Recent Progress
on
Small and Large Ramsey Numbers

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Small Ramsey Numbers

Dynamic Survey at the El-JC, 1994-2024

Revisions

1993, February	preliminary version, RIT-TR-93-009 [Ra2]
1994, July 3	posted on the web at the *ElJC*
1994, November 7	*ElJC* revision #1
1995, August 28	*ElJC* revision #2
1996, March 25	*ElJC* revision #3
1997, July 11	*ElJC* revision #4
1998, July 9	*ElJC* revision #5
1999, July 5	*ElJC* revision #6
2000, July 25	*ElJC* revision #7
2001, July 12	*ElJC* revision #8
2002, July 15	*ElJC* revision #9
2004, July 4	*ElJC* revision #10
2006, August 1	*ElJC* revision #11
2009, August 4	*ElJC* revision #12
2011, August 22	*ElJC* revision #13
2014, January 12	*ElJC* revision #14
2017, March 3	*ElJC* revision #15
2021, January 15	*ElJC* revision #16
2024, June 7	*ElJC* revision #17

Version #17: 133 pages, 965 references

Frank Plumpton Ramsey
22 February 1903 - 19 January 1930

Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color

- $2 - colorings \cong graphs, \quad R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist

Paul Erdős, Ronald Graham, Fan Chung (1986)

Asymptotics

bounds in diagonal cases

- Erdős (1947), Spencer (1975)

$$R(n,n) > \frac{\sqrt{2}}{e} 2^{n/2} n$$

- Sah (2020), Conlon (2010)

$$R(n+1,n+1) \leq \binom{2n}{n} e^{-c_2(\log n)^2} \leq \binom{2n}{n} n^{-c_1 \frac{\log n}{\log \log n}}$$

- Campos-Griffiths-Morris-Sahasrabudhe (2023)

$$R(n,n) \leq (4-\epsilon)^n$$

Asymptotics

Ramsey numbers avoiding K_3, K_4

- Shearer, upper bound (1983)
  Bohman-Keevash, lower bound (2009-2013-2021)
  Fiz Pontiveros-Griffiths-Morris, lower bound (2013-2020)

$$\left(\frac{1}{4} + o(1)\right)\frac{n^2}{\log n} \leq R(3, n) \leq (1 + o(1))\frac{n^2}{\log n}$$

- Mattheus-Verstraëte (2024)

$$c_1 \frac{n^3}{\log^4 n} \leq R(4, n) \leq c_2 \frac{n^3}{\log^2 n}$$

Values and bounds on $R(k, l)$

two colors, avoiding K_k, K_l

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 41	47 50	53 59	60 68	67 77	74 87
4		18	25	36 41	49 61	59 84	73 115	92 149	102 191	128 238	138 291	147 349	158 417
5			43 48	59 87	80 143	101 216	133 316	149 442	183 633	203 848	233 1138	267 1461	275 1878
6				102 165	115 298	134 495	183 780	204 1171	262 1804	294 2566	347 3703	5033	401 6911
7					205 540	219 1031	252 1713	292 2826	405 4553	417 6954	511 10578	15263	22112
8						282 1870	329 3583	343 6090	457 10630	16944	817 27485	41525	873 63609
9							565 6588	581 12677	22325	38832	64864		
10								798 23556	45881	81123			1313

