PATTERNS IN DIFFERENCES BETWEEN ROWS IN $k$-ZECKENDORF ARRAYS

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Abstract. For a fixed integer $k \geq 2$, we study the $k$-Zeckendorf array, $X_k$, based upon the $k$-th order recurrence $u_n = u_{n-1} + u_{n-k}$. We prove that the pattern of differences between successive rows is a $k$-letter infinite word generalizing the infinite Fibonacci.

1. Introduction and Background

Definition 1. The $k$-Zeckendorf array [3], $X_k = \{x_{r,c} \mid r, c \geq 0\}$, is a doubly subscripted array of positive integers. The first row begins with $x_{0,c} = c + 1$ for $0 \leq c < k$. For $i \geq k$,

$$x_{0,i} = x_{0,i-1} + x_{0,i-k}. \quad (1)$$

Subsequent rows are specified inductively as follows. For $r > 0$, $x_{r,0}$ is the smallest integer not in previous rows. Let the $k$-Zeckendorf representation (see Definition 2 below) of $x_{r,0}$ be $\sum_{i=0}^{m} d_i x_{0,i}$. Then for $c > 0$,

$$x_{r,c} = \sum_{i=0}^{m} d_i x_{0,i+c}. \quad (2)$$

Definition 2. The $k$-Zeckendorf representation of $n$ is $\sum_{i=0}^{m} d_i x_{0,i}$, where for all $i$, $d_i \in \{0, 1\}$ and every sequence $\{d_i, d_{i+1}, \ldots, d_{i+k-1}\}$ contains at most one 1. The upper limit $m$ in the sum is the largest integer such that $x_{0,m} \leq n$.

The well-known Zeckendorf theorem is for $k = 2$, and the sequence $\{x_{0,c}\}$ is the Fibonacci sequence $\{F_{c+2}\}$ (see [5, 8]). This generalizes easily to the $k$-Zeckendorf representation, which is also unique.

Definition 3. If the $k$-Zeckendorf representation of $n$ is $\sum_{i=0}^{m} d_i x_{0,i}$, the $k$-shift of $n$ is $S(n) = \sum_{i=0}^{m} d_i x_{0,i+1}$.

Taking $n = x_{r,0}$, we can write equation (2) as

$$x_{r,c} = S^c(x_{r,0}). \quad (3)$$

The arrays $X_k$ have several well-known properties [2, 3, 4]:

1. Every row of $X_k$ satisfies the recurrence $x_{r,c} = x_{r,c-1} + x_{r,c-k}$.
2. $X_k$ contains every positive integer exactly once.
3. $X_k$ is an interspersion [3]. If $x_{r,c} < x_{r',c'} < x_{r,c+1}$, then $x_{r,c+1} < x_{r',c'+1} < x_{r,c+2}$.

Portions of $X_2$, $X_3$, and $X_4$ are displayed below. $X_2$ is also known as the Wythoff array [3] and is given as OEIS # A035513 in [7]. We use precursion: $x_{r,n-k} = x_{r,n} - x_{r,n-1}$, to prepend $k$ columns to each $X_k$. We will establish later in Theorem 2 that $x_{r,-k} = r$ for $r \geq 0$ in all $X_k$, as shown by column $c = -k$ in these tables.

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1Clark Kimberling suggested the terminology at the 2010 Fibonacci Association conference in Mexico.
Sequences in \(X_k\) for rows 0 and columns 0 that are recorded in [7] are referenced here.

