12-vertex graphs $G \in \mathcal{F}_v(3,4;5)$ exist. By assuming that a 12-vertex graph $G \in \mathcal{F}_v(3,4;5)$, it is true that $\alpha(G) = 2$, which means that such a graph G is a subgraph of the graph P (see Figure 1 for the complement of this graph). The authors then showed that $P \in \mathcal{H}(3,4)$, thus proving that no such graphs G exist on 12 vertices and that $F_v(3,4;5) \geq 13$. Independently, and years later, Xu et al. [105] provided a computer assisted proof to show that the unique $(5,3)$-Ramsey graph G (see Figure 2) is the unique witnessing graph for $G \to (3,4;5)^v$, thus showing that $F_v(3,4;5) \leq 13$.

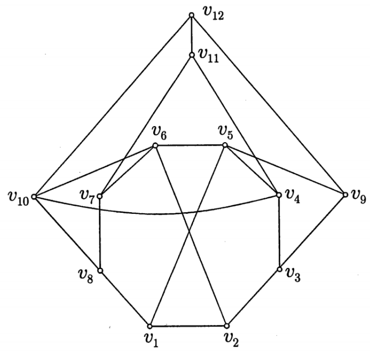

Figure 1: The complement of the graph $P \notin \mathcal{F}_v(3,4;5)$ [75].

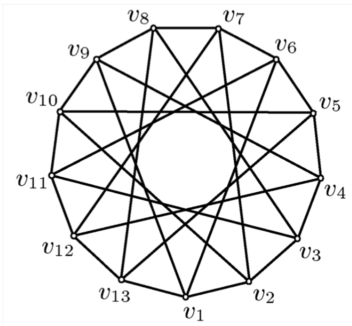

Figure 2: Complement of the critical Greenwood and Gleason graph [31].

$F_v(3,5;6)$

This is a particularly interesting result as it is the smallest known vertex Folkman number of the form $F_v(3,k;k+1)$. Furthermore, the exact value was determined over the span of three years using an analytical proof for the upper bound $F_v(3,5;6) \leq 16$ [96] and computational approach for the lower bound $F_v(3,5;6) \geq 16$ [99]. Prior to this result, Kolev and Nenov showed that $F_v(3,5;6) \leq 22$ in [46].

$F_v(3,6;7)$, $F_v(3,7;8)$, $F_v(3,8;9)$

Previous upper bounds of $F_v(3,6;7) \leq 26$ and $F_v(3,8;9) \leq 26$ were shown by Nenov and Kolev in [46]. Similarly, $F_v(3,7;8) \leq 27$ was shown by Xu et al. in [103]. These were all later reduced to $F_v(3,6;7) \leq 18$, $F_v(3,7;8) \leq 22$, and $F_v(3,7;8) \leq 23$ by Shao et al. in [96].

$F_v(4,4;6)$

The previous upper bound of $F_v(4,4;6) \leq 14$ shown by Nenov [75] was an improvement on the previous result of $F_v(4,4;6) \leq 35$, which was shown by Łuczak, Ruciński, and Urbański in [102], and was obtained with the witnessing graph $G = K_1 + Q$ (since $Q \to (3,4)^v$, see Figure 2). A previous lower bound of $F_v(4,4;6) \geq 13$ was proven by Nenov in [79], which soon led the establishment of $F_v(4,4;6) = 14$.

$F_v(4,4;5)$

Nenov proved an initial bound of $16 \leq F_v(4,4;5) \leq 35$ in [83]. Together with Kolev, they later reduced the upper bound to $F_v(4,4;5) \leq 26$ in [44]. This result was improved shortly thereafter by Kolev who pushed this bound down further to 25 in 2007 [49]. The lower bound of 16 was proved by Nenov in 2005 [84]. Both bounds were improved by Xu et al. [105] in 2010, who showed that $17 \leq F_v(4,4;5) \leq 23$. This was accomplished by proving that $17 \leq F_v(2,2,2,4;5) \leq F_v(2,3,4;5) \leq F_v(4,4;5)$ with the help of a computer and then presenting a witnessing graph $G \to (4,4;5)^v$ shown in Figure 3. An interesting byproduct of this work was the proof that the unique $(5,3)$-Ramsey graph is the unique K_5-free graph of order 13 in $\mathcal{H}(5,3)$, yielding the upper bound $F_v(3,4;5) \leq 13$.

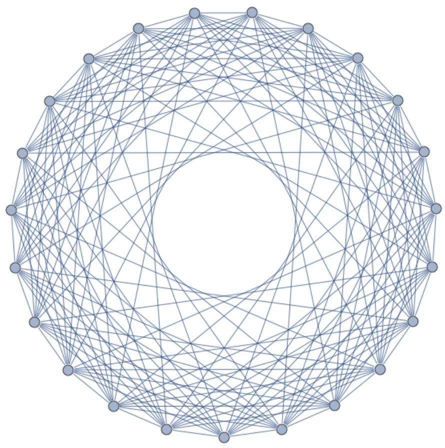

Figure 3: The witnessing graph G on 23 vertices for $G \to (4,4;5)^v$ found by Xu et al. in [105].

Table 1: Vertex Folkman numbers of the form $F_v(s,t;q)$.

s	t	q	$F_v(s,t;q)$	References
2	2	3	5	[73]
3	4	5	13	[75, 105]
3	3	4	14	[62, 89]
3	5	6	16	[46, 96, 99]
3	6	7	≤ 18	[46, 96]
3	7	8	≤ 22	[103, 96]
3	8	9	≤ 23	[46]
4	4	6	14	[77]
4	4	5	$[17, 23]$	[83, 84, 49, 105]

2.1.1 General Results

It is well known that $F_v(s,k;q) \leq F_v(s,k;k+1)$ for any positive integer $k < q$. Therefore, the most restrictive case for this class of Folkman numbers is $F_v(s,k;k+1)$. Nenov and others extensively studied $F_v(3,k;k+1)$ in [73, 46, 96]. In 2000, Nenov proved the following:

Theorem 1. [73] $2k+4 \leq F_v(3,k;k+1) \leq 4k+2$ for $k \geq 3$.

While not a two-color case, it was also proven that $2k+4 \le F_v(2,2,k;k+1) \le 4k+2$ (see Section 3.1 for more details). Furthermore, the upper bound of $4k+2$ follows immediately from the order of Γ_k (namely, $|V(\Gamma_k)| = 4k+2$). The class of graphs Γ_k was constructed by Nenov such that for each $G \in \Gamma_k$ it holds that $G \to \mathcal{F}_v(3,k;k+1)$. For completeness, Γ_k is defined as the extension of $\overline{C_{2k+1}}$ by adding pairwise independent vertices $u_1, \ldots, u_{2k+1}$, where each vertex u_i is adjacent to M_i ($M_i = \sigma^{i-1}(M_1)$, $\sigma(v_i) = v_{i+1}$ and $\sigma(v_{2k+1}) = v_1$, and $M_1 = V(C_{2k+1}) = \backslash\{v_1, v_{2k-1}, v_{2k-2})$. Note that in this way σ is an automorphism of C_{2k+1}. Figure 4 shows Γ_3. This result is captured in Theorem 2.

Theorem 2. [73] For any $k \ge 3$, $\Gamma_k \in \mathcal{F}_v(3,k;k+1)$.

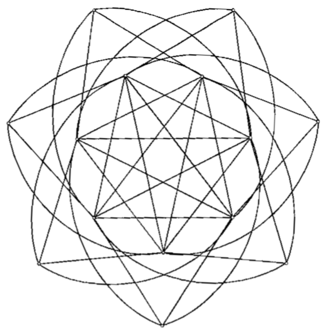

Figure 4: Γ_3, an instance of Γ_k where $k = 3$ [73].

Nenov [69] and Łuczak et al. [57] independently showed that for some c it is true that $F_v(2,k;k+1) \le c \cdot k! = \mathcal{O}(k!)$. Nenov [84, 44] also showed that $F_v(k+1,k+1;k+2) \le (k+1)F_v(k,k;k+1)$. Using the fact that $F_v(4,4;5) \le 25$ [45], Kolev [49] showed that $F_v(k,k;k+1) \le (25/24)k!$, where $k \ge 4$, which improved the upper bound of $F_v(k,k;k+1) \le \lfloor 2k!(e-1) \rfloor - 1$ that was presented by Łuczak et al. in 2001 [57]. This result is obtained by induction on k.

Corollaries resulting from this theorem are shown below.

Corollary 3. [46] Let $k \ge 4$ and $k = 4m + l$, $0 \le l \le 3$, then

$$F_v(3,k;k+1) \le (m-1)F_v(3,4;5) + F_v(3,4+l;5+l) \quad (1)$$

$$F_v(2,2,k;k+1) \le (m-1)F_v(2,2,4;5) + F_v(2,2,4+l;5+l) \quad (2)$$

Corollary 4. [46] $F_v(3,k;k+1) \le F_v(3,k-4;k-3) + F_v(3,4;5)$ and $F_v(2,2,k;k+1) \le F_v(2,2,k-4;k-3) + F_v(2,2,4;5)$ for $k \ge 8$.

Kolev and Nenov [46] found further inequalities after examining $k = 5,6,7$ and $F_v(3,4;5) = 13$ [75]. For $k = 5$, they used the values $F_v(2,2,4;5) = 13$ [74], $F_v(2,2,6;7) \le 22$ [76], $F_v(2,2,7;8) \le 28$ [76]. See Section 3.1 for more information on these multicolor results.

Conjecture 5. [46]

$$F_v(3,k;k+1) \le \frac{13k}{4} \text{ for } k \ge 4. \quad (3)$$

Remark 6. If Conjecture 5 is true for $k = 5,6,7$ then it is also true for $k \ge 4$ since $F_v(3,4;5) = 13$.

Conjecture 7. [46]

$$F_v(2,2,k;k+1) \le \frac{13k}{4} \text{ for } k \ge 4. \quad (4)$$

Remark 8. If Conjecture 7 is true for $k = 5,6,7$ then it is also true for $k \ge 4$ since $F_v(2,2,4;5) = 13$.

Kolev and Nenov lowered this bound to $F_v(3,k;k+1) \le (m-1)F_v(3,4;5)+F_v(3,4+l;5+l)$ for $k \ge 4$ and $k = 4m+l, 0 \le l \le 3$, and further classified the upper bound based on k, as shown in Corollarly 9.

Corollary 9. [46] For $k \ge 4$,

$$F_v(3,k;k+1) \le \begin{cases} 13/4 & : k \equiv 0 \mod 4 \\ (13k+23)/4 & : k \equiv 1 \mod 4 \\ (13k+26)/4 & : k \equiv 2 \mod 4 \\ (13k+29)/4 & : k \equiv 3 \mod 4 \end{cases}$$

Soon after, Xu et al. [103] reduced the upper bound to $F_v(3,k;k+1) \le (13k+17)/4$ for $k \equiv 3 \mod 4$. Subsequent work by Shao et al. [96] produced many new upper bounds for $F_v(3,k;k+1)$ for small k that were further generalized for $k \ge 5$ (see Theorem 10 below).