[*Small Ramsey Numbers*, revision #17], upper bounds until 2019

Small $R(k,l)$, references

k \ l	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR McZ	Ka2 GR	Ex5 GoR1	Ex20 GoR1	Kol1 Les	Kol1 GoR1	Kol2 GoR1	Kol2 GoR1
4	GG	Ka1 MR4	Ex19 ~~MR5~~	Ex3 ~~Mac~~	ExT ~~Mac~~	Ex16 ~~Mac~~	HaKr1 ~~Mac~~	ExT ~~Spe4~~	SuLL ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~	ExT ~~Spe4~~
5		Ex4 AnM	Ex9 ~~HZ1~~	CaET ~~HZ1~~	HaKr1 ~~Spe4~~	Kuz ~~Mac~~	ExT ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	ExT ~~HW+~~
6			Ka2 ~~Mac~~	ExT ~~HZ1~~	ExT ~~Mac~~	Kuz ~~Mac~~	Kuz ~~Mac~~	Kuz ~~HW+~~	Kuz ~~HW+~~	Kuz ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
7				She2 ~~Mac~~	XSR2 ~~HZ1~~	Kuz ~~HZ2~~	Kuz ~~Mac~~	XXER ~~HW+~~	XSR2 ~~HW+~~	XuXR ~~HW+~~	~~HW+~~	~~HW+~~
8					BurR ~~Mac~~	Kuz ~~Eol~~	Kuz ~~HZ2~~	~~HW+~~	~~HW+~~	XXER ~~HW+~~	~~HW+~~	2.3.h ~~HW+~~
9						She2 ~~ShZ1~~	XSR2 ~~Eol~~	~~HW+~~	~~HW+~~	~~HW+~~		
10							She2 ~~Shi2~~	~~HW+~~	~~HW+~~			2.3.h

2019-2023: an avalanche of improved upper bounds
after LP attack for higher k and l by Angeltveit-McKay

New upper bounds for $R(k, l)$

Angeltveit-McKay, 2019-2024

k \ l	5	6	7	8	9	10	11	12	13	14	15
4	25	40	58	79	105	135	170	210	256	307	364
5	46	85	133	193	282	381	511	672	860	1081	1341
6		160	270	423	651	944	1346	1855	2499	3301	4305
7			492	832	1368	2119	3197	4665	6653	9260	12635
8				1518	2662	4402	7040	10836			
9					4956	8675	14631				
10						16064					

60 Years of $R(5,5)$

year	reference	lower	upper	
1965	Abbott	38		quadratic residues in $\mathcal{Z}_{37}$
1965	Kalbfleisch		59	pointer to a future paper
1967	Giraud		58	LP
1968	Walker		57	LP
1971	Walker		55	LP
1973	Irving	42		sum-free sets
1989	Exoo	43		simulated annealing
1992	McKay-R		53	$(4,4)$-graph enumeration, LP
1994	McKay-R		52	more detailed LP
1995	McKay-R		50	implication of $R(4,5) = 25$
1997	McKay-R		49	long computations
2017	Angeltveit-McKay		48	large LP for $(\geq 4, \geq 5)$-graphs
2023	Angeltveit-McKay		46	massive LP for $(\geq 4, \geq 5)$-graphs

History of bounds on $R(5,5)$

Conjecture: $R(5,5) = 43$, McKay-R (1997)

One missing edge

$J_n = K_n - e$, asymptotics as for cliques

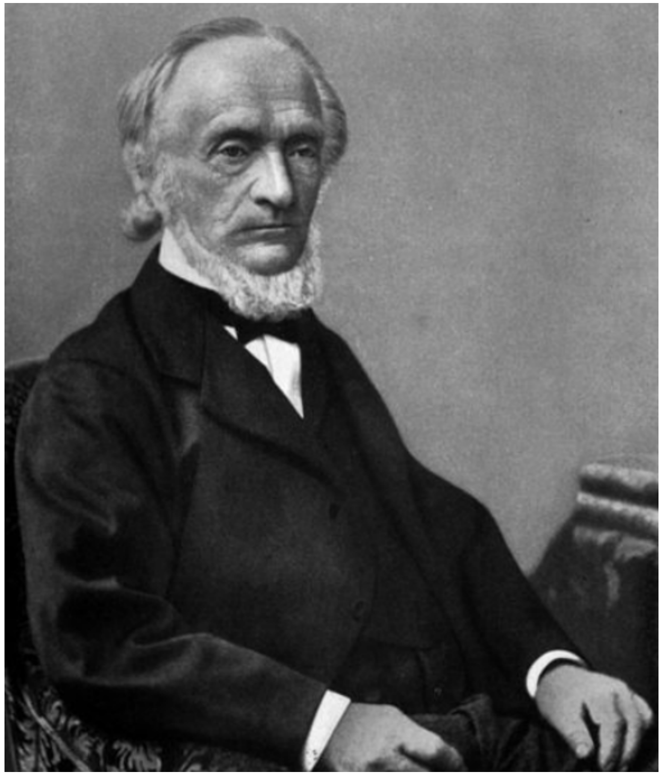
G \ H	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$	$K_{11}-e$
K_3-e	3	5	7	9	11	13	15	17	19
K_3	5	7	11	17	21	25	31	37	42 45
K_4-e	5	10	13	17	28	30 32	36 45	43 57	73
K_4	7	11	19	30	37 49	52 71	62 102	135	170
K_5-e	7	13	22	37	65	66 91	69 136	188	261
K_5	9	16	30 33	43 62	65 102	81 173	121 262	381	511
K_6-e	9	17	37	45 70	66 124	83 206	334	505	757
K_6	11	21	43 53	58 104	205	353	612	944	1346
K_7-e	11	28	65	66 124	247	432	761	1218	1964
K_7	13	28 29	65 82	80 184	370	716	1269	2119	3197