\[
\begin{array}{c|cccccccc}
X_2 & c & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
r : 0 & 0 & 1 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \\
1 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & 76 & 123 \\
2 & 2 & 4 & 6 & 10 & 16 & 26 & 42 & 68 & 110 & 178 \\
3 & 3 & 6 & 9 & 15 & 24 & 39 & 63 & 102 & 165 & 267 \\
4 & 4 & 8 & 12 & 20 & 32 & 52 & 84 & 136 & 220 & 356 \\
5 & 5 & 9 & 14 & 23 & 37 & 60 & 97 & 157 & 254 & 411 \\
6 & 6 & 11 & 17 & 28 & 45 & 73 & 118 & 191 & 309 & 500 \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
X_3 & c & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
r : 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 6 & 9 & 13 & 19 \\
1 & 1 & 3 & 4 & 5 & 8 & 12 & 17 & 25 & 37 & 54 & 79 \\
2 & 2 & 4 & 5 & 7 & 11 & 16 & 23 & 34 & 50 & 73 & 107 \\
3 & 3 & 5 & 7 & 10 & 15 & 22 & 32 & 47 & 69 & 101 & 148 \\
4 & 4 & 7 & 10 & 14 & 21 & 31 & 45 & 66 & 97 & 142 & 208 \\
5 & 5 & 9 & 13 & 18 & 27 & 40 & 58 & 85 & 125 & 183 & 268 \\
6 & 6 & 10 & 14 & 20 & 30 & 44 & 64 & 94 & 138 & 202 & 296 \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
X_4 & c & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
r : 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 7 & 10 & 14 \\
1 & 1 & 3 & 4 & 5 & 6 & 9 & 13 & 18 & 24 & 33 & 46 & 64 \\
2 & 2 & 4 & 5 & 6 & 8 & 12 & 17 & 23 & 31 & 43 & 60 & 83 \\
3 & 3 & 5 & 6 & 8 & 11 & 16 & 22 & 30 & 41 & 57 & 79 & 109 \\
4 & 4 & 6 & 8 & 11 & 15 & 21 & 29 & 40 & 55 & 76 & 105 & 145 \\
5 & 5 & 8 & 11 & 15 & 20 & 28 & 39 & 54 & 74 & 102 & 141 & 195 \\
6 & 6 & 10 & 14 & 19 & 25 & 35 & 49 & 68 & 93 & 128 & 177 & 245 \\
\end{array}
\]

2. Preliminaries

We focus our attention now on column zero of \(X_k\), \(\{x_{r,0} \mid r \geq 0\}\). The elements of column zero of \(X_k\) are those numbers whose \(k\)-Zeckendorf representation ends with the least significant portion (abbreviated LSP, with its complement MSP as the most significant portion) given by

\[
(d_{k-1}, d_{k-2}, \ldots, d_1, d_0) = (0, 0, \ldots, 0, 1).
\]

In this notation we mimic the usual binary number representation with bit strings. The rightmost bit is the coefficient of \(x_{0,0}\).

**Lemma 1.** \(x_{r,0} = S^k(r) + 1 = S \cdots S(r) + 1\).

**Proof.** This follows from the definition of \(X_k\) and equation (4). \(\square\)

The shift function preserves order, so \(x_{r,0} < x_{r+1,0}\) for all \(r \geq 0\).

In Theorem 1 below, we examine the pattern of the differences \(x_{r+1,0} - x_{r,0}\) and then the differences between successive rows of \(X_k\). This pattern is captured in the sequence of words \(W_k\), which for \(k = 2\) is the simplest Sturmian word (the Fibonacci word, see [1]), with its higher order generalizations.
Definition 4. $W_k = \{w_i\}_{i \geq 0}$ is an infinite sequence of words over the $k$-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. $w_i$ is a word of length $x_{0,i}$, as follows:

\[
\begin{align*}
  w_0 &= a_0, \\
  w_1 &= a_0a_1, \\
  w_2 &= a_0a_1a_2, \\
  &\vdots \\
  w_{k-1} &= a_0a_1a_2\cdots a_{k-1}, \\
  w_i &= w_{i-1}w_{i-k} \text{ for } i \geq k.
\end{align*}
\]

We define another infinite sequence of words $W'_k$, which is related to $W_k$. Each derivation has useful word properties, which will be highlighted in the table in Lemma 2.

Definition 5. $W'_k = \{w'_i\}_{i \geq 1}$ is an infinite sequence of words over the same $k$-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$, determined by the iterative algorithm:

\[
\begin{align*}
  w'_i &= \begin{cases} 
    a_i & \text{for } 1 \leq i < k, \\
    a_0 & \text{at } i = k,
  \end{cases} \\
  w'_i &= w'_{i-1}w'_{i-k} \text{ for } i > k.
\end{align*}
\]

$w'_i$ is a word of length $x_{0,i-k}$, where the word length is one for $0 < i < k$ according to precursion in row 0 of $X_k$. The shifted relationships between $w_i$ and $w'_i$ and between their respective lengths $|w_i|$ and $|w'_i|$ are given by

\[ w_i = w'_{i+k} \text{ and } |w_i| = |w'_{i+k}| = x_{0,i} \text{ for } i \geq 0. \]

The Fibonacci case ($k = 2$) has $|w_i| = x_{0,i} = F_{i+2}$ for $i \geq 0$, and $|w'_i| = x_{0,i-2} = F_i$ for $i \geq 1$.

