How Small Can the Most Wanted Folkman Graph Be ?

searching for a K_4-free graph which is not a union of two triangle-free graphs

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Abstract

We discuss a branch of Ramsey theory concerning edge Folkman numbers and how computer algorithms could help to solve some problems therein. We write $G \rightarrow (a_1, \ldots, a_k; p)^e$ if for every edge k-coloring of an undirected simple graph G not containing K_p, a monochromatic K_{a_i} is forced in color i for some $i \in \{1, \ldots, k\}$. The edge Folkman number is defined as $F_e(a_1, \ldots, a_k; p) = \min\{|V(G)| : G \rightarrow (a_1, \ldots, a_k; p)^e\}$. Folkman showed in 1970 that this number exists for $p > \max(a_1, \ldots, a_k)$.

In general, much less is known about edge Folkman numbers than the related and more studied vertex Folkman numbers, where we color vertices instead of edges. $F_e(3, 3; 4)$ involves the smallest parameters for which the problem is open, namely the question, "What is the smallest order N of a K_4-free graph, for which any edge 2-coloring must contain at least one monochromatic triangle?" This is equivalent to finding the order N of the smallest K_4-free graph which is not a union of two triangle-free graphs. It is known that $16 \leq N$ (an easy bound), and it is known through a probabilistic proof by Spencer (later updated by Hovey) that $N \leq 3 \times 10^9$. We suspect that $N \leq 127$.

This talk will present the background, overview some related problems, discuss the difficulties in obtaining better bounds on N, and give some computational evidence why it is very likely that even $N < 100$.

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Outline

- Arrowing
- Folkman numbers
- Story of $F_e(3,3;4)$
- Probabilistic upper bound on $F_e(3,3;4)$
- Some general known facts about edge- and vertex- Folkman numbers and bounds for specific small parameters
- Complexity of arrowing
- A very special graph G_{127}
- Can SAT-solvers help?

Graph notation

G - simple undirected loopless graph
$V(G)$ - vertex set of graph G
$E(G)$ - edge set of graph G

$R(s,t)$ - Ramsey number, the least n such that in any 2-coloring of the edges of K_n there is a monochromatic K_s in the first color or a monochromatic K_t in the second color.

$G(n,p)$ - random graph
n vertices, edge probability p

$\chi(G)$ - chromatic number of G

K_n, P_n, C_n - complete graph, path and cycle on n vertices

Arrowing - branch of Ramsey Theory

F, G, H - graphs, s, t, s_i - positive integers

Definitions

$F \rightarrow (s_1, ..., s_k)^e$ iff for every k-coloring of the edges of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \leq i \leq k$.

$F \rightarrow (s_1, ..., s_k)^v$ iff for every k-coloring of the vertices of F, F contains a monochromatic copy of K_{s_i} in color i, for some i, $1 \leq i \leq k$.

$F \rightarrow (G, H)^e$ iff for every red/blue edge-coloring of F, F contains a blue copy of G or a red copy of H.

Facts

$R(s, t) = \min\{n \mid K_n \rightarrow (s, t)^e\}$
$R(G, H) = \min\{n \mid K_n \rightarrow (G, H)^e\}$

Warming up

$G = K_6$ has the smallest number of vertices among graphs which are not a union of two K_3-free graphs, since $R(3,3) = 6$.

$$K_6 \to (K_3, K_3)^e, \text{ or } K_6 \to (3,3)$$

(picture proof)

Warming up

What if we want G to be K_6-free?
Graham (1968) proved that

- $G = K_8 - C_5 = K_3 + C_5 \rightarrow (K_3, K_3)$
clearly, G has no K_6

- $|V(H)| < 8 \wedge K_6 \not\subset H \Rightarrow H \not\rightarrow (K_3, K_3)$

(picture proof of)
$K_3 + C_5 \rightarrow (K_3, K_3)$

Folkman problems

edge Folkman graphs

$\mathcal{F}_e(s,t;k) = \{G \rightarrow (s,t)^e : K_k \not\subseteq G\}$

edge Folkman numbers

$F_e(s,t;k)$ = the smallest n such that there exists an n-vertex graph G in $\mathcal{F}_e(s,t;k)$

vertex Folkman graphs/numbers

2-coloring vertices instead of edges

Theorem 1. (Folkman 1970) For all $k > max(s,t)$, edge- and vertex- Folkman numbers $F_e(s,t;k)$, $F_v(s,t;k)$ exist.