Schläfli graph is $(J_4, J_7; 27)$-good. It is a jewel!

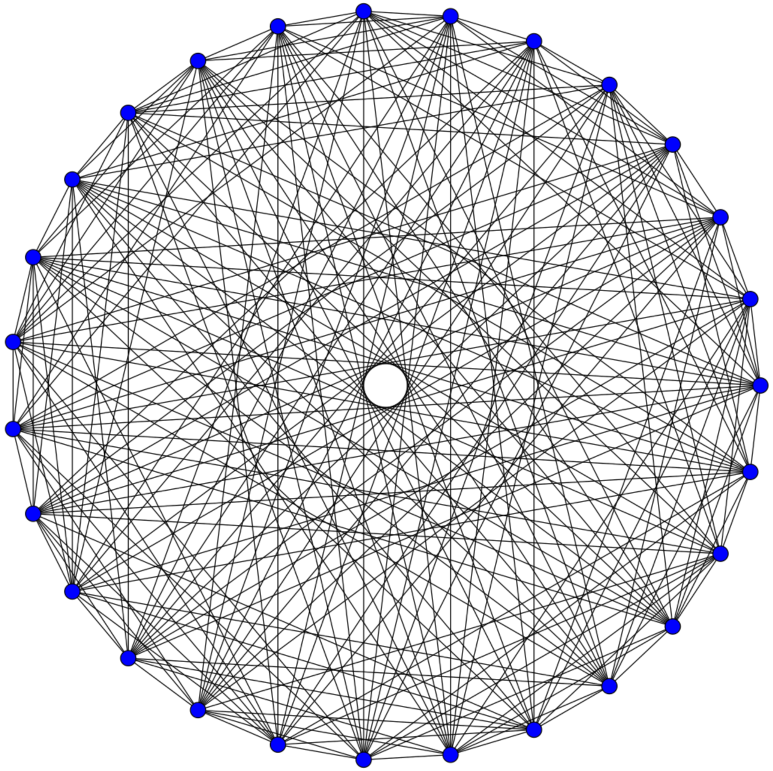
$R(J_5, J_7) = 65$, Goedgebeur-Van Overberghe (2022)

Schläfli $(J_7, J_4; 27)$-graph

building block of $(J_5, J_7; 64)$



[Wikipedia]



Ludwig Schläfli (1814-1895)