For $i \geq k$, $w_{i-1}$ is a prefix of $w_i$ and $w_{i-k} = w'_i$ is a suffix of $w_i$, yielding the infinite word

\[ w = \lim_{i \to \infty} w_i = \lim_{i \to \infty} w'_i = \{w(n)\}_{n \geq 0} = w(0)w(1)w(2)\cdots, \]

where $w(n) \in \Sigma$. At $k = 2$, the infinite Fibonacci word $w$ begins $101101101010\ldots$ with $a_0 = 1, a_1 = 0$.

Lemma 2. The least significant portion LSP of $k - 1$ coefficients of the $k$-Zeckendorf representation of the sequence of nonnegative integers follows the pattern of $W_k$.

Proof. We express the $k$-Zeckendorf representation using at least $k - 1$ bits. The LSP of length $k - 1$ may take on $k$ different patterns (composed of either all zeros or zeros with a single one). From these $k$ patterns, we create our $k$-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. The $k$-Zeckendorf representation of 0 is 0, and we assign $a_0$ to instances when the LSP is $0^{k-1}$ ($k - 1$ 0’s). For $1 \leq i \leq k - 1$, we have by Definition 1 that $i = x_{0,i-1}$. The $k$-Zeckendorf representation of $i$ consists of a single term $x_{0,i-1}$ with LSP of $0^{k-1-i}10^{i-1}$, to which we assign the letter $a_i$.

The following table for $k = 3$ illustrates the principle of this proof. In this example, we use \{c, a, b\} for \{a_0, a_1, a_2\}. 

MONTH YEAR 3
A table of \( k \)-Zeckendorf representations \((k = 3)\).

<table>
<thead>
<tr>
<th>( n )</th>
<th>MSP of ( k )-Zeck.</th>
<th>LSP of ( k )-Zeck.</th>
<th>letters</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0</td>
<td>c 0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0 1</td>
<td>a 1</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 0</td>
<td>b 2</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1 0 0</td>
<td>c 0</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1 0 0 0</td>
<td>c 0</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>1 0 0 1</td>
<td>a 1</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>1 0 0 0 0</td>
<td>c 0</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>1 0 0 0 1</td>
<td>a 1</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1 0 0 1 0</td>
<td>b 2</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>1 0 0 0 0 0</td>
<td>c 0</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>1 0 0 0 0 1</td>
<td>a 1</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>1 0 0 0 1 0</td>
<td>b 2</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>1 0 0 1 0 0</td>
<td>c 0</td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

The horizontal lines in the above table are placed over the values \( n \) when \( n = x_{0,c} \), the entries of \( X_3 \) in row 0 at \( c \geq 0 \) beginning with 1, 2, 3, 4, 6, 9, 13. The sequence of words in the letters column from the top of the table down to each horizontal line is the following:

\[ c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc, \ldots. \]

Thus from Definition 4, each successive word sequence \( w_i \) from \( i \geq 0 \) gives the letter sequence from the table top to the next horizontal line, and each word length is \( |w_i| = x_{0,i} \) as required.

The sequence of words in the letters column between consecutive horizontal lines in the table are the words \( w'_i \) for \( i \geq 1 \) listed in sequence as

\[ a, b, c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc, \ldots. \]

At \( k = 3 \), row 0 of \( X_3 \) for \( x_{0,c-k} \) at \( c \geq 1 \) beginning with 1, 1, 1, 2, 3, 4, 6, 9, 13 gives \( |w'_c| \) as the number of terms between the horizontal lines in the table.

To form such a table, we initialize the first \( k \) lines \((k = 3 \text{ in the illustration})\) straightforwardly: for \( 0 \leq i < k \), \( x_{0,i} = i + 1 \), so the \( k \)-Zeckendorf representation of \( i \) corresponds to the letter \( a_i \). Then proceed inductively. Having constructed the first \( x_{0,n-1} \) lines, the next \( x_{0,n-k} \) lines are built as follows. If \( x_{0,n-1} \leq m < x_{0,n} \), the \( k \)-Zeckendorf representation of \( m \) is the \( k \)-Zeckendorf representation of \( m - x_{0,n-k} \) augmented by \( x_{0,n-1} \). As bit strings, take the initial \( x_{0,n-k} \) lines of the table, pad their \( k \)-Zeckendorf representations with zeros on the left to make them all bit strings of length \( n - k \). Then prepend each of these strings with \( 10^{k-1} \), giving the new \( x_{0,n-k} \) lines, extending the table to \( x_{0,n} \) lines.