Theorem 10. [96] For $k \ge 5$,

$$F_v(3,k;k+1) \le \begin{cases} 23/8 & : k \equiv 0 \mod 8 \\ (23k+49)/8 & : k \equiv 1 \mod 8 \\ (23k+18)/8 & : k \equiv 2 \mod 8 \\ (23k+43)/8 & : k \equiv 3 \mod 8 \\ (23k+12)/8 & : k \equiv 4 \mod 8 \\ (23k+13)/8 & : k \equiv 5 \mod 8 \\ (23k+6)/8 & : k \equiv 6 \mod 8 \\ (23k+15)/8 & : k \equiv 7 \mod 8 \end{cases}$$

A similar class of Folkman numbers $F_v(k,k;k+1)$ was studied by Nenov [69], Łuczak et al. [57], and Xu et al. [103] [105]. Starting with an upper bound established by Nenov in 1985, these results are captured in the following theorems.

Theorem 11. [69] $F_v(k,k;k+1) \le \lfloor k!e \rfloor - 2 = \mathcal{O}(k!)$ for $k \ge 3$.

Theorem 12. [57] $F_v(k,k;k+1) \le \lfloor 2k!(e-1) \rfloor - 1$ for $k \ge 3$.

Later, Nenov provided a recurrent inequality for this upper bound for all values of $k \ge 2$, as shown in Theorem 13. One interesting corollary that followed from this result was that $F_v(k,k;k+1) \le 1.46k!$ for $k \ge 4$.

Theorem 13. [84] $F_v(k+1,k+1;k+2) \le (k+1)F_v(k,k;k+1)$ for $k \ge 2$.

Xu et al. [103] later improved the upper bound on this inequality, as shown in Theorem 14. A similar inequality is shown in Theorem 15.

Theorem 14. [103] $F_v(2k,2k;2k+1) \le kF_v(2k-1,2k-1;2k)+3k+1$ for $k \ge 2$.

Theorem 15. [103] Let $k \ge 2$ and H be a graph such that $H \subset \mathcal{F}_v(2k,2k;2k+1)$. Let $\{v_1\} \subset V(H)$, $A \subset V(H) \setminus \{v_1\}$, G_1 be a subgraph of H induced by A and G_2 be a subgraph of H induced by $V(H) \setminus \{v_1 \cup A\}$, where G_1 and G_2 are K_{2k}-free. If x is the order of the maximum isomorphic induced subgraphs of G_1 and G_2, then

$$F_v(2k+1,2k+1;2k+2) \le (k+1)F_v(2k,2k;2k+1) - x + 3k - 2.$$

Using constructive methods from asymptotic analysis, Shao et al. [104] proved a general upper bound for $F_v(k,k;k+1)$ for large values of k. Their main result is shown below in Theorem 16.

Theorem 16. [104] For any real number r such that $0 < r < \frac{1}{2}\log_2 3 - \frac{3}{4}$, there are $N(r) > 0$ and $c(r) > 0$ such that

$$F_v(k,k;k+1) \le c(r)(k-1)^{\frac{1}{4}\log_2(k-1)-r}$$

for any $k \ge N(r)$.

A more complete description of this same result is captured in Theorem 17.

Theorem 17. [104] Suppose $0 < r < \frac{1}{2}\log_2 3 - \frac{3}{4}$ and any k such that

$$k \geq a_0 \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}} \right)^2 \right\rceil.$$

It then holds that

$$F_v(k,k;k+1) \leq c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r},$$

where

$$c_0 = \max\left\{ \frac{F_v(i,i;i+1)}{(i-1)^{\frac{1}{4}\log_2(i-1)-r}} | a_0 \leq i \leq 2a_0 \right\}.$$

In addition, using Theorem 5 from [104], Xu et al. deduced Corollaries 18 and 19 below. Interestingly, the second result was obtained in conjunction with the fact that $F_v(2,2;3) = 5$.

Corollary 18. [104] For $a, b \leq 2$ such that $a \leq p$ and $b \leq q$,

$$F_v(ab, ab; pq+1) \leq F_v(a,a;p+1) \cdot F_v(b,b;q+1).$$

Corollary 19. [104] For $a \geq 2$,

$$F_v(2a, 2a; 2a+1) \leq 5F_v(a,a;a+1).$$

Later in 2010, Xu and Shao established an even tighter upper bound of $4k-1$ for $F_v(k,k;k+1)$, which is captured in the following theorem.

Theorem 20. [106] $F_v(k,k;k+1) \geq 4k-1$ for $k \geq 2$.

A variation of this class of vertex Folkman numbers, $F_v(2k+1, 2k+1; 2k+2)$ was studied by Xu et al. in [104]. Their main result is shown below in Theorem 21.

Theorem 21. [104] Let $k \geq 4$ and define the function $f(k) = \left\lceil \sqrt{\frac{k}{3}} \right\rceil$. Then, we have

$$F_v(2k+1, 2k+1; 2k+2) \leq (2f(k)+1)F_v(k,k;k+1) + 3F_v(k+1,k+1;k+2) + k + 2f(k).$$

Remark 22. This upper bound was proved by constructing a general class of graphs of order $(2f(k)+1)F_v(k,k;k+1) + 3F_v(k+1,k+1;k+2) + k + 2f(k)$.

2.2 $F_e(s,t;q)$

Edge Folkman numbers are much less studied than their vertex counterpart. Nevertheless, the depth of results pertaining to this flavor of numbers warrants its own discussion. As such, we first begin with the fundamental result proven by Folkman in 1970.

Theorem 23. [27] $F_e(s,t;q)$ exists iff $q > \max\{s,t\}$.

This can be generalized to state that $F_e(a_1, \ldots, a_r; q)$ exists iff $q > \max\{a_1, \ldots, a_r\}$, as was affirmatively shown by Nešetřil and Rödl in [88]. Furthermore, from this result, it should be clear that $G \rightarrow (s,t;q)^e$ implies that $\omega(G) \geq \max\{s,t\}$. Several known values of these two-color edge Folkman numbers have been found since then. Before discussing these results, we first note the important fact that if $q > R(s,t)$ then $F_e(s,t;q) = R(s,t)$. Conversely, when $k \leq R(s,t)$, very little is known about the bounds of $F_e(s,t;q)$. The data in Table 2 indicates that as the value of q decreases, the problem of finding $F_e(s,t;q)$ becomes increasingly difficult (i.e., the order of witnessing graphs become much larger). For this reason, exact values for $F_e(s,t;4)$ and $F_e(s,t;5)$ have been among the most well-studied problems. Other interesting results for numbers of the form $F_e(s,t;q)$ are given in Table 5. We provide a richer history and more background information for some of these Folkman numbers below.

Table 2: Edge Folkman number bounds for small values of q.

q	$F_e(3,3;q)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	[29]
5	15	659 graphs	[89]
4	≤ 786	G_{786}	[52]

$F_e(3,3;4)$

The number $F_e(3,3;4)$ has a particular intriguing history, which is captured in Table 3. The original upper bounds obtained via iterated towers in [27] and [88] after the original existential proofs by Folkman were still quite large. In particular, it was shown that

$$F_e(3,3;4) < 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}} .$$

Erdős followed this result by offering a \$100 prize (equivalent to 300 Swiss francs at the time) for determining if $F_e(3,3;4) < 10^{10}$. In 1988, Spencer [101] presented the first evidence of graphs below this bound using a probabilistic approach. Hovey later identified an error in his work and corrected it to obtain $F_e(3,3;4) < 3 \times 10^9$. In 2008, Lu [55] provided such a proof by showing that $F_e(2,3;4) \leq 9697$, and a weaker yet still positive result was obtained separately by Dudek and Rödl in [17]. The next significant improvement came with Dudek and Rödl's application of MAX-CUT to show that the graph $G_{941} = (\mathbb{Z}_{941}, \{(x,y)|x-y = \alpha^5(\mod 941) \text{ for some } \alpha\})$ witnesses $G \rightarrow (2,3;4)^e$ [16]. Lange et al. [52] further improved this bound to $F_e(3,3;4) \leq 786$ with the same technique using the Geomans-Williamson MAX-CUT approximation algorithm. The conjecture that $F_e(3,3;4) \leq 127$ proposed by Exoo and supported by Radziszowski et al. [90] is still an open problem, and is motivated by the intuition that $G_{127} = (\mathbb{Z}_{127}, \{(x,y)|x-y \equiv \alpha^3(\mod 127)\})$ contains many triangles and small dense subgraphs. Following in the footsteps in Erdős, Graham offered a \$100 prize for proving that $F_e(3,3;4) \leq 100$.

The lower bound of $F_e(3,3;4) \geq 10$ was first proved by Lin in 1972 [54]. Piwakowski et al. [89] improved this bound to $F_e(3,3;4) \geq 16$ by enumerating all graphs in $\mathcal{F}_e(3,3;5)$ and proving that all of them contain K_4's. In 2007, Radziszowski et al. [90] pushed this bound to $F_e(3,3;4) \geq 19$ with a construction technique that relied on the fact that $G \rightarrow (3,3;4)^v$ implies $G + x \rightarrow (3,3;5)^e$ and $F_e(3,3;5) = 15$.

Table 3: History of $F_e(3,3;4)$.

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[25]
1970	exist	Folkman	[27]
1972	≥ 10	Lin	[54]
1975	$\leq 10^{10}$?	Erdős offers \$100 for proof	
1986	$\leq 8 \times 10^{11}$	Frankl-Rödl	[28]
1988	$\leq 3 \times 10^9$	Spencer	[101]
1999	≥ 16	Piwakowski et al. (implicit)	[89]
2007	≥ 19, ≤ 127?	Radziszowski-Xu	[90]
2008	≤ 9697	Lu	[55]
2008	≤ 941	Dudek-Rödl	[13]
2012	≤ 786	Lange et al.	[52]
2012	≤ 100?	Graham offers \$100 for proof	

$F_e(3,3;5)$

This number also has a dated history, as shown in Table 4, starting with the existential question posed by Erdős et al. in 1967 [25]. However, the first proof of the existence of this number predates Erdős in an unpublished manuscript by Pòsa. A viable upper bound was first proven by Schăuble [95] in 1969, shortly after the proposition by Erdős et al., in which it was shown that $F_e(3,3;5) \leq 42$. Graham and Spencer [30] improved this bound to $F_e(3,3;5) \leq 23$ in 1971, and further conjectured that $F_e(3,3;5) = 23$ without supporting reasoning. This bound was subsequently reduced to 18 by Irving in 1973 [39].

At this point, work on the upper bound diverged. Hadziivanov and Nenov [34] were able to construct a 16-vertex graph in $\mathcal{F}_e(3,3;5)$, and Nenov [62] further improved this bound with a 15-vertex graph in $\mathcal{F}_e(3,3;5)$, thus proving that $F_e(3,3;5) \leq 15$. Hadziivanov and Nenov [35] found another such graph on 15 vertices in 1984. Years later, Erickson [26] found a 17-vertex graph in $\mathcal{F}_e(3,3;5)$ and subsequently conjectured that $F_e(3,3;5) = 17$. However, Bukor [3] disproved this conjecture by showing the same 16-vertex construction presented in [34]. In 1996, the upper bound of 15 was verified once more by Urbański [102] with a different construction of the 15-vertex graph in [62].