Known values/bounds for $F_e(3,3;k)$

Our goal $F_e(3,3;4)$

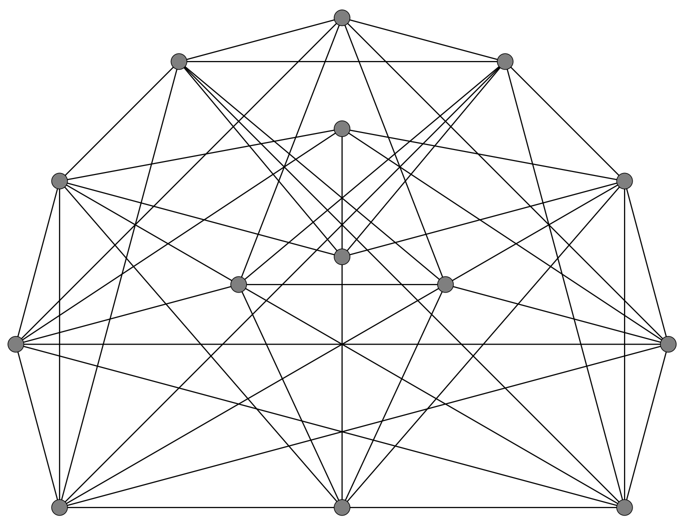
k	$F_e(3,3;k)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham'68
5	15	659 graphs	[PRU]'99
4	$\leq 3 \times 10^9$	probabilistic	'86,'88,'89

$k > R(s,t) \Rightarrow F_e(s,t;k) = R(s,t)$
$k \leq R(s,t)$, very little known in general

$F_e(3,3;5) = 15$ **and** $F_v(3,3;4) = 14$

153 out of 659 critical graphs in $\mathcal{F}_e(3,3;5)$ can be obtained from $\mathcal{F}_v(3,3;4)$ by using:

$$G \in \mathcal{F}_v(3,3;4) \Rightarrow G + x \in \mathcal{F}_e(3,3;5)$$



unique 14-vertex bicritical graph G

$$G \to (3,3;4)^v$$
$$G + x \to (3,3;5)^e$$

History of upper bounds on $F_e(3,3;4)$

- 1967 - Erdős, Hajnal state the problem
- 1970 - Folkman proves his theorem for 2 colors. VERY large bound for $F_e(3,3;4)$.
- 1975 - Erdős offers $100 (or 300 Swiss francs) for deciding if $F_e(3,3;4) < 10^{10}$
- 1986 - Frankl, Rődl give a probabilistic proof of $F_e(3,3;4) < 7 \times 10^{11}$
- 1988 - Spencer gives a probabilistic proof of $F_e(3,3;4) < 3 \times 10^{8}$
- 1989 - Hovey finds an error in Spencer's proof, bound up to $F_e(3,3;4) < 3 \times 10^{9}$
- 2005 - nothing better so far ...
- 2013 - "$F_e(3,3;4) < 100$" is decided (?)

History of lower bounds on $F_e(3,3;4)$

$10 \leq F_e(3,3;4)$ Lin (1972)

$16 \leq F_e(3,3;4)$ since $F_e(3,3;5) = 15$, all graphs in $\mathcal{F}_e(3,3;5)$ on 15 vertices are known, and all of them contain K_4's.