The list of LSPs follows the same pattern used to construct \( W_k \).

In the above, in the words in \( \{a, b, c\} \), \( a \) is always followed by \( b \) or \( c \), \( b \) is always followed by \( c \), and \( c \) is always followed by \( a \) or \( c \). For general \( k \), \( \Sigma = \{a_i \mid 0 \leq i < k\} \), \( a_i \) will always be followed by either \( a_{i+1} \) or \( a_0 \), with the latter occurring when there is a \textit{carry}, a replacement of \( x_i + x_{i+k-1} \) with \( x_{i+k} \). \( a_{k-1} \) is always followed by \( a_0 \).

The subscript \( i \) of \( a_i \) is simply the value of the length \( k - 1 \) LSP of the Zeckendorf representation of \( i \). In other words, if the \( k \)-Zeckendorf representation of \( r \) is \( \sum_{h=0}^{m} d_h x_{0,h} \), then \( i = \sum_{h=0}^{k-2} d_h x_{0,h} \).
3. Main Theorems

Theorem 1. The difference between two adjacent rows of \( X_k \) is a shift of row 0, where the difference \( \delta_r(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)} \) with \( 1 \leq j(r) \leq k \). Specifically the shift index \( j(r) \) is

\[
j(r) = \begin{cases} 
  i & \text{if } w(r) = a_i \text{ for } 0 < i < k, \\
  k & \text{if } w(r) = a_0.
\end{cases}
\]

\( w(r) \) is the letter of the infinite word \( w \) of equation (5), located at position \( r \).

Proof. By Lemma 1, \( x_{r,0} = S^k(r) + 1 \), so \( x_{r+1,0} - x_{r,0} = S^k(r + 1) - S^k(r) \).

Suppose \( w(r) = a_i \). Then \( w(r + 1) = a_i+1 \) or \( w(r + 1) = a_0 \), with the latter case occurring when there was one or more carries.

Case 1: \( w(r) = a_i \) with \( 0 < i < k \). Express the \( k \)-Zeckendorf representation of \( r \) as \( \sum d_n x_{0,n} \).

\( i - 1 \) is the smallest subscript such that \( d_{i-1} = 1 \), and \( i - 1 + k \) is the smallest subscript such that \( d_{i-1+k} = 1 \) in the \( k \)-Zeckendorf representation of \( S^k(r) \) as \( \sum d_n x_{0,n} \).

Case 1.1: Suppose \( w(r + 1) = a_{i+1} \) and the \( k \)-Zeckendorf representation of \( S^k(r + 1) \) is \( \sum d_n' x_{0,n} \). We have \( d_n' = d_n \) for \( n > i + k \), so \( S^k(r + 1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i} \).

Case 1.2: If \( w(r + 1) = a_0 \), we may execute a sequence of borrows reversing the above mentioned sequence of carries, to write \( S^k(r + 1) = \sum d_n'' x_{0,n} \) (which, after borrowing is not \( k \)-Zeckendorf). We have \( d_n'' = d_n' \) for \( n > i + 1 \), so again \( S^k(r + 1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i} \).

Case 2: \( w(r) = a_0 \). The two sub-cases, \( w(r + 1) = a_1 \) and \( w(r + 1) = a_0 \) follow a very similar argument to that given in Case 1. In both these sub-cases, \( S^k(r + 1) - S^k(r) = x_{0,k} - 0 = x_{0,k} \).

The above argument carries through identically for the general case

\[
\delta_r(c) = x_{r+1,c} - x_{r,c} = S^{c+k}(r + 1) - S^{c+k}(r) \]

which \( j(r) \) is given in equation (6).

For illustration, we show that the first 13 differences in column 0 of \( X_3 \) have the same pattern as the 3-letter word \( W_3 \). The first 13 terms of the sequences \( \{\delta_r(0)\} \), \( \{j(r)\} \) and \( \{w(r)\} \) for \( 0 \leq r \leq 12 \) are shown below. The substitution of \( \{c,a,b\} \) for \( \{a_0,a_1,a_2\} \) is made in the word \( \{w(r)\}_{r \geq 0} \).