The lower bound of $F_e(3,3;5)$ has much less history. In 1972 Lin [54] proved that $F_e(3,3;5) \geq 10$, which was subsequently improved by Nenov [61] to $F_e(3,3;5) \geq 11$, and then by Hadziivanov and Nenov [36] to $F_e(3,3;5) \geq 12$. The final value of $F_e(3,3;5) = 15$, proven by Piwakowski et al. in 1999 [89], was shown by constructing all 659 15-vertex graphs in $\mathcal{F}_e(3,3;5;15)$, where each such graph contains a K_4. In particular, the authors found that there exists exactly one bicritical graph $G \in \mathcal{F}_v(3,3;4;15)$, such that $G+e$ contains a K_5 and $G-e$ strips the Ramsey property. This was found by enumerating all nonisomorphic graphs of order 12, filtering all graphs H that do not satsify $K_5 \not\subseteq H$, $\chi(H) \geq 5$, and for every edge $\{u,v\} \in E(\overline{H})$ there are vertices $a,b \in V(H)$ such that $H[\{x,y,a,b\}]$ is isomorphic to $K_4 - e$. Then, with additional constraints, all of the $(+e, K_5)$-critical graphs from this set were found and then subsequently checked to see if $H - e \rightarrow (3,3)^e$. With this process, it was shown that $\delta(G) = 14$ due to some vertex v, and by removing v the result is a graph $G - v \in \mathcal{F}_v(3,3;4;14)$, as shown in Figure 6.

Table 4: History of $F_e(3,3;5)$.

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[25]
1969	≤ 42	Schăuble	[95]
1971	≤ 23	Graham and Spencer	[30]
1971	$= 23?$	Graham and Spencer	[30]
1972	≥ 10	Lin	[54]
1973	≤ 18	Irving	[39]
1979	≤ 16	Hadziivanov and Nenov	[34]
1980	≥ 11	Nenov	[61]
1981/84	≤ 15	Nenov, Hadziivanov and Nenov	[62, 35]
1985	≥ 12	Hadziivanov and Nenov	[36]
1993	≤ 17	Erickson	[26]
1993	$= 17?$	Erickson	[26]
1994	≤ 16	Bukor	[3]
1996	≤ 15	Urbański	[102]
1999	$\geq 15, = 15$	Piwakowski, Radziszowski, Urbański	[89]

$F_e(3,3;6)$

This is often touted as the one of the simpler and more elegant results for a Folkman number that comes from the graph $G = K_8 - C_5 = K_3 + C_5$. It is clear that $\omega(G) = 5$, so G is K_6-free, and by the pigeonhole principle it can be shown that coloring the edges of C_5 without a triangle is not possible (see Figure 5).

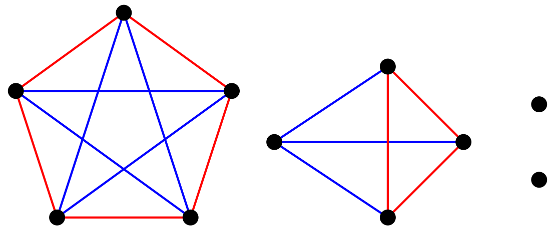

Figure 5: Figure (a) shows a coloring of K_5 that does not contain a monochromatic triangle, whereas Figure (b) shows a subset of a 2-edge coloring of $G = K_3 + C_5$ that shows how a triangle cannot be avoided. (Note that the entire vertex set of the C_5 graph is not shown for brevity.)

Table 5: Edge Folkman numbers of the form $F_e(s,t;q)$.

s	t	q	$F_e(s,t;q)$	Ref.
2	3	5	≤ 15	[62, 35]
3	3	≥ 7	6	[29]
3	3	6	8	[29]
3	3	5	15	[89]
3	3	4	≤ 786	[52]
3	4	10	≤ 9	[91]
3	4	9	14	[70]
3	4	8	16	[42]
3	4	5	≥ 22	[106]
3	5	14	16	*
3	5	8	≤ 21	[47]
3	7	22	≥ 27	*
4	4	18	20	*
4	4	17	≤ 25	[50]

$F_e(3,4;5)$
: A smaller lower bound of $F_e(3,4;5) \geq 21$ was proved by means of computation in [106] using Theorem 3 from [106], which states that $F_e(3,k;k+1) \geq F_v(k,k;k+1)+1$ for $k \geq 4$. Together with the fact that if $G \in \mathcal{F}_e(3,3;5;15)$ then $\chi(G) = 6$, it was proved that $F_e(3,4;5) \geq 22$ using a similar computational approach.

$F_e(3,5;8)$
: In [48] it was shown with the witnessing graph $G = K_8 + Q$ (where Q is the graph shown in Figure 7), which improved the upper bound of $F_e(3,5;8) \leq 24$ proven by Kolev and Nenov in 2008 [47]. Since $\omega(K_8 + Q) = \omega(K_8) + \omega(Q)$ and $\omega(Q) = 4$ [31], we have that $\omega(G) = 12$. Thus, since $|V(G)| = 21$ and $G \to (3,5)^e$, $F_e(3,5;13) \leq 21$ [47]. The best known lower bound for this number, $F_e(3,5;13) \geq 18$, was shown by Lin [54] in 1972. Nenov [67] showed that the equality $F_e(3,5;13) = 18$ is only possible by proving the arrowing $K_8 + C_5 + C_5 \to (3,5)^e$, but no one has been able to check this yet.

$F_e(3,4;9)$
: This is witnessed by the critical (3,4)-Ramsey graph $K_4 + C_5 + C_5 + C_5$.

$F_e(4,4;17)$
: This was the first result on the upper bound of this number, as previously the only known fact was that the number existed, as proved by Folkman [27].

Folkman numbers of the form $F_e(a_1,\ldots,a_r;R(a_1,\ldots,a_r))$, where cliques of size equal to the Ramsey number $R(a_1,\ldots,a_r)$ are avoided, turn out to be very difficult to find. As shown in Table 6, only a few results are known.

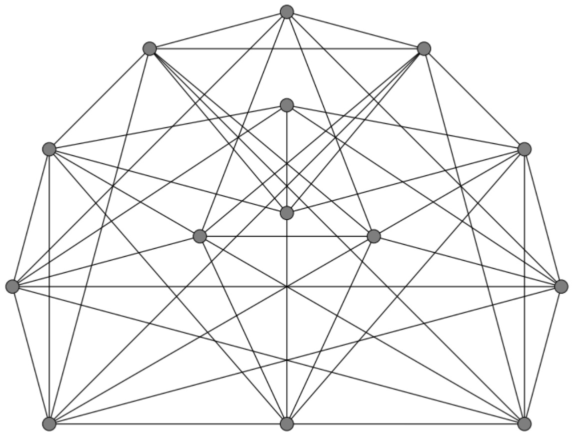

Figure 6: The unique bi-critical graph $G \in \mathcal{F}_v(3, 3; 4; 14)$ [89].

Table 6: Folkman numbers of the form $F_e(a_1, \ldots, a_r; R(a_1, \ldots, a_r))$.

$a_1, \ldots, a_r$	q	$F_e(a_1, \ldots, a_r; R(a_1, \ldots, a_r))$	Ref.
$3, 3$	6	7	[29]
$3, 4$	9	14	[70]
$3, 4$	8	≥ 14	[64]
$3, 5$	14	16	[54]
$4, 4$	18	20	[54]
$3, 3, 3$	17	19	[54]

2.3 Open Problems

In this section we list several open problems posed in the literature and from gaps in the results presented in the previous sections. Source for these problems are indicated where appropriate.

Problem 24. $F_v(2, 2, 3; 4) =?$

Problem 25. Does there exist an integer k such that $F_v(2, 2, k; k + 1) \neq F_v(3, k; k + 1)$?

Problem 26. [96] Is it true that

$$\lim_{k \to \infty} \frac{F_v(3, k; k + 1)}{k} = 2?$$

Problem 27. Is it true that $F_e(3, 3; 4) \leq 127$? If so, is it also true that $F_e(3, 3; 4) \leq 100$?

Problem 28. $F_e(3, 4; 7) =?$

Problem 29. $F_e(3, 5; 13) = 18?$

Problem 30. [25] Does there exists a K_4-free graph for which any coloring of its edges with countably many number colors yields a monochromatic copy of K_3.

3 Multicolor Problems

General Folkman numbers are typically unconstrained in the number of r colorings used in their specification. Research related to general Folkman numbers targets at finding bounds for a variety of Folkman number classes, often leading to explicit values for such classes. We will discuss such results in the following sections. First, however, we define some important terms and notation that will be used in this discussion.

Table 7: General results for multicolor vertex Folkman numbers of the form $F_v(a_1, \ldots, a_r; q)$.

$a_1, \ldots, a_r$	q	$F_v(a_1, \ldots, a_r; q)$	References
$2, 2, 4$	5	13	[74]
2_3	3	11	[58, 6]
2_4	4	11	[68]
2_4	3	22	[40]
2_5	4	16	[53]
2_r	4	11	[68, 72]
$2, 2, 3$	4	14	[8]
$2, 3, 3$	4	≥ 19	[97, 98]
$2, 3, 3$	5	12	[76]
$2_3, 3$	4	$[18, 30]$	[97]
$2_3, 3$	5	12	[76]
3_3	4	$[24, 66]$	[98, 11]
3_3	5	≤ 24	[11]
3_3	8	≤ 727	[11]
4_2	6	$[13, 14]$	[73, 75]
6_3	7	≤ 726	[11]

For vertex and edge Folkman numbers $F_v(a_1, \ldots, a_r; q)$ and $F_e(a_1, \ldots, a_r; q)$ we make use of the following useful quantities [56]:

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \tag{5}$$

$$p = \max\{a_1, \ldots, a_r\} \tag{6}$$

It is clear that $K_m \to (a_1, \ldots, a_r)^v$ and $K_{m-1} \not\to (a_1, \ldots, a_r)^v$, so if $q \geq m+1$ then $F_v(a_1, \ldots, a_r; q) = m$. Folkman [27] showed that $F_v(a_1, \ldots, a_k; q) = m$ for $q > m$ and $F_v(a_1, \ldots, a_k; q) = a + m$ for $q = m$.

A $(a_1, \ldots, a_r)$-vertex minimal graph G is one such that $G \in \mathcal{F}_v(a_1, \ldots, a_r; q)$ and $G - v \notin \mathcal{F}_v(a_1, \ldots, a_r; q)$ for all $v \in V(G)$. Similarly, a $(a_1, \ldots, a_r)$-edge minimal graph G is one such that $G \in F_e(a_1, \ldots, a_r; q)$ and $G - e \notin F_e(a_1, \ldots, a_r; q)$ for all $e \in E(G)$. Finally, since $F_v(a_1, a_2, \ldots, a_r; q) = F_v(a_2, \ldots, a_r; q)$ if $a_1 = 1$, we assume $a_i \geq 2$ for all $1 \leq i \leq r$ unless otherwise stated.

3.1 Vertex Colorings

To date, few exact values for $F_v(a_1, \ldots, a_r; m-1)$, are known. We present known results for multicolor vertex Folkman numbers in Table 7. Related comments about select results are given below.

$F_v(2_3; 3)$
: This was shown by the Grötzsch graph in Figure 8, which is also referred to as the Mycielski graph from [58].

$F_v(2_4; 4)$
: This means that the smallest 5-chromatic K_4-free graph has 11 vertices.

$F_v(2_4; 3)$
: This means that the smallest 5-chromatic K_4-free graph has 22 vertices. Furthermore, to date this is the unique known vertex Folkman number of the form $F_v(a_1, \ldots, a_r; q)$ for which $q \leq m - 2$.

