Huffman Coding (Ch. 4.8, K&T)

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Codes

Fixed Length Codes

Each symbol (e.g. letter) represented using same number of bits (e.g. ASCII)

Variable Length Codes

Symbols represented by different numbers of bits; normally reduces storage requirements

- Example: Morse code

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Codes, Cont'd

(Binary) Prefix Codes

Code such that the 0/1 prefix for each symbol is unique

- Example: a: 11 b: 01 c: 001 d: 10 e: 000
- May be represented by a function, gamma:

$$\gamma(a) = 11$$

- Can be decoded easily; as soon we read far ahead enough to match the code for a symbol (which is unique by def'n), we return the

Optimal Prefix Codes

We can represent the total length of our encoding in bits using the formula below

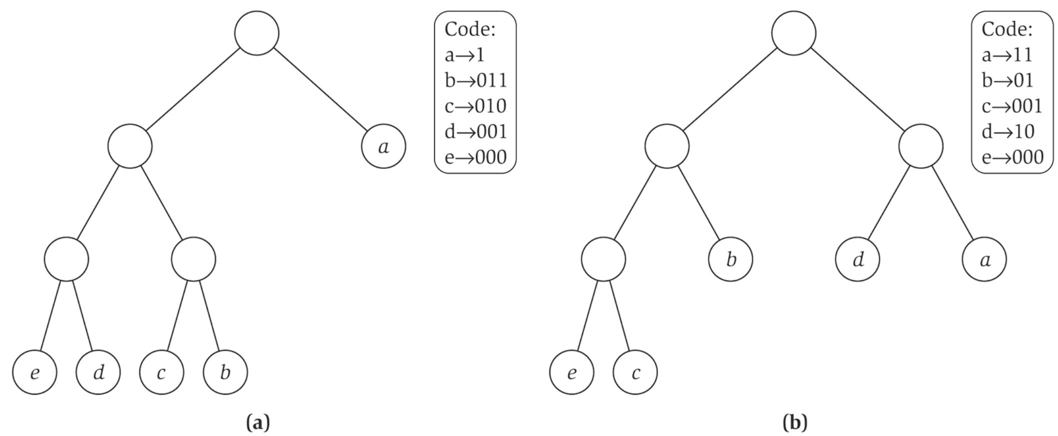
- n is # symbols in the original message, f_x the frequency of symbol x; S the symbol alphabet

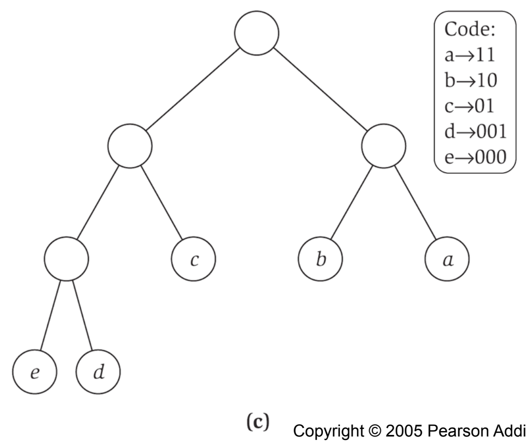
$$\sum_{x \in S} n f_x |\gamma(x)| = n \boxed{\sum_{x \in S} f_x |\gamma(x)|}$$

Optimization Problem

Given an alphabet and set of frequencies for letters, we want a prefix code that minimizes the average bits per letter (red box above)

Prefix Codes as Binary Trees





Note that symbols appear at leaves: prefixes are disjoint

Figure 4.16 Parts (a), (b), and (c) of the figure depict three different prefix codes for the alphabet $S = \{a, b, c, d, e\}$.

Reformulation

Restatement of Optimization Problem

Find a binary tree T & labeling of the leaves of this tree that minimizes average # bits/symbol:

$$\sum_{x \in S} f_x \cdot depth_T(x)$$

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An insight...

