11.4 The Pricing Method: Vertex Cover
Weighted vertex cover. Given a graph $G$ with vertex weights, find a vertex cover of minimum weight.

Weight = 2 + 2 + 4

Weight = 9
Weighted Vertex Cover

Pricing method. Each edge must be covered by some vertex $i$. Edge $e$ pays price $p_e \geq 0$ to use vertex $i$.

Fairness. Edges incident to vertex $i$ should pay $\leq w_i$ in total.

for each vertex $i$:  \[ \sum_{e=(i,j)} p_e \leq w_i \]

Claim. For any vertex cover $S$ and any fair prices $p_e$: \[ \sum_e p_e \leq w(S). \]

Proof.

\[ \sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S). \]

each edge $e$ covered by at least one node in $S$

sum fairness inequalities for each node in $S$
Pricing Method

Pricing method. Set prices and find vertex cover simultaneously.

```plaintext
Weighted-Vertex-Cover-Approx(G, w) {
    foreach e in E
        p_e = 0
    while (∃ edge i-j such that neither i nor j are tight)
        select such an edge e
        increase p_e without violating fairness
    } 

    S ← set of all tight nodes
    return S
}
```

\[ \sum_{e=(i,j)} p_e = w_i \]
Pricing Method

Figure 11.8
Theorem. Pricing method is a 2-approximation.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.

- Let $S$ = set of all tight nodes upon termination of algorithm. $S$ is a vertex cover: if some edge $i$-$j$ is uncovered, then neither $i$ nor $j$ is tight. But then while loop would not terminate.

- Let $S^*$ be optimal vertex cover. We show $w(S) \leq 2w(S^*)$.

\[
\begin{align*}
w(S) &= \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e = (i,j)} p_e \\ &= \sum_{i \in V} \sum_{e = (i,j)} p_e \\ &\leq 2 \sum_{e \in E} p_e \\ &\leq 2w(S^*).
\end{align*}
\]
11.6 LP Rounding: Vertex Cover
Weighted vertex cover. Given an undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a minimum weight subset of nodes $S$ such that every edge is incident to at least one vertex in $S$.

![Graph with weights]
Weighted vertex cover. Given an undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a minimum weight subset of nodes $S$ such that every edge is incident to at least one vertex in $S$.

Integer programming formulation.

- Model inclusion of each vertex $i$ using a 0/1 variable $x_i$.

$$
\begin{align*}
  x_i &= \begin{cases} 
    0 & \text{if vertex } i \text{ is not in vertex cover} \\
    1 & \text{if vertex } i \text{ is in vertex cover} 
  \end{cases} 
\end{align*}
$$

Vertex covers in 1-1 correspondence with 0/1 assignments:

$$
S = \{i \in V : x_i = 1\}
$$

- Objective function: maximize $\sum_i w_i x_i$.

- Must take either $i$ or $j$: $x_i + x_j \geq 1$. 

Weighted vertex cover. Integer programming formulation.

\[
(\text{ILP}) \quad \min \sum_{i \in V} w_i x_i \\
\text{s.t.} \quad x_i + x_j \geq 1 \quad (i,j) \in E \\
x_i \in \{0,1\} \quad i \in V
\]

Observation. If \( x^* \) is optimal solution to (ILP), then \( S = \{i \in V : x^*_i = 1\} \) is a min weight vertex cover.
Integer Programming

**INTEGER-PROGRAMMING.** Given integers $a_{ij}$ and $b_i$, find integers $x_j$ that satisfy:

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \geq b$$
$$x \quad \text{integral}$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad 1 \leq i \leq m$$
$$x_j \geq 0 \quad 1 \leq j \leq n$$
$$x_j \quad \text{integral} \quad 1 \leq j \leq n$$

**Observation.** Vertex cover formulation proves that integer programming is NP-hard search problem.

even if all coefficients are 0/1 and at most two variables per inequality
Linear Programming

Linear programming. Max/min linear objective function subject to linear inequalities.

- Input: integers $c_j, b_i, a_{ij}$.
- Output: real numbers $x_j$.

\[
(P) \quad \text{max } c^T x \\
\text{s.t. } Ax \geq b \\
x \geq 0
\]

Linear. No $x^2, xy, \arccos(x), x(1-x)$, etc.


LP Feasible Region

LP geometry in 2D.

The region satisfying the inequalities:
\[ x_1 \geq 0, \ x_2 \geq 0 \]
\[ x_1 + 2x_2 \geq 6 \]
\[ 2x_1 + x_2 \geq 6 \]
Weighted Vertex Cover: LP Relaxation

**Weighted vertex cover.** Linear programming formulation.

\[
\text{(LP) } \min \sum_{i \in V} w_i x_i \\
\text{s. t. } x_i + x_j \geq 1 \quad (i,j) \in E \\
x_i \geq 0 \quad i \in V
\]

**Observation.** Optimal value of (LP) is \( \leq \) optimal value of (ILP).

**Pf.** LP has fewer constraints.

**Note.** LP is not equivalent to vertex cover.

**Q.** How can solving LP help us find a small vertex cover?

**A.** Solve LP and round fractional values.
Weighted Vertex Cover

**Theorem.** If $x^*$ is optimal solution to (LP), then $S = \{i \in V : x^*_i \geq \frac{1}{2}\}$ is a vertex cover whose weight is at most twice the min possible weight.

**Pf.** [S is a vertex cover]
- Consider an edge $(i, j) \in E$.
- Since $x^*_i + x^*_j \geq 1$, either $x^*_i \geq \frac{1}{2}$ or $x^*_j \geq \frac{1}{2} \Rightarrow (i, j)$ covered.

**Pf.** [S has desired cost]
- Let $S^*$ be optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \geq \sum_{i \in S} w_i x^*_i \geq \frac{1}{2} \sum_{i \in S} w_i$$

LP is a relaxation $x^*_i \geq \frac{1}{2}$
Weighted Vertex Cover

**Theorem.** 2-approximation algorithm for weighted vertex cover.

**Theorem.** [Dinur-Safra 2001] If $P \neq NP$, then no $\rho$-approximation for $\rho < 1.3607$, even with unit weights.

$10 \sqrt{5} - 21$

**Open research problem.** Close the gap.