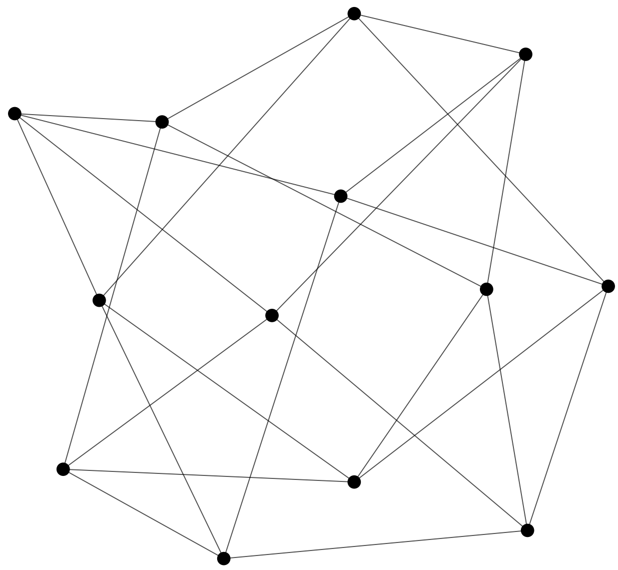

Figure 7: The graph Q from [48], where $\omega(Q) = 4$, $\alpha(Q) = 2$, and $Q \to (3,4)^v$.

$F_v(2_5; 4)$

The only witnessing graphs of the upper bound $F_v(2_5; 4) \le 16$ are the two $R(4,4)$ graphs on 16 vertices. This was an improvement on the previously known bound of $12 \le F_v(2_5; 4) \le 16$ proved by Nenov in [85].

$F_v(3_3; 4)$

While the lower bound of $F_v(3_3; 4) \ge 24$ is established in [98], the upper bound has received considerably less attention. However, Shao et al. suggest a cyclic graph G of order 91 that may witness an upper bound of $F_v(3_3; 4) \le 91$. This candidate graph G is the graph with vertex set $\mathbb{Z}_{91}$ and edges between vertices u and v iff $\min\{|u-v| : 91 - |u-v|\} \in \{1, 2, 4, 7, 8, 14, 16, 17, 23, 27, 28, 32, 34, 37, 45\}$. Interestingly, Harborth and Krause [37] use this graph G to prove that $R(4, 10) \ge 92$.

A more general result for the multicolor case $F_v(3_r; q)$, shown in Theorem 31, was established by Nenov in 2003 [81].

Theorem 31. [81] $F_v(3_r; 2r) = 2r + 7$ for $r \ge 3$.

Remark 32. This was a very important result for $F_v(3_r) = \min\{|V(G)| : G \to (3_r) \text{ and } \omega(G) < 2r\}$. It was an improvement on the bounds of $2r + 5 \le F_v(3_r; 2r) \le 2r + 10, r \ge 4$, proven by Luczak et al. [57]. A similar bound of $2r + 6 \le F_v(3_r; 2r) \le 2r + 8, r \ge 3$, was proven by Nenov in [73].

3.1.1 $F_v(a_1, \ldots, a_r; m)$

In 2001, Łuczak et al. [57] proved that $F_v(a_1, \ldots, a_r; q) \ge 2m - q + 1$. Computing the vertex Folkman number becomes quite interesting when $q = m$, and the authors went on to construct very large classes, and in certain cases, infinitely many graphs that satisfy this property. Let $k \in \{a_r, a_r + 1, \ldots, m - 1\}$, n be an integer such that $n > 2k$, $s = \gcd\{s, k\}$, and $t = k/s$. Using the graph $G = G(n,k) = C_n^{k-1} + K_{m-k-1}$, Łuczak et al. also showed that $G \in \mathcal{F}_v(a_1, \ldots, a_r; m)$ iff

$$\sum_{i=1}^{r} \lfloor (a_i - 1)/t \rfloor < s.$$

Note that $\omega(G) = m - 1$ because

$$\omega(C_n^{k-1}) + \omega(K_{m-k-1}) = m - k - 1 + k = m - 1,$$

so G is K_m-free. This construction is unique in that it enables the construction of infinitely many vertex Folkman graphs. In particular, if $\gcd\{n, k\} = 1$, then $s = 1$ and $t = k$, so

$$\sum_{i=1}^{r} \lfloor (a_i - 1)/t \rfloor = 0 < s = 1,$$

which means that $G \in \mathcal{F}_v(a_1, \dots, a_r; m)$ [57]. Clearly, since there exists infinitely many integers n, k that satsify this construction criteria, an infinite number of graphs can be produced.

It is also interesting to note that not all such graphs are minimal with respect to their membership in $\mathcal{F}_v(a_1, \dots, a_r; m)$. In fact, G is only $(a_1, \dots, a_r)$-vertex minimal if $k = p$, $n \geq 2k + 1$, and of course, if $\gcd\{n, k\} = 1$.

Theorem 33. [56, 57, 73] $F_v(a_1, \dots, a_r; m \geq p + 1) = m + p$ for positive integer $p \geq p + 2$ and for $r \geq 2$.

Remark 34. A weaker bound of $F_v(a_1, \dots, a_r; m) \leq m + p$ still holds with the exception of the unique distinguishing graph $G = K_{m-p-1} + \overline{C_{2p+1}}$ in $\mathcal{F}_v(a_1, \dots, a_r; m)$ with vertex order $m + p$. This is the only distinguishing graph in $\mathcal{F}_v(a_1, \dots, a_r; m)$. Furthermore it was shown that if $G \to (a_1, \dots, a_r)$, $\omega(G) < m$, and $|V(G)| = m + p$, then $G = K_{m+p} - C_{2p+1}$. Stated another way, $F_v(a_r, \dots, a_r; m) = m + p$ is witnessed by the critical graph $G = K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C_{2p+1}}$.

3.1.2 $F_v(a_1, \dots, a_r; m - 1)$

It is well known that $F_v(a_1, \dots, a_r; m) < F_v(a_1, \dots, a_r; m - 1)$, and furthermore, that $\mathcal{F}_v(a_1, \dots, a_r; m-1)$ is non-empty iff $m \geq p+2$. To date, there are few vertex Folkman numbers of the form $F_v(a_1, \dots, a_r; m - 1)$. In 1955 Mycielski [58] found an 11-vertex graph G (see Figure 8) such that $G \to (2_3)^v$ and $\omega(G) = 2$. This proved that $F_v(2_3; 3) \leq 11$. The lower bound of $F_v(2_3; 3) = 11$ was shown to be exact by Chavátal in 1974 [6].

Theorem 35. [73] $F_v(a_1, \dots, a_r; m - 1) \geq m + p + 2$ for $r \geq 2$ and positive integer p such that $m \geq p + 2$.

Remark 36. Equality of $F_v(a_1, \dots, a_r; m - 1) = m + p + 2$ only occurs when $p = 2$ and $m \geq 5$ [33, 57, 67].

Theorem 37. [80] Let $a_1, \dots, a_r$ be positive integers and $m \geq p + 2$. If G is a graph such that $G \to (a_1, \dots, a_r)$ and $\omega(G) < m - 1$, then $|V(G)| \geq m + p + \alpha(G) - 1$ and furthermore, if $|V(G)| = m + p + \alpha(G) - 1$, then $|V(G)| \geq m + 3p$.

$F_v(2_4; 4) = 11$ was proven disjointly in time by Nenov in 1983 [67, 60], who first showed that $F_v(2_4; 4) \leq 11$ (see [72] as well), and a year later by showing that $F_v(2_4; 4) \geq 11$ [68].

Theorem 38. [73]

$$F_v(2_r; m - 1) = \begin{cases} 11, & r = 3, 4, \\ r + 5, & r \geq 5 \end{cases}$$

Remark 39. The exact value of $F_v(2_r; m - 1) = r + 5, r \geq 5$ was proven in [67, 60, 57, 33]. Furthermore, it was shown by Nenov [67, 81] that $K_{r-5} + C_5 + C_5$ is the only witnessing graph in $\mathcal{F}_v(2_r; m - 1)$. Interestingly, $|\mathcal{F}_v(2_4; 4)| = 56$, where each graph in this set is of order 11 [40].

Few other numbers of the form $F_v(a_1, \dots, a_r; m-1)$ are known. One particularly interesting case is $F_v(3, 3; 4) = 14$, which was proved over the span of almost two decades by Nenov [62] and Piwakowski et al. [89]. Nenov showed that $F_v(3, 3; 4) \leq 14$ via existential constructions, and Piwakowski et al. showed that $F_v(3, 3; 4) \geq 14$ using computer programs. Other numbers include $F_v(3, 4; 5) = 13$ [75], $F_v(2, 2, 4; 5) = 13$ [73], and $F_v(4, 4; 6) = 14$ [59].

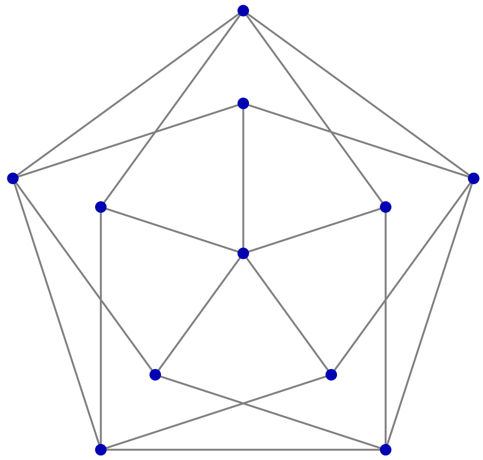

Figure 8: The 11-vertex Mycielski graph [58].

In 2002, Nenov [78] showed that $F_v(2_3, 4; 6) = F_v(2, 3, 4; 6) = 14$. To do this, he also showed that if $G \to (a_1, \dots, a_r)^v$ and there exists one $a_i \geq 2$, then $G \to (a_1, \dots, a_{i-1}, 2, a_i - 1, a_{i+1}, \dots, a_r)$. Thus, if $G \to (2, 3, 4)^v$, for example, then by this fact we also have that $G \to (2_3, 4)^v$.

Theorem 40. [43] $F_v(a_1, \dots, a_r; m-1) \leq m + 3p$ for $p \geq 3$.

Remark 41. This improved the upper bound of $F_v(a_1, \dots, a_r; m-1) \leq m+p^2$ when $m \geq 2p+2$ shown by Łuczak et al. [57]. This bound is exact for $F_v(2, 2, 3; 4) = 14$ [8], though a looser bound of $10 \leq F_v(2, 2, 3; 4) \leq 14$ was proved by Nenov in 2000 [73], and $F_v(3, 3; 4) = 14$ [89]. The value $F_v(2, 2, 3; 4) = 14$ is among the known $F_v(a_1, \dots, a_r; m-1)$ such that $p \leq 4$. Coles et al. obtained this result by constructing all Folkman graphs G on 14 vertices from three-vertex extensions of smaller K_4-free graphs G' on 11 vertices, and for all such graphs G checking to see if $G-v \in \mathcal{F}_v(2, 2, 3; 4)$, where $|V(G-v)| = 13|$. Large-scale computations were used to exhaustively show that no such graphs on 13 vertices exist, and thus the exact value of $F_v(2, 2, 3; 4) = 14$ holds. Łuczak et al. also claimed that $F_v(a_1, \dots, a_r; m-1) \leq 3p^2 + p - mp + 2m - 3$, for $p + 3 \leq m \leq 2p + 1$, without proof.