<table>
<thead>
<tr>
<th>( \delta_{0,...,12}(0) )</th>
<th>( j(0,...,12) )</th>
<th>( w(0,...,12) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,2,3,4,4,2,4,2,3,4,2,3,4</td>
<td>3,1,2,3,3,1,2,3,1,2,3</td>
<td>cabccacabcabc</td>
</tr>
</tbody>
</table>

Theorem 2. For all \( r, x_{r,-k} = r \).

Proof. \( x_{r+1, -k} - x_{r, -k} = x_{0, -k+j(r)} = x_{0,n} \). Because \( 1 \leq j(r) \leq k \), we have \( 1 - k \leq n \leq 0 \).

It follows from Definition 1 and a simple precursion argument that \( x_{0,n} \) is always 1, and that \( x_{0,-k} = 0 \).

Theorem 3. For all \( r \geq 0 \), \( w(x_{r,c}) = a_{c+1} \) if \( 0 \leq c \leq k - 2 \), and \( w(x_{r,c}) = a_0 \) if \( c > k - 2 \).

Proof. This follows from: the definition of \( X_k \), Definition 3, equation (3), Lemma 1, equation (4), and Lemma 2.

By Lemma 2, the LSP of \( 0^{k-1-i}10^{i-1} \) assigns \( w(n) = a_i \) if \( 1 \leq i \leq k - 1 \) and \( w(n) = a_0 \) if \( i = 0 \) or \( i \geq k \). Likewise by Lemma 1, column 0 has LSP of \( 0^{k-1-i}11 \); and by Definition 3 (equation (3)), column \( c \) has LSP of \( 0^{k-1-c}10^{c-1} \).
Some examples of Theorem 3 include:

1. \( w(x_{r,0}) = a_1 \) for \( r \geq 0, k \geq 2 \),
2. \( w(x_{r,1}) = a_2 \) for \( r \geq 0, k \geq 3 \),
3. \( w(x_{r,k-1}) = a_0 \) for \( r \geq 0 \).

The following results deal with how the numbers in a column \( c \) of \( X_k \) punctuate (i.e., break into factors) the infinite word \( w \). By Theorem 1, we know there are \( k \) different intervals \([x_{r,c}, x_{r+1,c})\), and the lengths of these intervals are \( k \) successive numbers of the sequence \( \{x_{0,i}\} \).

**Definition 6.** Let the sequence

\[
v_{r,c} = \langle w(i) \mid x_{r,c} \leq i < x_{r+1,c} \rangle
\]

be a factor of the infinite word \( w \). Its length is

\[
|v_{r,c}| = \delta_v(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)}.
\]

**Definition 7.** Let \( \sigma : \Sigma^* \to \Sigma^* \) be a homomorphism defined on the letters of \( \Sigma \) by

\[
\sigma(a_0) = a_0a_1, \quad \sigma(a_i) = a_{i+1} \quad \text{for} \quad 1 \leq i \leq k - 1.
\]

As above, take \( a_k = a_0 \).

**Lemma 3.** The infinite word \( w \) is the fixed point of \( \sigma \).

**Proof.** A straightforward inductive argument shows \( \sigma(w_i) = w_{i+1} \) for all \( i \geq 0 \). \( \square \)

**Definition 8.** Denote by \( \psi_\lambda(c) \), for \( 0 \leq \lambda < k \) words, as follows. For \( c = 0 \) we have the following

\[
\psi_0(0) = a_1 a_0, \\
\psi_\lambda(0) = a_1 \cdots a_{\lambda+1} a_0 \quad \text{for} \quad 1 \leq \lambda \leq k - 2, \\
\psi_{k-1}(0) = a_1 \cdots a_{k-1} a_k a_0.
\]

For \( c > 0 \), let \( \psi_\lambda(c) = \sigma(\psi_\lambda(c - 1)) \).