$srg(27, 16, 10, 8)$

$51840 = 2^7 3^4 5$ automorphisms

C_4 versus stars and wheels

star $S_n = K_{1,n}$, wheel $W_{n+1} = K_1 + C_n$

n	$R(C_4, K_{1,n})$	reference	n	$R(C_4, K_{1,n})$	reference
2	4	ChH2	22	28	SunSh
3	6	ChH2	23	29	Par5
4	7	Par3	24	30	WuSZR
5	8	Par3	25	31	Par3
6	9	FRS4	26	32	Par3
7	11	FRS4	27	33	Boza8
8	12	Tse1	28	35	Boza8
9	13	Par3/Tse1	29	36	Boza8
10	14	Par3/Tse1	30	36-37	WuSR/DyDz2
11	16	Tse1	31	38	Boza8
12	17	Tse1	32	39	Boza8
13	18	DyDz2	33	39-40	WuSR/DyDz2
14	19	DyDz2	34	41	WuSR
15	20	Law2/DyDz2	35	42	WuSR
16	21	Par3/DyDz2	36	43	WuSR
17	22	Par3	37	44	Boza8
18	23	ZhaBC1	38	45	ZhaCC2
19	24	WuSR	39	46-47	WuSR/DyDz2
20	25	WuSR	40	47	ZhaCC2
21	27	Par5	41	49	Boza8

Coolly, we have $R(C_4, K_{1,n}) = R(C_4, W_{n+1})$ for $n \geq 6$.

Zhang-Broersma-Chen (2014)

$R(K_m, C_n)$

Conjecture: Erdős-Faudree-Rousseau-Schelp (1976)
$R(K_m, C_n) = (m-1)(n-1)+1$ for all $n \geq m \geq 3$, except $m = n = 3$.

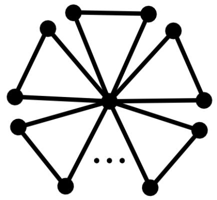
	C_3	C_4	C_5	C_6	C_7	C_8	C_9	...	C_n for $n \geq m$
K_3	6 2.1.a	7 ChaS	9 ...	11	13	15	17	... ...	$2n-1$ ChaS
K_4	9 GG	10 ChH2	13 He4/JR4	16 JR2	19 YHZ1	22 ...	25	... ...	$3n-2$ YHZ1
K_5	14 GG	14 Clan	17 He2/JR4	21 JR2	25 YHZ2	29 BolJY+	33 ...	... ...	$4n-3$ BolJY+
K_6	18 Kéry	18 Ex2-RoJa1	21 JR5	26 Schi1	31 ...	36	41	... ...	$5n-4$ Schi1
K_7	23 Ka2-GrY	22 RaT-JR1	25 Schi2	31 CheCZN	37 CheCZN	43 JaBa/Ch+	49 Ch+	... ...	$6n-5$ Ch+
K_8	28 GR-McZ	26 RaT	29 LidP	36 ChenCX	43 ChenCZ1	50 JaAl/ZZ3	57 BatJA	... ...	$7n-6$ conj.
K_9	36 Ka2-GR	30 RaT-LaLR	33-36 LidP	41 LidP	49 BanAA	56 BanAA	65 conj.	... ...	$8n-7$ conj.
K_{10}	40-41 Ex5-Ang	36 LaLR						...	$9n-8$ conj.
K_{11}	47-50 Ex20-GoeR1	40-43 VO-BoRa						...	$10n-9$ conj.

- New insights since 2019: Keevash-Long-Skokan, Liu-li, Bohman-Keevash, Conlon-Mattheus-Mubayi-Verstraëte
- The first open case is $33 \leq R(K_9, C_5) \leq 36$

Fans

high activity since 2020

F_n



5.7. Fans, fans versus other graphs

The fan graph F_n is defined by $F_n = K_1 + nK_2$,
and generalized fan $F_{k,n}$ is defined by $F_{k,n} = K_1 + nK_k$.

$R(F_2, F_2) = 9$ [Bu4]
$R(F_3, F_3) = 14$ [ZhaoW]