ANY proof technique improving on 16 very likely will be of interest

Probabilistic construction

Frankl, Rődl, Spencer, Hovey
used graph G^* constructed as follows:

Construction

1: input an integer n, and probability p
2: $G \leftarrow G(n,p)$
3: remove random edge from each K_4 in G
4: output G^*, the result of step 3

Sometimes $G^* \rightarrow (3,3)^e$

Frankl, Rődl:
very difficult probabilistic graph theory
$n = 7 \times 10^{11}$

Spencer/Hovey:
difficult probabilistic graph theory
$n = 3 \times 10^9$, $p = 6n^{-1/2} \approx 1/9129$

Probabilistic construction

main proof steps

Let

$$U(G) = \{(x, xyz) \mid \triangle xyz \ \ in \ \ G\}$$

$$U^* = U(G^*)$$

For each $x \in V(G)$, define (maximum over all partitions $N(x) = T \cup B,\ T \cap B = \emptyset$)

$$A(x) = \max |\{yz \in E(G) \mid y \in T \wedge z \in B\}|$$

Theorem 2. (Spencer)

$$\sum_{x \in V(G)} A(x) < \frac{2}{3}|U^*|$$

holds with positive probability for $n = 3 \times 10^9$, $p \approx 0.00011$, and $|E(G)| \approx 4 \times 10^{14}$.

Probabilistic construction

main counting trick

Theorem 3.
If

$$\sum_{x \in V(G)} A(x) < \frac{2}{3}|U^*|$$

then

$$G^* \in \mathcal{F}_e(3,3;4).$$

Proof.
G has no K_4 by construction.
Suppose f colors $E(G^*)$ in $\triangle$-free way.

Count marked triangles (x, xyz) such that $f(xz) \neq f(xy)$. It is $2|U^*|/3$, but also bounded by $\sum_{x \in V(G)} A(x)$. Contradiction. ∎

General facts on $F_e(s,t;k)$

- $G \in \mathcal{F}_e(s,t;k) \Rightarrow \chi(G) \geq R(s,t)$
  no k in the bound!, easy

- $\mathcal{F}_e(s,t; > R(s,t)) = R(s,t)$
- $\mathcal{F}_e(s,t; R(s,t)) = R(s,t) + c$
  in most cases c is small (2, 4, 5)
- $\mathcal{F}_e(s,t; < R(s,t)) \geq R(s,t) + 4$

- $G \in \mathcal{F}_v(R(s-1,t), R(s,t-1); k-1) \Rightarrow$
  $G + x \in \mathcal{F}_e(s,t;k)$, or equivalently
- $G + x \not\rightarrow (s,t)^e \Rightarrow$
  $G \not\rightarrow (R(s-1,t), R(s,t-1))^v$,
  and clearly $cl(G+x) = cl(G) + 1$

Special cases (other than $F_e(3,3;4)$)

$F_e(3,4;\geq 10) = 9$, K_9 since $R(3,4) = 9$

$F_e(3,4;9) = 14$, $K_4 + C_5 + C_5$, Nenov (1991)

$14 \leq F_e(3,4;8) \leq 314$,
Łuczak, Ruciński, Urbański (2002)

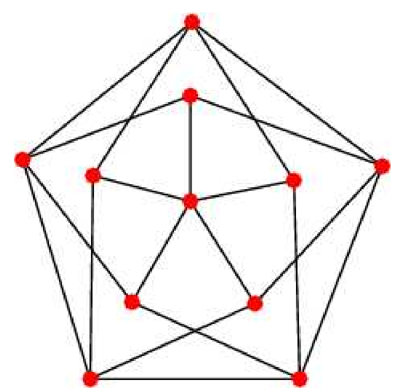
$F_e(3,5;14) = 16$
$F_e(4,4;18) = 20$
$F_e(3,7;22) \geq 27$
$F_e(3,3,3;17) = 19$
$F_e(3,3,3;16) = 21$

forbidden K_k in the above items has
$k = R(s,t)$ or $k = R(s,t) - 1$

several critical graphs have the form
$K_p + C_q$, $K_p + \overline{C_q} + C_r$, or $K_p - C_q$

Vertex Folkman numbers pearls

$F_v(2,2,2;3) = 11$
the smallest 4-chromatic triangle-free graph



Mycielski (1955) [mathworld.wolfram.com]

$F_v(2,2,2,2;4) = 11$
the smallest 5-chromatic K_4-free graph has 11 vertices, Nenov (1984)

$F_v(2,2,2,2;3) = 22$
the smallest 5-chromatic triangle-free graph has 22 vertices, Jensen/Royle (1995)

Vertex Folkman numbers pearls

Theorem 4. (ancient folklore)
$F_v(\underbrace{2, \cdots, 2}_{r}; r) = r + 5$ for $r \geq 5$

Sketch of the proof
for the upper bound consider as
the critical graph $K_{r-5} + C_5 + C_5$

for the lower bound take any
K_r−free graph G on $r + 4$ vertices, then
assemble matchings in $\overline{G}$ to show $\chi(G) \leq r$ ∎

Theorem 5. (Nenov 2002)
$F_v(\underbrace{3, \cdots, 3}_{r}; 2r) = 2r + 7$, for $r \geq 3$.