The boundary case for $F_v(a_1, \dots, a_r; m-1 = p+1)$ for $p \geq 5$ was studied by Kolev and Nenov in 2006 [46]. With the constraint on m, only two such vertex Folkman numbers exist; namely, $F_v(2, 2, p; p+1)$ and $F_v(3, p; p+1)$. For a graph G, if $G \to (3, p)^v$, then $G \to (2, 2, p)^v$.

Theorem 42. [46] $F_v(2, 2, p; p+1) \leq F_v(3, p; p+1)$.

Remark 43. Using Theorem 42, the following inequalities immediately follow.

$$F_v(3, p; p+1) \leq 4p + 2$$

$$F_v(2, 2, p; p+1) \leq 4p + 2$$

Theorem 44. Let $a_1, \dots, a_r$ for $r \geq 2$ be positive integers and $a_r = b_1 + \cdots + b_s$, where b_i are positive integers and $b_i \geq a_{r-1}, i = 1, \dots, s$. Then

$$F_v(a_1, \dots, a_r; a_r + 1) \leq \sum_{i=1}^{s} F_v(a_1, \dots, a_{r-1}, b_i; b_i + 1).$$

Theorem 44 is further generalized in Theorem 45.

Theorem 45. [103] Let $a_1, \ldots, a_r$ for $r \geq 2$ be positive integers and $p_i \geq \max\{a_r, b_i\}, i = 1, \ldots, s$, where p_i and b_i are positive integers. Then,

$$F_v(a_1, \ldots, a_r, \sum_{i=1}^{l} b_i; \sum_{i=1}^{l}(p_i - 1) + 1) \leq \sum_{i=1}^{l} F_v(a_1, \ldots, a_r, b_i; p_i).$$

Lemma 46. Let $a_1, \ldots, a_r$ be positive integers and m and p satisfy Equations 5 and 6. If G is a graph such that $\omega(G) < m - 1$, $G \to (a_1, \ldots, a_r)^v$ and $N(u) \subseteq N(v)$ for some $u, v \in V(G)$, then $|V(G)| \geq m + p + 3$.

Lemma 47. Let $a_1, \ldots, a_r$ be positive integers and m and p satisfy Equations 5 and 6. If G is a graph such that $\omega(G) < m - 1$, $G \to (a_1, \ldots, a_r)^v$, and $\alpha(G) \neq 2$, then $|V(G)| \geq m + p + 3$.

Lemma 48. Let n and p be positive integers and $p \geq 2$. Let G be a graph such that $b_1, \ldots, b_s \in \mathbb{N}$, $1 \leq b_1 \leq b_s \leq p$, and $\sum_{i=1}^{s}(b_i - 1) + 1 = n$, then $G \to (b_1, \ldots, b_2)^v$.

In 2002, Nenov [79] presented a variety of significant results on the vertex Folkman number $F_v(a_1, \ldots, a_r; m - 1)$. We begin with three of his preliminary theorems.

Theorem 49. [79] Let $p \geq 3$ such that $F_v(2, 2, p; p + 1) \geq 2p + 5$. Then, for each $t \geq 2$, $F_v(2_t, p; t + p - 1) \geq t + 2p + 3$.

Theorem 50. [79] For positive integers $a_1, \ldots, a_r$, $p \geq 3$, and $m \geq p + 2$, if $F_v(2, 2, p; p + 1) \geq 2p + 5$, then $F_v(a_1, \ldots, a_r; m - 1) \geq m + p + 3$.

Remark 51. Nenov [81] also proved the special case of this theorem with $a_1 = \cdots = a_r = 3$, $r \geq 3$, showing that $F_v(3, \ldots, 3; m - 1) = 2r + 7$ (see Theorem 31).

Theorem 52. [79] Let $m \geq 6$. Then, the following hold:

$$F_v(a_1, \ldots, a_r; m - 1) = \begin{cases} m + 6 & : p = 3, \\ m + 7 & : p = 4 \end{cases}$$

Remark 53. Equality was shown for $p = 4$ by proving the upper and lower bounds of $F_v(a_1, \ldots, a_r; m - 1)$. The lower bound of $F_v(a_1, \ldots, a_r; m - 1) \geq m + 7$ is a direct result from $F_v(2, 2, 4; 5) = 13$ [74] and Theorem 50. The upper bound was witnessed with the critical Greenwood and Gleason graph shown in Figure 2.

3.1.3 $F_v(2_r; q)$ and $F_v(2_r, p; p + r - 1)$

The vertex Folkman number $F_v(2_r; r + 1)$ has received considerable attention in recent years, mainly because it is directly related to the chromatic number of a graph. In particular, it is well known that $G \to (2_r)^v$ iff $\chi(G) \geq r + 1$. If less than $r + 1$ colors could be used to color the vertices of G, then clearly there exists a color i that is not contained in an r-coloring of the vertices. We begin with some foundational results for this class of vertex Folkman numbers.

Theorem 54. [12] For a graph G with $\chi(G) \geq r + 1$ and $\omega(G) \leq r$ that $|V(G)| \geq r + 3$ and, furthermore, that $G = K_{r-3} + C_5$ is the only graph that witnesses equality in this bound.

Remark 55. This means that for positive integers $r \geq 2$ such that $F_v(2_r; r + 1) = r + 3$, $K_{r-3} + C_5$ is the only graph in $\mathcal{F}_v(2_r; r + 1)$.

From Theorem 23 it follows that $F_v(2_r; r - 1)$ exists iff $r \geq 4$. Similarly, $F_v(2_r; r - 2)$ exists iff $r \geq 5$. Nenov extended these results in the following theorems.

Theorem 56. [67] $F_v(2_r; r - 1) = r + 7$ if $r \geq 8$.

Theorem 57. [85] Let r be a positive integer such that $r \geq 4$. Then, the following hold:

1. $F_v(2_r; r - 1) \geq r + 7$;

2. $F_v(2_r; r-1) = r+7$ if $r \geq 6$;

3. $F_v(2_5; 4) \leq 16$.

Theorem 58. [85] Let $r \geq 5$ be a nonnegative integer. Then, the following hold:

1. $F_v(2_r; r-2) \geq r+9$;

2. $F_v(2_r; r-2) = r+9$ if $r \geq 8$

Remark 59. The numbers $F_v(2_r; r-2)$ for $5 \leq r \leq 7$ are unknown.

Theorem 60. [85] Let $G \in \mathcal{F}_v(2_r; q)$, $q \geq 3$ and $|V(G)| = F_v(2_r; q)$, then

1. G is a vertex-critical $(r+1)$-chromatic graph, and

2. if $q < r+3$ then $\omega(G) = q-1$.

Jensen and Royle [40] proved that $F_v(2_4; 3) = 22$, which is a special case of this result.

Nenov [85, 86] proves many interesting results for the upper bound of $F_v(2_r; q)$. Using a modified construction of the graph P, whose complement is shown in Figure 2, he was able to show new constructions that place a tighter upper bound on this vertex Folkman number. The result of this construction is summarized in Theorem 61

Theorem 61. [85] Let r and s be non-negative integers and $r \geq 3s+6$, then $F_v(2_r; r-s-1) \leq r+2s+7$.

Nenov provided a similar construction to improve the upper bound of $F_v(2_r; r-s-2)$ for non-negative integer s, captured below in Theorem 62.

Theorem 62. [85] Let r and s be non-negative integers such that $r \geq 4s+8$, then $F_v(2_r; r-s-8) \leq r+2s+9$.

It is known that $K_{r+1} \rightarrow (2_r)^v$ and $K_r \not\rightarrow (2_r)^v$, which therefore means that $F_v(2_r; q) = r+1$ if $q \geq r+2$. In 2009, Nenov [86] studied such numbers where $k \geq -1$, since $k \leq -2$ will not fall within this bound (i.e., $q < r+2$). As an immediate result from Folkman [27], it is clear that $F_v(2_r; q)$ exists if an only if $q \geq 3$. Therefore, for the following results on $F_v(2_r; r-k+1)$, it is required that $r \geq k+2$.

Theorem 63. [86] Let r and k be integers such that $-1 \leq k \leq 5$ and $r \geq k+2$. Then,

$$F_v(2_r; r-k-1) \geq r+2k+3,$$

and

$F_v(2_r; r-k-1) = r+2k+3$ if $k \in \{0,2,3,4,5\}$ and $r \geq 2k+2$ or $k \in \{-1,1\}$ and $r \geq 2k+3$.

Remark 64. The $k=0$ case was proved much earlier by Dirac in 1956 [12], where it was also shown that $K_{r-2}+C_5 \in \mathcal{F}_v(2_r; r+1)$ for $r \geq 2$ is the only minimal graph witnessing this arrowing. The cases of $k \in \{1,2\}$ were also proved by Nenov in 1983 [67], where $K_{r-5}+C_5+C_5 \in \mathcal{F}_v(2_r; r)$ for $r \geq 5$. Finally, the case of $k=3$ was proved slightly earlier by Nenov again in 1981 [66]. Also, when $r \geq 8$, $G = K_{r-8}+C_5+C_5+C_5 \in \mathcal{F}_v(2_r; r-1)$ and G is minimal, which is a result first proved in [67].

Theorem 65. [85] Let $r \geq 8$ be a natural number. Then,

1. $F_v(2_r; r-5) \geq r+14$ and $F_v(2_r; r-5) = r+14$ iff $r \geq 13$;

2. $F_v(2_r; r-6) \geq r+16$ if $r \geq 9$ and $F_v(2_r; r-6) = r+16$ if $r \geq 15$;

3. $F_v(2_r; r-7) \geq r+17$, $r \geq 10$ and $F_v(2_r; r-7) = r+17$ iff $r \geq 16$;

4. $F_v(2_r; r-8) \geq r+18$, $r \geq 11$ and $F_v(2_r; r-8) = r+18$ iff $r \geq 17$;

5. $F_v(2_r; r-9) \geq r+20$, $f \geq 12$ and $F_v(2_r; r-9) = r+20$ if $r \geq 19$.

Theorem 66. [86] Let $r \geq 13$ be a natural number. Then,

1. $F_v(2_r; r-10) \geq r+21$ and $F_v(2_r; r-10) = r+21$ if $R(10,3) > 41$ and $r \geq 20$;
2. If $R(10,3) \leq 41$ then $F_v(2_r; r-10) \geq r+22$ and $F_v(2_r; r-10) = r+22$ if $r \geq 21$.

Remark 67. To date, the exact value of $R(10,3)$ is unknown. For more information, see the dynamic survey on small Ramsey numbers maintained by Radziszowski [91].

Theorem 68. [86] Let r and k be natural numbers such that $r \geq k+2$ and $k \geq 12$. Then,

1. $F_v(2_r; r-k+1) \geq r+k+11$;
2. If $k = 12$ and $r \geq 22$ then $F_v(2_r; r-11) = r+23$.

Nenov extended this theorem by using the following result from [82]:

$$G \to (a_1, \ldots, a_r)^v \implies \chi(G) \geq m$$

Since $G \to (2_r)^v \Leftrightarrow \chi G \geq r+1$, then we have that a graph $G \in \mathcal{F}_v(2_{m-1}; q)$ if G is a minimal graph in $\mathcal{F}_v(a_1, \ldots, a_r; q)$.

Similar classes of vertex Folkman numbers, $F_v(3_r, p; 2r+p-1)$, were also studied by Nenov in the same work [73]. In this, he proved the following theorems.