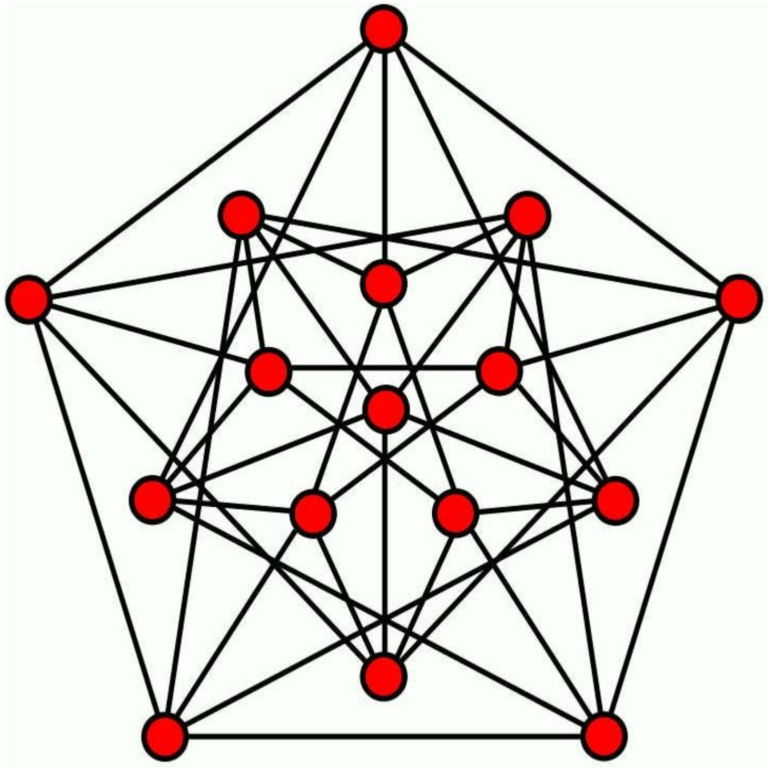
$R(F_1, F_n) = R(K_3, F_n) = 4n+1$ for $n \geq 2$, and bounds for $R(F_m, F_n)$ [LiR2, GuGS]

$R(F_2, F_n) = 4n+1$ for $n \geq 2$, and
$R(F_m, F_n) \leq 4n+2m$ for $n \geq m \geq 2$ [LinLi1],
$R(F_m, F_n) = 4n+1$ for n sufficierntly larger than m [LinLD].

$9n/2-5 \leq R(F_n, F_n) \leq 11n/2+6$ for all $n \geq 1$ [ChenYZ],
upper bound growth improved to $R(F_n, F_n) \leq 31n/6+15$ for all $n \geq 1$ [DvoMe].

$R(K_4, F_n) = 6n+1$ for $n \geq 3$ [SuBB3]
$R(K_5, F_n) = 8n+1$ for $n \geq 5$ [ZhaCh]
$R(K_6, F_n) = 10n+1$ for $n \geq 6$ [KaOS]

$R_r(m)$



[Wikipedia]
Alfred Clebsch (1833-1872)

For 3 colors, $R(3,3,3) = 17$
2 critical coloring, all colors in both are Clebsch graph.

For 4 colors, attend *Lightning Talks* tomorrow.

$R_r(m)$
lower bounds

r \ m	3	4	5	6	7	8	9	10
3	17 GG	128 HiIr	454 Ex23	1106 Row3	3214 XuR1	7174 Row5	15041 Row5	23094 Row5
4	51 Chu1	634 XXER	4073 Row3	23502 Row5	94874 Row5	182002 Row5	719204 Row5	
5	162 Ex10	4176 Row1	41626 Row5	258506 Row5				
6	538 FreSw	32006 Row1	441606 Row5					
7	1698 Row4	160024 Row1						
8	5288 Row3							
9	17805 AgCP+							

Fred Rowley, linear patterns constructions (2017-2023),
builds-up on Giraud (1960's), followed by Ageron et al. (2022+)

Diagonal Multicolorings for Cycles

k-colors, bounds on $R_k(C_m)$ as in SRN 2024

k \ m	3	4	5	6	7	8
3	17	11	17	12	25	16
4	51 62	18	33 77	18 20	49	20
5	162 307	27 29	65	26	97	28
6	538 1838	34 38	129		193	

Three jewels:

- $R_3(C_7) = 25$, Faudree-Schelten-Schiermeyer (2003)
- $R_4(C_4) = 18$, Exoo (1987)/Sun-Yang-Lin-Zheng (2007)
- $R_5(C_6) = 26$, Sun-Yang-Jiang-Lin-Shi/Sun-Yang-Wang (2008)

Columns:

- 3 - just triangles, Schur numbers, L_3 story
- 4 - relatively well understood, thanks geometry!
- 5 - poorly understood

Recent books

...