(4.31) There is an optimal prefix code (T*) in which the two lowest frequency letters are siblings in T*

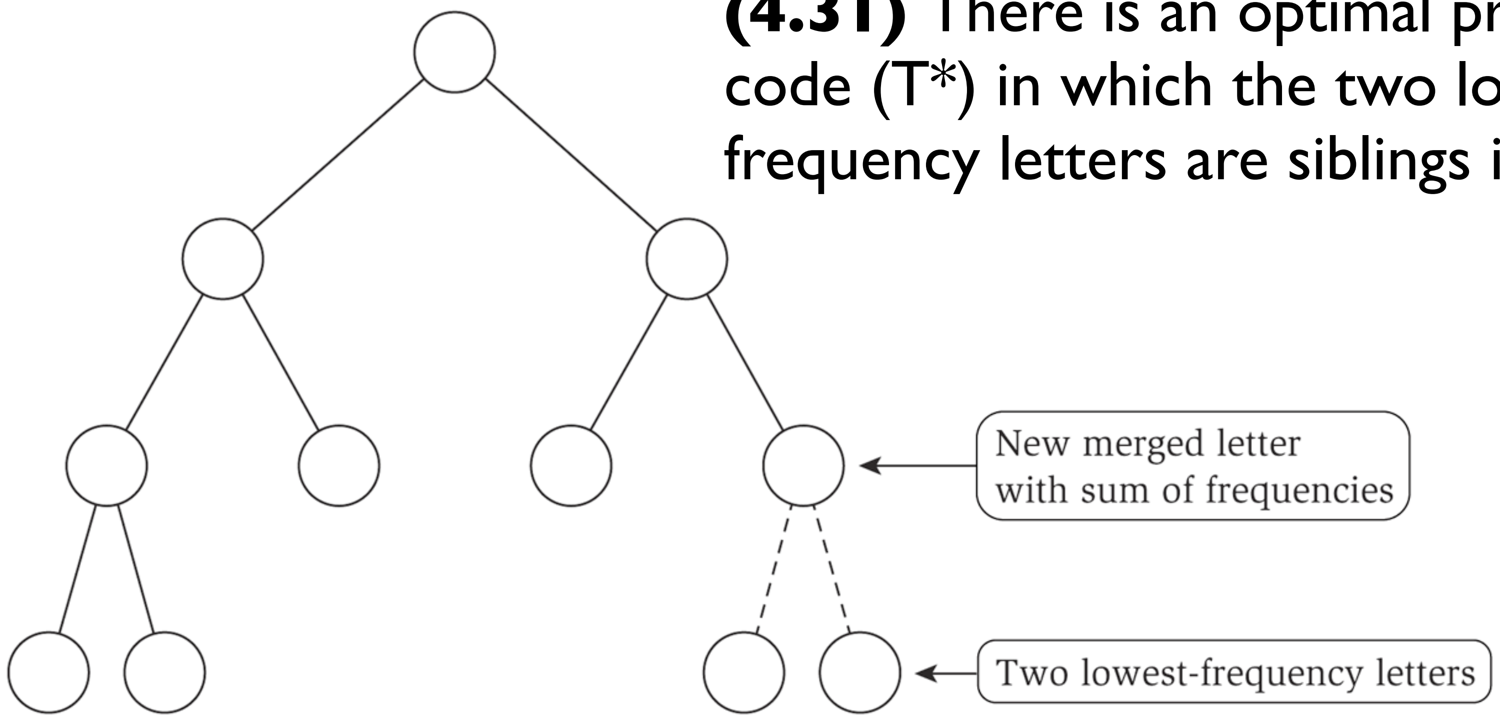

Figure 4.17 There is an optimal solution in which the two lowest-frequency letters label sibling leaves; deleting them and labeling their parent with a new letter having the combined frequency yields an instance with a smaller alphabet.

Huffman's Algorithm

To construct a prefix code for an alphabet S, with given frequencies:

  If S has two letters then

    Encode one letter using 0 and the other letter using 1

  Else

    Let y^* and z^* be the two lowest-frequency letters

    Form a new alphabet S' by deleting y^* and z^* and

      replacing them with a new letter ω of frequency $f_{y^*} + f_{z^*}$

    Recursively construct a prefix code γ' for S', with tree T'

    Define a prefix code for S as follows:

      Start with T'

      Take the leaf labeled ω and add two children below it

        labeled y^* and z^*

  Endif

Optimality of Huffman's Algorithm

(4.32) ABL(T') = ABL(T) - f_w

(4.33) Huffman code for a given alphabet achieves the minimum average number of bits per letter of any prefix code

See course text for proofs
of these properties:
(4.32 - by definition)
(4.33 - by induction, base case for 2 symbols;
>= 3 symbols by contradiction)

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Analysis of Run Time

Naive implementation ($O(k^2)$)

k-1 recursive calls, finding low frequency symbols if O(k)

Priority Queue Implementation (O(k log k))

Store symbols in queue using frequency as the key: can insert and extract symbols in O(log k) time

Each iteration: 2 deletions (min frequ. symbols), one insertion (add w for combined symbol): O(log k)