Theorem 69. [73] Let $G \in \mathcal{F}_v(3, p; p+1)$, then for all $r \geq 1$ it is true that $K_{2r-2} + G \in \mathcal{F}_v(3_r, p; 2r+p-1)$.

Theorem 70. [73] Let $p \geq 3$ and $r \geq 1$, then

$$2p + 2r + 2 \leq F_v(3_r, p; 2r+p-1) \leq 4p + 2r.$$

Theorem 71. [73] $2r+10 \leq F_v(3_r, 4; 2r+3) \leq 2r+11$ for $r \geq 1$.

Theorem 72. [73] $F_v(3_{r+1}; 2r+2) \leq 2r+10$ for $r \geq 2$.

3.1.4 Avoiding Smaller Cliques

Obtaining bounds on Folkman numbers becomes much more difficult as the forbidden clique size decreases. Łuczak et al. [57] obtained the bound of $F_v(2_r; r+1-k) \leq r+2k+3$ when $0 \leq k \leq (r-2)/3$. This bound is exact when $k \in \{0,1\}$.

For a graph G with girth $g(G) \geq 2p$ and $|V(G)| \geq 2m-1$, Łuczak et al. observed that $\overline{G} \to (a_1, \ldots, a_r)^v$, since any r-coloring of G will yield a vertex set S of $2a_i - 1$ vertices for some $i \in \{1, \ldots, r\}$. Also, since $g(G) \geq 2p$, it is clear that $\overline{G[S]}$ does not contain a cycle, and thus is bipartite, meaning that $\alpha(\overline{G[S]}) \leq \lceil |S|/s \rceil = a_i$. Therefore, if $\alpha(\overline{G[S]}) < w$, then G does not contain a clique of size w, and therefore $G \in \mathcal{F}_v(a_1, \ldots, a_r; w)$ [57]. This observation led to the theorem by Łuczak et al. [57] stating that if G is a graph such that $g(G) \geq 2p$, $\alpha(G) < l$, $w \geq l$, and $w - l + \frac{1}{2}|V(G)| \geq m$, then $G = K_{w-l} + \overline{G} \in \mathcal{F}_v(a_1, \ldots, a_r; w)$.

Theorem 73. [57] Let $q = 2m - w \geq e^{e^{e^2}}$ and $B = 2q(\log\log q)/\log q + 2\log\log q(\log q)^{2p-1}$. If $w \geq B$ and $p \leq \log q/\log\log q$, then $F_v(a_1, \ldots, a_r; w) \leq q + B$.

Remark 74. This bound was obtained using a probabilistic construction with the Galois circulant graph $G(n,r)$.

Given that $F_v(a_1, \ldots, a_r; m) < F_v(a_1, \ldots, a_r; m-1)$, it might be intuitive to attempt to prove the existence of Folkman numbers using a construction based on recurrence relation between these two Folkman numbers. In fact, this is exactly what Łuczak et al. [57] did to study the most restrictive case of the vertex Folkman numbers (when $w = p+1$), as shown in the following theorem.

Theorem 75. [57] For all $r \geq 2$ and $2 \leq a_1 \leq \cdots \leq a_r$, the following recurrence inequality holds:

$$F_v(a_1, \ldots, a_r; p+1) \leq 1 + (1 + (r-1)(F_2 - 1)) \cdot F_1 + \binom{1 + (r-1)(F_2 - 1)}{F_2} F_2,$$

where $F_1 = F_v(a_1 - 1, \ldots, a_r - 1; p)$ and $F_2 = F_v(a_2, \ldots, a_r; p+1)$.

Remark 76. This type of construction is actually a modification of Folkman's original existential proof [27].

Corollary 77. [57] $F_v(k, l; l+1) \leq 2\sum_{i=0}^{k-1} \frac{l!}{(l-i)!} - 1$.

Remark 78. This was proved by induction on k using the fact that $F_v(a_1, \ldots, a_r; m) \leq p + m$, which implies that $F_v(2, l; l+1) \leq 2l + 1 = 2\sum_{i=0}^{1} \frac{l!}{(l-i)!} - 1$ for all $l \geq 2$. When $k = l$, the upper bound of

$$2\left(\sum_{i=0}^{k-1} \frac{l!}{(l-i)!} - 1\right)$$

collapses to $\lfloor 2k!(e-1) \rfloor - 1$, where e is the number of edges in the graph G. Also, this bound implies that $F_v(3, 3; 4) \leq 19$, which supports the result from Piwakowski et al. [89] that $F_v(3, 3; 4) = 14$.

In 2007, Kolev [49] considered the product of two vertex Folkman numbers $F_v(a_1, \ldots, a_r; s+1)$ and $F_v(b_1, \ldots, b_r; t+1)$, for positive integers s and t. The product of these numbers was shown to bound $F_v(a_1, \ldots, a_r; st+1)$, meaning that

$$F_v(a_1, \ldots, a_r; st+1) \leq F_v(a_1, \ldots, a_r; s+1) \times F_v(b_1, \ldots, b_r; t+1).$$

This result can be generalized to

$$F_v(kl_r; kl+1) \leq F_v(k_r; k+1) \times F_v(l_r; l+1)$$

if we let $a_i = s = k$, $b_i = t = l$.

In 2008, Xu et al. [104] studied the upper bounds of vertex Folkman numbers based on compositions. We include their main results in the following theorems.

Theorem 79. [104] Let $a_1, \ldots, a_k, b_1, \ldots, b_k, p, q$ be positive integers such that $\max\{a_1, \ldots, a_k\} \leq p$ and $\max\{b_1, \ldots, b_k\} \leq q$. It then holds that

$$F_v(a_1b_1, \ldots, a_kb_k; pq+1) \leq F_v(a_1, \ldots, a_k; p+1) \cdot F_v(b_1, \ldots, b_k; q+1).$$

Efforts to improve Folkman's theorem using the induced subgraph constraint and without controlling the forbidden clique size were started in 1991 by Brown et al. [2], in which they proved the following theorem.

Theorem 80. For every $r \in \mathbb{N}$ there exists constants C and c such that for every graph G of order n,

$$cn^2 \leq \max_G \left\{ \min_H \left\{ |V(H)| : H \underset{\text{ind}}{\rightarrow} (G)_r^v \right\} \right\} \leq Cn^2 \log^2 n.$$

Remark 81. This is equivalent to saying

$$cn^2 \leq \max\{F_v(r, G)\} \leq Cn^2 \log^2 n.$$

After Nešetřil and Rödl proved the existence of Folkman numbers $F_v(r, \mathcal{G})$ for general classes of graphs $\mathcal{G}$ in 1976 [88], Dudek and Rödl [14] [18] asked the general question of determining $F_v^{ind}(r, \mathcal{G})$, the minimal order of a graph H such that $\omega(H) = \omega(\mathcal{G})$ and for every r-coloring of the vertices of H there exists a monochromatic, induced copy of $\mathcal{G}$. This is equivalent to finding a minimal graph $H \underset{\text{ind}}{\rightarrow} (G)_b^r$ Clearly, the induced subgraph requirement places a tighter constraint on the structure of H (i.e. $F_v^{ind}(\cdot) \geq F_v(\cdot)$). Critical results are captured in the following theorems.

Theorem 82. [18] $F_v(r, n; n+1) \leq cn^2(\log n)^4 = \mathcal{O}(n^2(\log n)^4)$ for some constant $c = c(r)$.

Remark 83. This was proven with a construction for graphs H of order $cn^2 \log^4 n, c = c(\alpha)$, such that $\omega(H) = n$ and for all induced subgraphs $H[V]$, where $V \subset V(H)$ and $|V| = \alpha|V(H)|$, $H[V]$ contains a copy of K_n, and thus avoids a clique of size $n+1$. These graphs were randomly constructed from the vertex set $\mathcal{V}$ of projected planes $PG(2, q)$, where q is prime.

Looser bounds for this Folkman number were obtained in [20], in which Dudek and Rödl studied the following function f presented by Erdős and Rogers: given integers $2 \leq s < t$, let $f_{s,t}(n) = \min\{\max\{|S| : S \subset V(H) \text{ and } H[S] \text{ contains no } K_s\}\}$, where the minimum is taken over all K_t-free graphs H of order n. In their work, Dudek and Rödl established the bounds shown in the following lemmas.

Lemma 84. [20] For every integer $s \geq 2$ there exists a positive constant $c = c(s)$ such that for every integer n it is true that $f_{s,s+1}(n) \leq cn^{2/3}$.

Lemma 85. [20] For any arbitrarily small $\epsilon > 0$ and a given integer $k \geq 2$ there is a constant $s_0 = s_0(\epsilon, k)$ such that for every $s \geq s_0$ and every n,

$$c_1 n^{1/(1+(\frac{s}{s-1})^{k-1})} \leq f_{s,s+k}(n) \leq c_2 n^{(\frac{k+1}{2k+1})+\epsilon}$$

If the asymptotic of these lemmas is taken in r, the following upper bounds on $F_v(r, n, n+1)$ and $F_v(r, n, n+k)$ are established.

Theorem 86. [20] For every integer s there is a positive constant $c = c(s)$ such that for every integer r it is true that

$$F_v(r, n, n+1) \leq cr^3.$$

Theorem 87. [20] For any arbitrarily small $\epsilon > 0$ and a given positive integer k there exists a constant $s_0 = s_0(\epsilon, k)$ such that for every $s \geq s_0$ and every integer r it is true that

$$F_v(r, n, n+k) \leq cr^{2+\frac{1}{k}+\epsilon}.$$

Another interesting byproduct of studying the Erdős-Rogers function is the observation that if $f_{s,t}(n) < u$ for some non-negative integer u, then $F_v(\lfloor n/u \rfloor, s, t) \leq n$. Similarly, if $n < F_v(r, s, t)$, then $\lceil n/r \rceil \leq f_{s,t}(n)$. While these two facts are not the converse of one another, one may now see the clear relationship between the vertex Folkman number and the Erdős-Rogers function.

Theorem 88. [18] $F_v(r, n, \lceil (2+\epsilon)n \rceil) \leq cn$ for some $r \in \mathbb{N}$, some arbitrarily small constant $\epsilon > 0$, and constant $c = c(r, \epsilon)$.

Remark 89. This is a direct result of Theorem 82, which arises when cliques of size bigger than $q = (2 + \mathcal{O}(1))$ are forbidden. This result is also complementary to the results obtained by Łuczak et al. [57] and Kolev et al. [43], who found that

$$F_v(r, n, r(n-1)) \leq r(n-1) + n^2 + 1$$

and

$$F_v(r, n; r(n-1)) \leq r(n-1) + 3n + 1,$$

respectively. The bound obtained by Kolev et al. is significantly tighter than that of Łuczak et al. as n tends towards infinity. The proof was shown by probabilistic construction from random graphs $G = G(m, 1 - \frac{c}{m})$, where $c = c(r, \epsilon)$, such that $\omega(G) < (2 \log c)/cm$ and every subset of vertices $U \subset V(G), |U| = \lceil \alpha m \rceil$ induces a clique of size at least $[2 \log c]/[(2+\epsilon)c]c$. These criteria were then used to show that

$$F(r, c, \lceil (2+\epsilon)n \rceil) \leq F\Big(r, n, \Big\lceil \frac{2 \log c}{c} m \Big\rceil\Big) \leq m \leq \frac{(2+\epsilon)c}{2 \log c} n.$$

In 2010, Dudek and Rödl [18] showed that for some $c = c(r)$ it is true that $F_v(r, k, k+1) \leq ck^2 \log^4 k$, an almost quadratic upper bound. Dudek and Rödl note that improving this bound would be of significant progress. Łuczak et al. [57] showed that $F_v(r, k, r(k-1)) \leq r(k-1)+k^2+1$, and later Kolev and Nenov independently showed that $F_v(r, k, r(k-1)) \leq r(k-1)+3k+1$ [43].