For $r = 2$, a small but hard case,
$F_v(3, 3; 4) = 14$ (PRU 1999)

$F_v(2, 2, 3; 4) = 14$ (Coles MS-CS 2005)

Complexity of arrowing

- Testing whether $F \to (3,3)^e$ is **coNP**-complete (Burr 1976).
- Determining if $R(G,H) < m$ is **NP**-hard (Burr 1984).
- For any fixed 3-connected graphs G and H, testing whether $F \not\to (G,H)^e$ is **NP**-complete (Burr 1990).
- For any fixed G on at least 3 vertices, testing whether $F \mapsto (G,G)^v$ is **coNP**-complete (Achlioptas 1997).
- Testing whether $F \to (G,H)^e$ is $\mathbf{\Pi_2^p}$-complete (Schaefer 2001).

Testing whether $F \to (K_2, K_n)^e$ is the same as checking $K_n \subset F$, so it is NP-hard.

Complexity of (edge) arrowing

Compendium of arrowing complexity including contributions by Cook (1971), Burr (1976, 1984, 1990), Rutenburg (1986) and Schaefer (2001)

Problem	Fixed	Complexity
$F \rightarrow (G, H)$		Π_2^p-complete
$F \rightarrow (G, H)$	G, H	in coNP
$F \rightarrow (K_2, H)$		NP-complete
$F \rightarrow (K_2, H)$	H	NP-complete
$F \rightarrow (T, K_n)$	$T,\ e(T) \geq 2$	Π_2^P-complete
$F \rightarrow (G, H)$	$G, H \in \Gamma_3$	coNP-complete
$F \rightarrow (P_4, P_4)$		coNP-complete
$F \rightarrow (kK_2, H)$	$k,\ H$	P
$F \rightarrow (K_{1,n}, K_{1,m})$		P
$K_n \rightarrow (G, H)$		NP-hard

[from recent ECS/NSF grant application]

Tools in complexity of arrowing

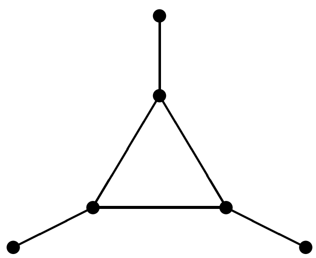
(G, H)-enforcers, -signal senders, -cleavers, -determiners are the tools (gadgets) used in reductions (Burr, Schaefer).

Such gadgets permit to construct F for which we are in control of whether $F \rightarrow (G, H)$.

Definition (Grossman 1983)
F is a (G, G)-cleaver iff there exists unique coloring of F witnessing $F \not\rightarrow (G, G)$.

Cleavers

P_4 cleaved graph F, $F \not\rightarrow (P_4, P_4)$,
but there is only one witness coloring.



graph F

Known K_3-cleaved graphs contain K_4.
K_5 is not C_5-cleaved, P_3 cleaves C_{2n}.

$G_{127} \rightarrow (3,3)^e$?

Exoo suggested to look at a well known $(4,4,4)$- and $(4,12)$-Ramsey graph (Hill, Irving 1968), defined by:

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x,y) | x - y = \alpha^3 \pmod{127}\}$$

- 127 vertices, 2667 edges, 9779 triangles
- regular of degree 42
- independence number 11, no K_4's !
- vertex- and edge-transitive
- 5334 ($= 127 * 42$) automorphisms
- $(127, 42, 11, \{14, 16\})$ - regularity, almost strongly regular graph
- K_{127} can be partitioned into three G_{127}'s

When to expect $G \to (3,3)^e$?