(2018) *Ramsey Theory, Unsolved Problems and Results* by Xiaodong Xu, Meilian Liang and Haipeng Luo [XuLL].

(2021) *The Discrete Mathematical Charms of Paul Erdős: A Simple Introduction,* by Vašek Chvátal [Chv2].

(2022) *Elementary Methods of Graph Ramsey Theory,* an extensive presentation of the area by Yusheng Li and Qizhong Lin [LiLin].

(2024) *The New Mathematical Coloring Book, Mathematics of Coloring and the Colorful Life of Its Creators,* greatly extended and revised first edition of the 2009 coloring book [Soi1] by Alexander Soifer [Soi3].

Journal paper counts

9.2. Journal paper counts

Out of 965 references gathered in this survey, most are papers which appeared in more than 100 different periodicals (in addition to books, conference proceedings, arXiv postings, and personal communications). The most popular periodicals were:

Discrete Mathematics	101
Journal of Combinatorial Theory (old, Series A and B)	64
Journal of Graph Theory	63
Electronic Journal of Combinatorics	53
Graphs and Combinatorics	43
Ars Combinatoria	32
Journal of Combinatorial Mathematics and Combinatorial Computing	32
European Journal of Combinatorics	31
Discrete Applied Mathematics	24
Australasian Journal of Combinatorics	23
Combinatorica	21
Utilitas Mathematica	18
Combinatorics, Probability and Computing	16
SIAM Journal on Discrete Mathematics	16
Congressus Numerantium	12
Discussiones Mathematicae Graph Theory	11
Random Structures and Algorithms	9
Mathematica Applicata	8
Applied Mathematics Letters	7
arXiv preprints	35

Exclusions

9.1. Exclusions

This compilation does not include much information on numerous variations of Ramsey numbers, nor related topics, like

- anti-Ramsey numbers,
- ascending Ramsey index,
- avoiding sets of graphs in some colors,
- barycentric Ramsey numbers,
- bipartite Ramsey numbers,
- blowup Ramsey numbers,
- chromatic Ramsey numbers,
- class Ramsey numbers,
- complementary Ramsey numbers,
- connected Ramsey numbers,
- cover Ramsey numbers,
- defective Ramsey numbers,
- degree Ramsey numbers,
- directed Ramsey numbers,
- distance Ramsey numbers,
- edge-chromatic Ramsey numbers,
- edge-ordered Ramsey numbers,
- Folkman numbers,
- Gallai-Ramsey numbers,
- generalized Ramsey numbers,
- induced Ramsey numbers,
- irredundant Ramsey numbers,
- list Ramsey numbers,
- local Ramsey numbers,
- k-Ramsey numbers,
- mixed Ramsey numbers,
- multipartite Ramsey numbers,
- online Ramsey numbers,
- ordered Ramsey numbers,
- ordered size Ramsey numbers,
- oriented Ramsey numbers,
- oriented size Ramsey numbers,
- planar Ramsey numbers,
- poset Ramsey numbers,
- potential Ramsey numbers,
- proper Ramsey numbers,
- quasi-Ramsey numbers,
- rainbow Ramsey numbers,
- Ramsey equivalence,
- Ramsey game numbers,
- Ramsey games,
- Ramsey multiplicities,
- Ramsey-Turán numbers,
- Ramsey sequences of graphs,
- restricted online Ramsey numbers,
- restricted size Ramsey numbers,
- Schur numbers,
- semi-algebraic Ramsey numbers,
- set-coloring Ramsey numbers,
- signed Ramsey numbers,
- singular Ramsey numbers,
- size Ramsey numbers,
- size multipartite Ramsey numbers,
- star-critical Ramsey numbers,
- sub-Ramsey numbers,
- Van der Waerden numbers,
- weakened Ramsey numbers,
- zero-sum Ramsey numbers,
- or coloring graphs other than complete.

Thanks for listening!