The order of graphs satisfying this (induced) arrow property was further constrained by Dudek and Rödl [18], who added the additional constraint that $\omega(H) = \omega(G)$. In doing so, they were only able to achieve an upper bound of $\mathcal{O}(n^3 \log^3)$, as shown in the following theorem.

Theorem 90. [18] For a given nonnegative integer r, there exists a constant $c = c(r)$ such that for every graph G of order n

$$\min\left\{|V(H)| : H \underset{\text{ind}}{\rightarrow} (G)^v_r \text{ and } \omega(H) = \omega(G)\right\} \leq Cn^3 \log^3 n = \mathcal{O}(n^3 \log^3 n).$$

Remark 91. This is the same as saying that $F^{ind}_v(r, G) \leq cn^3 \log^3 n$. In [18] Dudek and Rödl also show that $F^{ind}_v(r, K_n) \leq cn^2 \log^4 n$ for some constant $c = c(r)$.

This upper bound was improved in [21] as follows:

Theorem 92. [21] $F^{ind}_v(r, G) \leq \frac{cn^3}{\omega(G)}(\log n)^c$ for some constant $c = \mathcal{O}(r)$.

Theorem 93. This theorem was actually proved with the explicit case of $c = 5$. The optimal value of c was not found.

This bound can be tightened even further by considering the graph $G = K_{n/2} \cup \overline{K_{n/2}}$, which means that $\Omega(n^2) = F_v(r, G) = \mathcal{O}(n^2 \log^5 n)$ [23]. Dudek then made the generalization that $F_v(r, G) = \Theta(n^{2+o(1)})$ for any graph G of order n with $\omega(G) = \Theta(n)$, as is the case for $G = K_{n/2} \cup \overline{K_{n/2}}$.

3.2 Edge Colorings

Many lower bounds for edge Folkman numbers $F_e(a_1, \ldots, a_r; q)$ rely on Lin's inequalities given in [54], which are as follows:

$$F_e(a_1, \ldots, a_r; R) \geq R + 2$$
$$F_e(a_1, \ldots, a_r; R-1) \geq R + 4$$

Nenov improved these bounds in [65]. Let $M(a_1, \ldots, a_r)$ be the Ramsey multiplicity of the corresponding Ramsey number $R(a_1, \ldots, a_r)$. Nenov showed that if $M(a_1, \ldots, a_r) = 1$ then $F_e(a_1, \ldots, a_r; R) \geq R+3$. Furthermore, it was shown that if K_R has an r-coloring with only one monochromatic a_i-clique of the ith color ($a_1 \geq 4$) and no monochromatic a_j-clique of the j-th color for any $i \neq j$, then $F_e(a_1, \ldots, a_r; R) \geq R + 4$.

Years later, using a similar approach, Nenov [71] showed that if $M(a_1, \ldots, a_r) = 1$ and the graph $G \notin \mathcal{F}_e(a_1, \ldots, a_r; R)$ for $G = K_{R-4} + \overline{C_7}$, then $F_e(a_1, \ldots, a_r; R-1) \geq R+4$. It was also shown that if K_R has an edge coloring an r-coloring with only one monochromatic a_i-clique of the ith color ($a_1 \geq 4$) and no monochromatic a_j-clique of the jth color for any $i \neq j$, and $G \notin \mathcal{F}_e(a_1, \ldots, a_r; R)$ for $G = K_{R-5}+\overline{C_9}$, then $F_e(a_1, \ldots, a_r; R-1) \geq R+5$. In the special two-color case where $R = R(3, 4)$, the bound $F_e(3, 4; 9) \geq 14$ is shown. This bound is in fact exact due to a prior result by Nenov in [64]. In this work Nenov proved that for any two nonnegative numbers r, s, $C_{2r+1} + C_{2s+1} + K_4$ is in the set $\mathcal{F}_e(3, 4; R)$. When $r = s = 2$, it follows that $F_e(3, 4; 9) \leq 14$, thus proving equality in the bound.

In 1983, Nenov [67] showed that for integer colors $a_1, \ldots, a_r$, where and $a_i \geq$ for all $1, \ldots, r$, $r \geq 2$, then $F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r)-2) \geq R(a_1, \ldots, a_r)+6$. The Folkman number $F_e(3_3; 15) = 23$ is one such number of this kind [63]. Since $a_i \geq 3$ and $r \geq 2$, $R(a_1, \ldots, a_r) > 2+\max\{a_1, \ldots, a_r\}$, which implies that such Folkman numbers exist. The exact value of
$F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r)-2) = R(a_1, \ldots, a_r)+6$ was shown to be true if an only if $K_{R-7}+Q \rightarrow (a_1, \ldots, a_r)^e$ or $K_{R-9} + C_5 + C_5 + C_5 \rightarrow (a_1, \ldots, a_r)^e$, where $R = R(a_1, \ldots, a_r)$. In this case, $\overline{Q}$ is the graph shown in Figure 7. Based on this result, Nenov showed that $F_e(3, 5; 12) \geq 12$

and $F_e(4,4;16) \geq 24$. In combination with the bound $F_e(3,4;7) \geq 15$ and $F_e(3,4;8) = 16$ (found in [42]), the bound $F_e(3,4;7) \geq 17$ emerges. Of further interest, the Folkman number $F_e(3_3;15) = 23$ proves that $K_8 + C_5 + C_5 + C_5 \to (3,3,3)^e$ [87]. Another known result due to Kolev et al. is $F_e(3_3;13) \leq 30$ [51]. $F_e(3_3;15)$ was proven by Nenov in [63], in which he showed that $K_8 + C_{2p+1} + C_{2q+1} + C_{2r+1}$ is a critical $(3,3,3)$-Folkman graph, and when $r = s = q = 2$ the bound $F_e(3_3;15) \leq 23$ is obtained. Together, this proves the exactness of the bound.

General upper bounds for edge Folkman numbers have been found to be closely related to certain Ramsey numbers. In particular, Kolev [50] considered the following relationship. Let $a, \alpha \in \mathbb{Z} \cup \{0\}$, such that $R(3,a) = R(3,a-1) + a - \alpha, \alpha \geq 4$. Under these conditions, given the existence of a graph U such that $\omega(U) = a-1$, $U \to (a-1, a-2)^v$, and $U \to (3, a-3_r)^v$, and $R(3,a) - 3a + \alpha + 5 \geq R(3,a-2)$, then $F_e(3,a;R-a+\alpha+4) \leq R(3,a) - 2a + \alpha + 4 + |V(U)|$. Searching for such graphs U is very much a nontrivial problem, and to date there have been no such graphs that serve as appropriate candidates for this theorem.

3.3 Connections Between Edge and Vertex Colorings

Given the Ramsey number $R(a_1,\ldots,a_r)$, it is clear that $F_e(a_1,\ldots,a_r;q)$ for all $q > R(a_1,\ldots,a_r)$, simply because the Ramsey number already attains this lower bound on the size of the graph n. Given this relation, the following lemma emerges.

Lemma 94. [57] Let $R_i = R(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_r)$ for all $i \in \{1,2,\ldots,r\}$. If $H \in \mathcal{F}_v(R_1,R_2,\ldots,R_r;w)$, then $H + v \in \mathcal{F}_v(a_1,\ldots,a_r;w+1)$.

Remark 95. Clearly, this enables one to derive bounds on edge Folkman numbers using known bounds on related vertex Folkman numbers that are increased by one.

Corollary 96. [57] For $k, l \geq 3$, let $M = \max\{R(k-1,l), R(k,l-1)\}$ and $m = \min\{R(k-1,l), R(k,l-1)\}$, then it is true that $F_e(k,l;M+2) \leq 2\sum_{i=0}^{m-1} \frac{M!}{(M-i)!}$.

Remark 97. This bound collapses to $F_e(k,k;M+2) \leq \lfloor 2M!(e-1) \rfloor$ when $k = l$. Furthermore, this is a very loose upper bound. The bound obtained when $k = l = 3$ is $F_e(3,3;5) \leq 20$ is close to the true value of 15, but other values for k and l are much farther off. However, it is still very useful for closing the gap between the original existential question and true values.

Theorem 98. [86] Let $a_1,\ldots,a_r$ be nonnegative integers where $a_i \geq 2$ for $i \in \{1,\ldots,r\}$ and $m = \sum_{i=1}^{r}(a_i - 1) + 1$, and let k be an integer k such that $m - k > \max\{a_1,\ldots,a_r\}$. Then, the following inequalities hold:

$$\begin{aligned}
F_v(a_1,\ldots,a_r;m-k) &\geq m+2k+2 \text{ if } -1 \leq k \leq 5\\
F_v(a_1,\ldots,a_r;m-6) &\geq m+13\\
F_v(a_1,\ldots,a_r;m-7) &\geq m+15\\
F_v(a_1,\ldots,a_r;m-8) &\geq m+16\\
F_v(a_1,\ldots,a_r;m-9) &\geq m+17\\
F_v(a_1,\ldots,a_r;m-10) &\geq m+19\\
F_v(a_1,\ldots,a_r;m-11) &\geq m+20\\
F_v(a_1,\ldots,a_r;m-11) &\geq m+21 \text{ if } R(10,3) \leq 41\\
F_v(a_1,\ldots,a_r;m-k) &\geq m+k+10 \text{ if } k \geq 12
\end{aligned}$$

In order to prove Theorem 98, Nenov used Lemmas 99 and 101, shown below.

Lemma 99. [86] Let $q \geq 4$ be an integer and G be a minimal graph in $\mathcal{F}_v(2_r;q-1)$. It then holds that

$$F_v(2_r;q-1) \geq F_v(2_r;q) + \alpha(G) - 1.$$

An immediate corollary from this Lemma is as follows.

Corollary 100. [86] Let q and r be integers such that $4 \leq q < r+3$. The following then hold:

- $F_v(2_r; q-1) \geq F_v(2_r; q) + 1$, and
- If $F_v(2_r; q) + 1 \geq R(q-1, 3)$ then the equality in (1) is strict.

Lemma 101. [86] Let m and k be positive integers such that $m \geq k+3$ and $2m-1 < R(m-k, 3)$, and let $F_v(2_r; r-k+1) \geq r+m$ for any $r \geq m-1$. It then holds that $F_v(2_r; r-k+1) = r+m$ if $r \geq m-1$.

Theorem 102. [73] Let $G \in \mathcal{F}_v(2, 2, p; p+1)$, then for any $r \geq 2$, $K_{r-2}+G \in \mathcal{F}_v(2_r, p; p+r-1)$.

Theorem 103. For any $r \geq 2$, it is true that $r + 10 \leq F_v(2_r, 4; 3 + r) \leq r + 11$.

Remark 104. The complement Q of the Greenwood and Gleason graph shown $\overline{Q}$ in Figure 2 is a member of $\mathcal{F}_V(2, 2, 4; 5)$. Using Theorem 102 we obtain that $K_{r-2} + Q \in \mathcal{F}_v(2_r, 4; 3 + r)$, and since $|V(Q)| = 11$, the upper bound is established. The lower bound follows trivially from Theorem 72.