G has a large number of triangles

G has many small dense subgraphs

Unfortunately, $A(x)$ used in the proof by Spencer is very far from being useful for G_{127}

Conjecture: $G_{127} \to (3,3)^e$

If $G_{127} \to (3,3)^e$ then it gives 23,622,047-fold improvement over Spencer/Hovey bound.

Proving $G \rightarrow (3,3)^e$

First, solve a simpler task: find a small subgraph H, embedded in G in many places, such that there is a small number of colorings witnessing $H \not\rightarrow (3,3)^e$

Second, try to extend all (not many) colorings for $H \not\rightarrow (3,3)^e$ to whole G,

or, if this is too expensive ...

go via SAT ...

Reducing $\{G \mid G \not\to (3,3)^e\}$ to 3-SAT

edges in $G \longmapsto$ variables of ϕ_G
each (edge)-triangle xyz in $G \longmapsto$ add to ϕ_G

$$(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z})$$

Clearly,

$$G \not\to (3,3)^e \iff \phi_G \quad \text{is satisfiable}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 3-clauses, 2 for each of the 9779 triangles.

Note: By taking only the positive clauses, we obtain a reduction to ϕ'_G in NAE-3-SAT with half of the clauses.

Algorithms for 3-SAT

Randomized algorithms finding a satisfying assignment to n-variable 3-SAT in expected time

$$O(c^n)$$

Between 1997 and 2004, c was sliding down from 1.782 to 1.324 (Iwama, Tamaki - 2004) in a dozen of papers.

8-authors TCS 2002 paper presenting a deterministic algorithm for k-SAT running in time

$$\left(2 - \frac{2}{k+1}\right)^n$$

DIMACS format

$\phi_{G_{127}}$ in standard DIMACS format:

```
p cnf 2667 19558
1 2 43 0
-1 -2 -43 0
1 6 44 0
-1 -6 -44 0

       ...
<19550 lines deleted>
       ...

2656 2657 2664 0
-2656 -2657 -2664 0
2659 2660 2666 0
-2659 -2660 -2666 0
```

SAT-solvers - enhanced/tuned Davis-Putnam Algorithm

zChaff
Well known solver since 2001, winner of competitions. EE Princeton group: Fu, Mahajan, Zhao, Zhang, Malik, joined by Madigan (MIT), Moskewicz (UC Berkeley).

BerkMin561
New contender since 2003, strong "industrial" performance, Goldberg (Cadence Berkeley), Novikov (BAS Mińsk)

Satzoo
New contender since 2003, strong for combinatorial/handmade instances, Eén and Sörensen (Chalmers U., Sweden)

SAT-solvers

SAT 2004 Competition
awards in nine categories

(random, crafted, industrial)
$\times$ (SAT, UNSAT, ALL)

March-eq - winner of 2004 competition
in the category (crafted, UNSAT)
Heule and van Maaren, Delft Un. of Tech.

Adaptnovelty, Kcnfs, Jerusat 1.3
other recent less known SAT-solvers

GRASP'99, SATO'97, POSIT'95
other older more known SAT-solvers

zChaff experiments on $\phi_{G_{127}}$

- Pick $H = G_{127}[S]$ on $m = |S|$ vertices. Use zChaff to split H:
  - $m \leq 80$, H easily splittable
  - $m \approx 83$, phase transition ?
  - $m \geq 86$, splitting H is very difficult

- G_{127} has many small dense subgraphs, but no K_3-cleaved graphs among them.

- $\#(clauses)/\#(variables) = 7.483$ for G_{127}, far above conjectured phase transition ratio $r \approx 4.2$ for 3-SAT. It is known that

$$3.52 \leq r \leq 4.596$$

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SAT solvers

ZCHAFF
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MARCH_EQ
Marijn Heule and Hans van Maaren, March_eq SAT-solver, 2004. Available at `http://www.isa.ewi.tudelft.nl/sat/march_eq.htm`.

Links to other SAT-solvers can be easily found on the web.

Revisions

Revision #1, October 28, 2004
presented at MCCCC'04, Rochester NY

Revision #2, February 7, 2005
presented at the University of Rochester, Rochester NY

Revision #3, June 7, 2013
presenting solution to the G_{127} problem, Playa Azul, Cozumel QR

ThanX