**Lemma 4.** \( |\psi_\lambda(c)| = x_{0,c+\lambda+1} \) for \( c \geq 0 \).

**Proof.** For \( 0 \leq \lambda \leq k - 1 \), the word \( \psi_\lambda(0) \) has length \( x_{0,\lambda+1} \) and is a permutation of the word \( w_{\lambda+1} \). The result then follows from the observation that \( \sigma(w_i) = w_{i+1} \) for all \( i \geq 0 \) (see Lemma 3) and that \( \sigma \) is a homomorphism. \( \square \)

**Theorem 4.** For \( c \geq 0 \), the \( \psi_\lambda(c) \) are the only strings among the words \( v_{r,c} \).

**Proof.** From Definition 8, each string \( \psi_\lambda(0) \) begins with the letter \( a_1 \), and it contains no other instance of \( a_1 \). In the infinite word \( w \), every factor \( \psi_\lambda(0) \) is followed by the letter \( a_1 \), beginning the next factor. The infinite string \( w \), therefore, has the following descriptions:

\[
w = w_0, \langle \psi_{j(r)+1}(0) \mid r = 0, 1, 2, \ldots \rangle, \\
\sigma(w) = w_1, \langle \psi_{j(r)+1}(1) \mid r = 0, 1, 2, \ldots \rangle,
\]

\[
\vdots \\
\sigma^c(w) = w_c, \langle \psi_{j(r)+1}(c) \mid r = 0, 1, 2, \ldots \rangle.
\]

Because \( w \) is the fixed point of \( \sigma \), the result follows. \( \square \)
We illustrate the first few iterations from (8) using $k = 3$ with $\{a_0, a_1, a_2\} = \{c, a, b\}$.

$$w = c, abcc, ac, abcc, abcc, ac, abcc, ac, abcc, ac, abc, abcc, abcc, ac, abc, abcc,\ldots$$

$$\sigma(w) = ca, becaca, bca, becaca, becaca, bca, becaca, bca, becaca, bca, becaca,\ldots$$

$$\sigma^2(w) = cab, ccacabcb, cab, ccacabcb, ccacabcb, cab, ccacabcb, cab, ccacabcb, ccacabcb,\ldots$$

Finally for $k = 3$, we display the $k$ distinct strings $\psi^\lambda(c)$ in the first few columns $c \geq 0$.

<table>
<thead>
<tr>
<th>$\psi^\lambda(c)$</th>
<th>$c : 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_c$</td>
<td>c</td>
<td>ca</td>
<td>cab</td>
<td>cab</td>
<td>cabcc</td>
</tr>
<tr>
<td>$\lambda : 0$</td>
<td>abc</td>
<td>bca</td>
<td>ccab</td>
<td>cacabc</td>
<td>ccabcabca</td>
</tr>
<tr>
<td>1</td>
<td>abc</td>
<td>bca</td>
<td>ccab</td>
<td>cacabc</td>
<td>ccabcabcca</td>
</tr>
<tr>
<td>2</td>
<td>abcc</td>
<td>becca</td>
<td>ccacab</td>
<td>ccacababc</td>
<td>ccacabccabaceb</td>
</tr>
<tr>
<td>3</td>
<td>abce</td>
<td>beccaca</td>
<td>ccacababcc</td>
<td>cacabcabccabca</td>
<td>ccacabccabca</td>
</tr>
</tbody>
</table>

We note that the strings, $\psi^\lambda(c)$ and $\psi^{\lambda-1}(c + 1)$, have the same length and are a rotation permutation of each other. We thus define a rotation operator $\rho$ such that $\rho(a_i x) = x a_i$ for $a_i \in \Sigma$ and $a_i x \in \Sigma^*$, and we present without proof:

**Proposition 1.** $\psi^\lambda(c) = \rho \cdots \rho(w_{c + \lambda + 1}) = \rho^{\left|w_{c}\right|}(w_{c + \lambda + 1})$.

Taking $k = 3$ and $c = 1$, Proposition 1 gives $\psi_1(1) = \rho^{\left|w_{2}\right|}(w_{3})$ with $\psi_0(1) = \rho^{2}(cab) = bca$, $\psi_1(1) = \rho^{2}(cab) = bca$, and $\psi_2(1) = \rho^{3}(ccacabcb) = ccacabcb$. As noted for diagonals, at $\lambda + c = 3$ we get $\psi_3(1) = \rho^{3}(ccacabcb) = ccacabcb$, $\psi_2(1) = \rho^{3}(ccacabcb) = ccacabcb$, and $\psi_0(3)$ having equal lengths $|\psi^\lambda(c)| = |w_4| = x_{0.4} = 6$ as required by Proposition 1, Theorem 1, and Lemma 4.

**Acknowledgment:** Thanks to Jenny Fayette (an “epsilon”) for suggesting this topic to us.

**References**


AMS Classification Numbers: 11B34, 11B37, 11B39, 68R15.

Printed Thursday 22nd September, 2011 16:45

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