Theorem 105. [86] Let $a_1, \ldots, a_r$ be non-negative integers such that $a_i \geq 2$ for all $1 \leq i \leq r$, and let $R - k > \max\{a_1, \ldots, a_r\}$, where $k \geq -1$ is an integer and $R = R(a_1, \ldots, a_r)$. Then, the following inequalities hold:

$$\begin{aligned}
F_e(a_1, \ldots, a_r; R-k) &\geq R + 2k + 2 \text{ if } -1 \leq k \leq 5\\
F_e(a_1, \ldots, a_r; R-6) &\geq R + 13\\
F_e(a_1, \ldots, a_r; R-7) &\geq R + 15\\
F_e(a_1, \ldots, a_r; R-8) &\geq R + 16\\
F_e(a_1, \ldots, a_r; R-9) &\geq R + 17\\
F_e(a_1, \ldots, a_r; R-10) &\geq R + 19\\
F_e(a_1, \ldots, a_r; R-11) &\geq R + 20\\
F_e(a_1, \ldots, a_r; R-11) &\geq R + 21 \text{ if } R(10, 3) \leq 41\\
F_e(a_1, \ldots, a_r; R-k) &R + k + 10 \text{ if } k \geq 12
\end{aligned}$$

3.4 Open Problems

Problem 106. $F_v(2_5; 4) =?$

Problem 107. Determine $F_v(2_r; r-2)$ for $5 \leq r \leq 7$.

Problem 108. [19] Given an integer $r \geq 2$, is it true that

$$\lim_{k\to\infty} \frac{F_v(r, k, k+1)}{k} = \infty?$$

Problem 109. [19] Is it true that for each $\epsilon > 0$ and integer $r \geq 2$

$$\lim_{k\to\infty} \frac{F_v(r, k; (1+\epsilon)k)}{k} < \infty?$$

Problem 110. [21] Is it true that

$$\lim_{n\to\infty} \max_G \{\frac{F_v^{ind}(r, G)}{n^2}\} = \infty?$$

Problem 111. [19] $F_e(3_3; 4) \leq 3^{3^4}$?

4 Going Further with Hypergraphs

We begin our discussion with the induced Folkman number $F_v^{\mathsf{ind}}(r, \mathcal{G})$ for k-uniform hypergraphs $\mathcal{G}$ of order n, which is the minimum order of a k-uniform hypergraph $\mathcal{H}$ such that $\omega(\mathcal{H}) = \omega(\mathcal{G})$ and that for every r-coloring of the vertices of $\mathcal{H}$ there exists a monochromatic, induced copy of $\mathcal{G}$. Dudek et al. [22] showed that $F_v(r, \mathcal{G})$ for hypergraphs $\mathcal{G}$ is *almost* quadratic, with the bound $F_v^{\mathsf{ind}}(r, \mathcal{G}) \leq cn^2(\log n)^2$ for any k-uniform hypergraph $\mathcal{G}$ on n vertices. In doing so, they also showed that for every pair of positive integers k, n there exists a constant c such that $F_v(r, \mathcal{G}) \geq cr^2$ for a hypergraph $\mathcal{G}$ on n vertices and any non-negative number of colors r. This result is characterized in Theorem 112 below.

Theorem 112. [22] For all $n \geq 1$ and $k \geq 3$ there are constants c and C such that

$$cn^2 \leq \max\{F_v^{\mathsf{ind}}(r, \mathcal{G})\} \leq cn^2(\log n)^2,$$

where the maximum is taken over all k-uniform hypergraphs $\mathcal{G}$ of order n.

Interestingly, when considered asymptotically, this upper bound is superior to the one for regular graphs. Dudek et al. [24] showed a special case of this bound with $\mathcal{G} = \mathcal{K}_n^k$, the complete hypergraph on n vertices which is k-uniform, captured below in Theorem 113.

Theorem 113. [24] $F_v(r, \mathcal{K}_n^k) \leq cr(\log r)^{\frac{1}{k-2}}$ for some constant $c = \mathcal{O}(k, n)$.

It may be intuitive to assume that hypergraphs with small clique numbers have linear or sub-quadratic Folkman numbers. However, Dudek et al. [22] have shown this is not the case with Theorem 114.

Theorem 114. [22] For all natural numbers $r \geq 1$ and $k \geq 3$ there are constants c and $d = \mathcal{O}(k)$ such that for every n there exists a k-uniform hypergraph $\mathcal{G}$ of order n and clique number $\omega(\mathcal{G}) \leq d$ such that

$$cn^2 \frac{\overbrace{\log\log\ldots\log\log n}^{k-1}}{\underbrace{\log\log\cdots\log n}_{k-2}} \leq F_v(r, \mathcal{G}).$$

It is not known whether or not this same bound holds for simple graphs. In addition, this result can be restated in terms of the asymptotic value of r, as is done in Theorem 115.

Theorem 115. [22] For every k and n there is a constant $c = C(n, k)$ such that for any k-uniform hypergraph $\mathcal{G}$ of order n and any number of colors r

$$F_v(r, \mathcal{G}) \leq cr^2.$$

Remark 116. For simple graphs, this upper bound is cubic in r, yet again showing a tighter bound for hypergraphs. Mubayi and Dudek [24] showed that when $\mathcal{G}$ is the complete k-uniform hypergraph the following holds true:

$$F_v(r, \mathcal{K}_n^k) \leq cr(\log r)^{\frac{1}{k-2}},$$

for a constant $c = C(k, n)$. The analogous result for simple graphs is again cubic in r.

Dudek also went on to show results for 3-uniform hypergraphs, drawing on results for hypergraph Ramsey numbers $R_k(s, t)$, which is the minimum integer n such that every 2-coloring of $\mathcal{K}_n^k$ contains a monochromatic $\mathcal{K}_s^k$ or $\mathcal{K}_t^k$. In particular, using the result of Conlon, Fox, and Sudakov [9], which states that $R_e(4, t) \geq 2^{t\log t}$ for some constant c, Dudek proved Theorem 117.

Theorem 117. [22] $F_v^{\mathsf{ind}}(r, \mathcal{G}_n^3 = \mathcal{R}_{n/2}^3 \cup \overline{\mathcal{K}_{n/2}^3}) = \Theta(n^{2+o(1)})$.

Dudek et al. [22] further studied k-uniform hypergraphs for $k \geq 4$. After applying the stepping-up lemma from Conlon, Fox, and Sudakov [9] to the bound $R_3(4,t) \geq 2^{ct \log t}$ for 3-uniform hypergraphs, one obtains that $s = \lceil (5k/2) \rceil - 3$. This means that for k-uniform hypergraphs, $k \geq 4$, the following lower bound holds:

$$R_k(s,t) \geq 2^{2^{\cdot^{\cdot^{\cdot^{2^{ct \log t}}}}}},$$

where the tower is repeated $k-2$ times. As a direct result, there exists some k-uniform hypergraph $\mathcal{G}_n^k = \mathcal{R}_{n/2}^k \cup \overline{\mathcal{K}}_{n/2}^k$ such that $\omega(\mathcal{G}_n^k) = \mathcal{O}(1)$ and

$$\alpha(\mathcal{G}_n^k) < \mathcal{O}\left(\frac{\log^{k-2}(n)}{\log^{k-1}(n)}\right).$$

4.1 Open Problems

Problem 118. [22] Is there a family of hypergraphs $\{\mathcal{G}_n\}$ for which $F_v(r, \mathcal{G}_n)$ is asymptotically larger than n^2?

Problem 119. [22] Is there a hypergraphs $\mathcal{G}$ of a fixed order n such that $F_v(r, \mathcal{G}) = \Omega(r^2)$?

Problem 120. [22] Do hypergraphs with small clique numbers have linear, or much smaller than quadratic, (vertex) Folkman numbers?

Problem 121. [22] Tighten the bound of $F_v(r,k;k+1) = \mathcal{O}(n^2 \log^4 n)$ for simple graphs, and try to extend these results to edge Folkman numbers.

Problem 122. [22] Tighten the bound of $F_v(r,G) = \Omega O(n^4)$ for (hyper)graphs, and try to extend these results to edge Folkman numbers.

Problem 123. [22] Determine if

$$\max\{F_v^{\mathsf{ind}}(r, \mathcal{G})\} = \Omega\left(n^2 \frac{\log^{k-1}(n)}{\log^{k-2}(n)}\right)$$

is true for all k-uniform hypergraphs $\mathcal{G}_n^k$, $k \geq 4$, with $\omega(\mathcal{G}_n^k) = s - 1 = k$, which implies that $s \geq \lceil \frac{5}{2}k \rceil - 3$.

5 Complexity and Computability

The central concept behind Folkman numbers, and more generally, graph Ramsey Theory, is that of arrowing. The complexity of deciding the boolean value of a particular arrowing instance has been well studied from the context of complexity theory, leading to several known results and implications on the difficulty of computing Folkman numbers. The general arrowing problem, i.e. given graphs F, G, and H, does $F \to (G,H)$?, is Π_2^p-complete [94], which is equivalent to $coNP^{NP}$, the class of problems whose complements are solvable by a nondeterministic polynomial-time Turing machine with access to an NP oracle. As Folkman-related problems do not fix the structure of G or H, the problem of determine Folkman graph set membership clearly also lies in Π_2^p, simply because determining whether a graph F is a member of the (edge or vertex) Folkman graph set $\mathcal{F}(G,H;I)$ is a generalization of the standard arrowing problem $F \to (G,H)$. However, computing upper bounds for (edge or vertex) Folkman numbers of the form $F(G,H;q)$, i.e. determining if $F(G,H;q) \leq n$, is in Σ_3^p and is NP-hard.

If G and H are fixed, however, the problem reduces to $coNP$. There are many cases where fixed structures for G or H can reduce the complexity, as shown in Table 8. Remarks on these results follow as needed. We use the notation kG to denote the graph with k disjoint copies of G. Similarly, we use the notation T_3 to denote the class of 3-connected graphs together with K_3.

Table 8: Known complexity results for special classes of arrowing problems.

Problem	*Fixed*	*Complexity*
$F \to (G, H)$		Π_2^p-complete [94]
$F \to (T, K_n)$	tree T of order ≥ 3	Π_2^p-complete [94]
$F \to (K_2, H)$		NP-complete [10]
$F \to (G, H)$	G, H	coNP
$F \to (G, H)$	$G, H \in T_3$	coNP-complete [5]
$F \to (kK_2, H)$	k, H	P [5]
$F \to (K_2, H)$	H	P
$F \to (K_{1,2}, K_{1,m})$		P [5]
$K_n \to (G, H)$		NP-hard [4]
$F \to (\underbrace{G, \ldots, G}_{k})^v$	G, $\lvert E(G)\rvert \geq 1$	coNP-complete [93]
$F \to (\underbrace{G, \ldots, G}_{k})^v$	G	coNP-complete [1]

$G = K_2$

The predicate $F \to (K_2, H)$ is only true iff every red-blue edge coloring of F that does not contain a red edge (i.e. a K_2)) contains a blue H, which is equivalent to the SUBGRAPH ISOMORPHISM PROBLEM, one of Cook's famous 21 NP-complete problems.

$F \to (\underbrace{G, \ldots, G}_{k})^v$

$F \to (P_3, P_3, P_3)$ is one instance of this class of arrowing problem which is coNP-complete since it is the complement of the 3-edge-colorability [38].

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